EXTRA LECTURE 5

GRA 6035

MATHEMATICS

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PLAN:

1. DIFFERENTIAL EQUATIONS
2. OTHER BASIC PROBLEMS : DIFFERENCE EQUATIONS
3. CONSTRINED OPTIMIZATION : ADVANCED TOPICS

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1. OK
2. DIFFERENCE EQUATIONS

FIRST ORDER LINEAR (CONSTANT COEFFICIENTS)

Ex:

$$y_{t+1} - y_t = r \cdot y_t - 1000$$

$$y_{t+1} - (1+r) y_t = -1000$$

General form

$$\boxed{y_{t+1} - r\, y_t = f_t}$$

$$y_t = y_t^h + y_t^p$$

$$= C_1 \cdot (1+r)^t + \frac{1000}{r}$$

y_t^h: $y_t^h = C_1 \cdot (1+r)^t$

y_t^p: $y_t = A$

$$A - (1+r) \cdot A = -1000$$

$$\cancel{A} - \cancel{A} - rA = -1000$$

$$A = \frac{1000}{r}$$

$$y_t^p = \frac{1000}{r}$$

Ex:

$$y_{t+1} + 2y_t = 5$$

$r + 2 = 0$
$r = -2$

$$y_t = y_t^h + y_t^p = C \cdot (-2)^t + 5/3$$

$y_t^p = A$: $A + 2A = 5$
$3A = 5 \Rightarrow A = 5/3$

LINEAR SECOND ORDER (WITH CONSTANT COEFF.)

General form:

$$y_{t+2} + a y_{t+1} + b y_t = f_t$$

Ex: $y_{t+2} - y_{t+1} - y_t = 0$

$$y_t = y_t^h + y_t^p$$
$$= C_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^t + C_2 \left(\frac{1-\sqrt{5}}{2}\right)^t$$

$y_t^p = 0$ since it is homog.

y_t^h: $r^2 - r - 1 = 0$

$$r = \frac{1 \pm \sqrt{1 - 4 \cdot (-1)}}{2}$$
$$= \frac{1 \pm \sqrt{5}}{2}$$

$$y_t^h = C_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^t + C_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right)^t$$

Ex:

i) $y_{t+1} + 5y_t = t$

ii) $y_{t+2} - 4y_{t+1} + 3y_t = t + 1$

iii) $y_{t+2} + 6y_{t+1} + 9y_t = 32$

Solution:

i) $y_{t+1} + 5y_t = t$

$$y_t = y_t^h + y_t^p = C \cdot (-5)^t + y_t^p$$
$$= \cancel{C \cdot (-5)^t - \frac{1}{4}t - \frac{1}{16}}$$
$$= C \cdot (-5)^t + \frac{1}{6}t - \frac{1}{36}$$

y_t^h:

$r + 5 = 0 \Rightarrow r = -5$

y_t^p: $\left.\begin{array}{l} f_t = t \\ f_{t+1} = t+1 \end{array}\right\}$ try $At + B$

$y_{t+1} = A(t+1) + B = At + A + B$

$y_t = At + B$

$(At + A + B) + 5(At + B) = t$

$(\cancel{4}6A)t + (A + \cancel{4}6B) = t$

$A = 1/6 \Leftarrow \cancel{4}6A = 1$

$B = -1/36$ $\quad A + \cancel{4}6B = 0$

$\cancel{A = 1/4}$ $\cancel{B = A/4 = -1/16}$

ii) $y_{t+2} - 4y_{t+1} + 3y_t = t+1$

y_t^h: $r^2 - 4r + 3 = 0$
$r = 3, 1$ $\Rightarrow$ $y_t^h = C_1 \cdot 3^t + C_2 \cdot 1^t$
$= \underline{C_1 \cdot 3^t} + C_2$

y_t^p: $\left.\begin{array}{l} f_t = t+1 \\ f_{t+1} = t+2 \\ f_{t+2} = t+3 \end{array}\right\}$ try $\underline{At+B}$

$y_t = At + B$
$y_{t+1} = A(t+1) + B$
$y_{t+2} = A(t+2) + B$

$\big(A(t+2) + B\big) - 4\big(A(t+1) + B\big) + 3(At+B) = t+1$

$\ldots + 0 \cdot At = t+1$
impossible!

Try $y_t = (At+B) \cdot t = \underline{At^2 + Bt}$
$y_{t+1} = A(t+1)^2 + B(t+1)$
$y_{t+2} = A(t+2)^2 + B(t+2)$

$\left[A(t+2)^2 + B(t+2)\right] - 4\left[A(t+1)^2 + B(t+1)\right]$
$+ 3\left[At^2 + Bt\right] = t+1$

$\left(At^2 - 4At^2 + 3At^2\right)$ $= 0t^2$
$+ \left(4At + Bt\right) - 8At - 4Bt + 3Bt$ $+ 1 \cdot t$
$+ 4A + 2B - 4A - 4B$ $+ 1$

$-4At - 2B = t+1$

$-4A = 1$ $\quad A = -1/4$
$-2B = 1$ $\quad \underline{B = -1/2}$

$$\underline{\underline{y_t = C_1 \cdot 3^t + C_2 - \tfrac{1}{4}t^2 - \tfrac{1}{2}t}}$$

iii) $y_{t+2} + 6y_{t+1} + 9y_t = 32$

y_t^h: $r^2 + 6r + 9 = 0$

$r = -3$ (double root) $\Rightarrow$ $y_t^h = c_1(-3)^t + c_2 t(-3)^t$

y_t^p: $y_t = A$

$A + 6A + 9A = 32$

$16A = 32$

$A = 2$

$$y_t = y_t^h + y_t^p = c_1 \cdot (-3)^t + c_2 t(-3)^t + 2$$

3 Constrained optimalization problems.

Typical examples:

a) max xy when $x+4y=16$ Equality constr.
b) max xy when $x+4y\leq 16$ Ineq. constr.

Lagrange problems: Equality constraints

max xy when $x+4y=16$ <u>max = global max</u>

i) Find candidates for max: F.O.C. + Constraint

$$\mathcal{L} = xy - \lambda\cdot(x+4y \quad)$$

$$\boxed{\begin{aligned} \mathcal{L}'_x &= y - \lambda\cdot 1 = 0 \\ \mathcal{L}'_y &= x - \lambda\cdot 4 = 0 \end{aligned}} \leftarrow \text{F.O.C.}$$

$$\boxed{x + 4y = 16} \leftarrow \text{Constraint}$$

$$\left.\begin{aligned} y &= \lambda \\ x &= 4\lambda \end{aligned}\right\} \quad (4\lambda)+4(\lambda)=16, \quad 8\lambda=16, \quad \lambda=2 \qquad \left.\begin{aligned} x&=8 \\ y&=2 \end{aligned}\right\}$$

One candidate:

$$(x,y;\lambda) = (8,2;2)$$

ii) Check if $(x,y)=(8,2)$ is max.

If $\mathcal{L}(x,y;\lambda=2)$ is concave, then $(x,y)=(8,2)$ is a global max.

$\mathcal{L}(x,y;2) = xy - 2\cdot(x+4y) = xy - 2x - 8y$

$H(\mathcal{L}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad D_1 = 0$, $D_2 = -1$ $\quad$ not concave, (not convex)

If $x + 4y = 16$, then $x = 16 - 4y$

and

$$xy = (16-4y)y = 16y - 4y^2 = g(y)$$

$$g'(y) = 16 - 8y = 0$$
$$\Rightarrow y = 2$$
$$g''(y) = -8$$

This proves that

$y=2$, $x=8$

is a global max.

If $(x^*, y^*, z^*; \lambda^*)$ satisfy FOC + Constraints in a Lagrange problem, then:

If $\mathcal{L}(x,y,z;\lambda^*)$ is concave, then (x^*, y^*, z^*) is max

If $\mathcal{L}(x,y,z;\lambda^*)$ is convex, then (x^*, y^*, z^*) is min

Problems:

(a) Envelope thms

i) Consider $\max \; -a^3 x^4 + 15x^3 - e^a x^2 + 17$ around $a=1$.

How is this problem affected by variation of the parameter a?

ii) What is the effect of a unit increase in parameter a on

$$\max \; f(x;a) = -x^2 + 2ax + 4a^2$$

(b) Lagrange problems:

i) max xy when $x + 4y = 16$

ii) max $x^2 y$ " $2x^2 + y^2 = 3$

iii) max xyz " $\begin{cases} x^2 + y^2 = 1 \\ x + z = 1 \end{cases}$

iv) min $x^2 + y^2$ " $x^2 + xy + y^2 = 3$

v) min $x^2 + y^2 + z^2$ " $\begin{cases} 3x + y + z = 5 \\ x + y + z = 1 \end{cases}$

vi) max/min $x + y + z^2$ " $\begin{cases} x^2 + y^2 + z^2 = 1 \\ y = 0 \end{cases}$

vii) max $yz + xz$ " $\begin{cases} y^2 + z^2 = 1 \\ xz = 3 \end{cases}$

viii) max $x^2 y^2 z^2$ " $x^2 + y^2 + z^2 = 3$

c) Bordered Hessian:

i) Consider $\max\ x^2y^2z^2$ when $x^2+y^2+z^2=3$

Find admissible points satisfying first order conditions, and use the bordered Hessian to check if any of them are local max.

What about global max?

d) Kuhn-Tucker problems:

i) $\max\ xy$ when $x+4y \leq 16$

ii) max/min x^2y when $2x^2+y^2 \leq 3$

iii) $\max\ xyz$ when $\begin{cases} x^2+y^2 \leq 1 \\ x+z \geq 1 \end{cases}$

iv) $\max\ xy$ when $x^2+y^2 \leq 1$

v) $\max\ xyz$ " $\begin{cases} x+y+z \leq 1 \\ x \geq 0 \\ y \geq 0 \\ z \geq 0 \end{cases}$

vi) $\max\ xyz+z$ " $\begin{cases} x^2+y^2+z \leq 6 \\ x \geq 0 \\ y \geq 0 \\ z \geq 0 \end{cases}$