EXTRA LECTURE 2:

GRA 6035

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Plan:

1. BASICS OF OPTIMIZATION: (continued)
   - (d) Determine if a function is convex / concave
   - (e) Find points that satisfy first order conditions + constraints in constrained optimalization problems.

2. MATRICES / LIN. ALGEBRA:
   - (a) Compute determinants and ranks
   - (b) Finding eigenvalues / eigenvectors
   - (c) Determine if a matrix is diagonalizable

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(d) Convex / Concave

We use the Hessian matrix of $f(x,y,z)$,

$$H(f) = f'' = \begin{pmatrix} f''_{xx} & f''_{xy} & f''_{xz} \\ f''_{xy} & f''_{yy} & f''_{yz} \\ f''_{xz} & f''_{yz} & f''_{zz} \end{pmatrix}$$

Criteria:

$f''(x,y,z)$ pos. semidefinite for all x,y,z $\iff$ f convex

$f''(x,y,z)$ neg. semidefinite for all x,y,z $\iff$ f concave

ⓐ Derivation, Hessian matrix.

Find f'_x, f'_y, f'_z and the Hessian matrix in these cases:

i) $f = xy + xz - yz$

ii) $f = x^2 + y^2 + z^2 + z^3 + 2yz - 2x + 12y$

iii) $f = x^2 + 4xy + 4y^2 + e^y - y$

iv) $f = x^2 + y^2 + y^4 + yz - 1$

v) $f = \ln(x+1) + \ln(y+1) - \ln(z-1)$

vi) $f = z \cdot \sqrt{x^2+y^2}$

vii) $f = e^{xyz}$

Solution:

i) $f'_x = y+z$
$f'_y = x-z$
$f'_z = x-y$

$$H(f) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

ii) $f'_x = 2x-2$
$f'_y = 2y+2z+12$
$f'_z = 2z+3z^2+2y$

$$H(f) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2+6z \end{pmatrix}$$

iii) $f'_x = 2x+4y$
$f'_y = 4x+8y+e^y-1$
$f'_z = 0$

$$H(f) = \begin{pmatrix} 2 & 4 & 0 \\ 4 & 8+e^y & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Check if f is convex or concave (or both) in cases i) – vii)

<u>Solution:</u>

i)

$$f'' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \qquad \begin{matrix} D_1 = 0 \\ D_2 = -1 \end{matrix} \Rightarrow \text{indefinite} \Rightarrow \left\{ \begin{matrix} \text{not convex} \\ \text{not concave} \end{matrix} \right.$$

ii)

$$f'' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2+6z \end{pmatrix} \qquad \begin{matrix} D_1 = 2 \\ D_2 = 4 \\ D_3 = 24z \end{matrix}$$

$D_3 = 24z$ – can be both pos. and neg. $\Rightarrow$ ~~indefinite~~ $\left\{ \begin{matrix} \text{not convex} \\ \text{not concave} \end{matrix} \right.$

iii)

$$f'' = \begin{pmatrix} 2 & 4 & 0 \\ 4 & 8+e^y & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{matrix} D_1 = 2 \\ D_2 = 2e^y > 0 \\ D_3 = 0 \end{matrix} \qquad \begin{matrix} \Delta_1 = 2,\ 8+e^y > 0,\ 0 \\ \Delta_2 = 2e^y > 0,\ 0,\ 0 \\ \Delta_3 = 0 \end{matrix}$$

$\Downarrow$

pos. semidefinite

$\Downarrow$

convex,
not concave

iv)

$$f'' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2+12y^2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \left. \begin{matrix} D_1 = 2 \\ D_2 = 4+24y^2 > 0 \\ D_3 = -2 < 0 \end{matrix} \right\} \Rightarrow \begin{matrix} \text{not convex} \\ \text{not concave} \end{matrix}$$

v) $$f'' = \begin{pmatrix} -\frac{1}{(x+1)^2} & 0 & 0 \\ 0 & -\frac{1}{(y+1)^2} & 0 \\ 0 & 0 & \frac{1}{(z-1)^2} \end{pmatrix}$$

$$D_1 = -\frac{1}{(x+1)^2} < 0$$

$$D_2 = +\frac{1}{(x+1)^2(y+1)^2} > 0$$

$$D_3 = +\frac{1}{(x+1)^2(y+1)^2(z-1)^2} > 0$$

$\Downarrow$

not concave
not convex

vi) $$f'' = \begin{pmatrix} \frac{y^2 z}{(x^2+y^2)^{3/2}} & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$$

$$D_1 = \frac{y^2 z}{(x^2+y^2)^{3/2}}$$

can be pos. and neg.

$\Downarrow$

not concave
not convex

vii) $$f'' = \begin{pmatrix} y^2 z^2 & z(xyz+1) & y(xyz+1) \\ z(xyz+1) & x^2 z^2 & x(xyz+1) \\ y(xyz+1) & x(xyz+1) & x^2 y^2 \end{pmatrix} e^{xyz}$$

$$D_1 = y^2 z^2 e^{xyz} \geq 0 \qquad \Delta_1 = x^2 z^2 e^{xyz},\ x^2 y^2 e^{xyz} \geq 0$$

$$D_2 = x^2 y^2 z^4 - z^2 (xyz+1)^2 = -2xyz^3 - z^2 = -z^2(1+2xyz)$$

can be both pos. and neg.

$\Downarrow$

not convex
not concave

(e) Find all points that satisfy first order conditions (FOC), constraints (C) and, in case of inequality constraints, the complementary slackness conditions (CSC).

Ex: max/min $f(x,y) = \ln(x+1) + \ln(y+1)$ subj. to $\begin{cases} y \leq 5 \\ x+y \leq 2 \end{cases}$

ADMISSIBLE PTS:
"
PT that satisfy the constraint (C)

$$\boxed{\begin{array}{l} y \leq 5 \\ x+y \leq 2 \end{array}}$$

FIRST ORDER CONDITIONS:
(FOC)

$$\mathcal{L} = \ln(x+1) + \ln(y+1) - \lambda_1 \cdot (y) - \lambda_2 (x+y)$$

$$\boxed{\begin{array}{l} \mathcal{L}'_x = \dfrac{1}{x+1} - \lambda_2 \cdot 1 = 0 \\ \mathcal{L}'_y = \dfrac{1}{y+1} - \lambda_1 \cdot 1 - \lambda_2 \cdot 1 = 0 \end{array}}$$

COMPL. SLACKNESS COND:
(CSC)

$$\boxed{\begin{array}{l} \lambda_1 \geq 0 \\ \lambda_2 \geq 0 \\ \text{If } y < 5 \text{ then } \lambda_1 = 0 \\ \text{If } x+y < 2 \text{ then } \lambda_2 = 0 \end{array}}$$

Exercises: Find the pts that satisfy FOC + C (+ CSC)

i) max/min $f(x,y,z) = 12x - 9y^2 + 2z^3$ subj. to $\begin{cases} z - x = 0 \\ y - z = 0 \end{cases}$

ii) max/min $f(x,y) = \ln(x+1) + \ln(y+1)$ subj. to $\begin{cases} y \leq 5 \\ x + y \leq 2 \end{cases}$

iii) max/min $f(x,y,z) = 2z$ — " — $\begin{cases} x^2 + y^2 = 2 \\ x + y + z = 1 \end{cases}$

Solution:

i) FOC: $\mathcal{L} = 12x - 9y^2 + 2z^3 - \lambda_1(z - x) - \lambda_2(y - z)$

$$\boxed{\begin{aligned} \mathcal{L}'_x &= 12 + \lambda_1 = 0 \\ \mathcal{L}'_y &= -18y - \lambda_2 = 0 \\ \mathcal{L}'_z &= 6z^2 - \lambda_1 + \lambda_2 = 0 \end{aligned}}$$

C: $\boxed{\begin{aligned} z - x = 0 \\ y - x = 0 \end{aligned}}$

No csc's since equality constraints

$\Downarrow$

$x = y = z$

$12 + \lambda_1 = 0 \Rightarrow \lambda_1 = -12$

$-18y - \lambda_2 = 0 \Rightarrow \lambda_2 = -18y$

$6z^2 - \lambda_1 + \lambda_2 = 0 \Rightarrow 6y^2 + 12 - 18y = 0$ abc

$y = 1, 2$

$y = 1$: $x = y = z = 1$, $\lambda_1 = -12$, $\lambda_2 = -18$

$y = 2$: $x = y = z = 2$, $\lambda_1 = -12$, $\lambda_2 = -36$

ii) FOC:
$$\boxed{\begin{aligned} &\frac{1}{x+1} - \lambda_2 = 0 \\ &\frac{1}{y+1} - \lambda_1 - \lambda_2 = 0 \end{aligned}}$$

See above

C:
$$\boxed{\begin{aligned} y &\le 5 \\ x+y &\le 2 \end{aligned}}$$

CSC:
$$\boxed{\begin{aligned} &\lambda_1 \ge 0 \\ &\lambda_2 \ge 0 \\ &y < 5 \Rightarrow \lambda_1 = 0 \\ &x+y<2 \Rightarrow \lambda_2 = 0 \end{aligned}}$$

Cases:

A) $\left.\begin{aligned} y &= 5 \\ x+y &= 2 \end{aligned}\right\}$ $y = \underline{5}$, $x = 2 - y = \underline{-3}$ $\quad \lambda_2 = \frac{1}{-3+1} = -1/2$ impossible $\Rightarrow$ no soln in A)

B) $\left.\begin{aligned} y &= 5 \\ x+y &< 2 \end{aligned}\right\} \Rightarrow \lambda_2 = 0 \Rightarrow \frac{1}{x+1} = 0$ impossible $\Rightarrow$ no soln in B)

C) $\left.\begin{aligned} y &< 5 \\ x+y &= 2 \end{aligned}\right\} \Rightarrow \lambda_1 = 0 \Rightarrow \lambda_2 = \frac{1}{x+1} = \frac{1}{y+1} \Rightarrow x = y$

$\Rightarrow x = y = 1$, $\lambda_2 = 1/2$, $\frac{1}{2} = \lambda_1 + \lambda_2 = \lambda_2 \Rightarrow \lambda_2 = 1/2$

One soln: $x = y = 1$, $\lambda_1 = 0$, $\lambda_2 = 1/2$

D) $\left.\begin{aligned} y &< 5 \\ x+y &< 2 \end{aligned}\right\}$ $\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= 0 \end{aligned}$ $\Rightarrow \frac{1}{x+1} = 0$ impossible $\Rightarrow$ no soln. in D

Concl: Solution $(x,y) = (1,1)$, $\lambda_1 = 0$, $\lambda_2 = 1/2$

(iii) FOC: $\mathcal{L} = 2z - \lambda_1(x^2+y^2) - \lambda_2(x+y+z)$

$$\boxed{\begin{aligned} \mathcal{L}'_x &= -\lambda_1 \cdot 2x - \lambda_2 = 0 \\ \mathcal{L}'_y &= -\lambda_1 \cdot 2y - \lambda_2 = 0 \\ \mathcal{L}'_z &= 2 - \lambda_2 = 0 \end{aligned}}$$

C: $\boxed{\begin{aligned} x^2+y^2 &= 2 \\ x+y+z &= 1 \end{aligned}}$

$\lambda_1 \neq 0$ since $\lambda_1 = 0$ is impossible

$$\left.\begin{aligned} &\lambda_2 = 2 \\ &-2\lambda_1 x - 2 = 0 \Rightarrow \lambda_1 x = -1 \\ &-2\lambda_1 y - 2 = 0 \Rightarrow \lambda_1 y = -1 \end{aligned}\right\} \Rightarrow x = -\frac{1}{\lambda_1} = y$$

$$\Downarrow$$

$$x = y$$

$x^2+y^2 = 2x^2 = 2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

$\underline{x=1}$: $y = 1$, $z = 1-x-y = -1$, $\lambda_1 = -\frac{1}{x} = -1$, $\lambda_2 = 2$

$\underline{x=-1}$: $y=-1$, $z = 1-x-y = 3$, $\lambda_1 = -\frac{1}{x} = 1$, $\lambda_2 = 2$

$$\Downarrow$$

Solutions: $(x, y, z; \lambda_1, \lambda_2) = (1, 1, -1; -1, 2)$

$(-1, -1, 3; 1, 2)$