Solutions:	GRA 60353 Mathematics	
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Permitted examination support material:	A bilingual dictionary and BI-approved calculator TEXAS INSTRUMENTS BA II Plus	
Answer sheets:	Squares	
	Counts 80% of GRA 6035	The subquestions are weighted equally
Re-sit exam		Responsible department: Economics

Question 1.

(a) We compute the partial derivatives $f'_x = 7y + 4(z-x)^3$, $f'_y = 7x + 10y$ and $f'_z = -4(z-x)^3$. The stationary points are given by the equations

$$7y + 4(z-x)^3 = 0, \quad 7x + 10y = 0, \quad -4(z-x)^3 = 0$$

The last equation gives $x = z$ and the first then gives $y = 0$. From the second equation, we get that $x = 0$, hence $z = 0$. The stationary points are therefore given by $(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$.

(b) We compute the second order partial derivatives of f and form the Hessian matrix

$$f'' = \begin{pmatrix} -12(z-x)^2 & 7 & 12(z-x)^2 \\ 7 & 10 & 0 \\ 12(z-x)^2 & 0 & -12(z-x)^2 \end{pmatrix}$$

We see that the matrix has second leading principal minor $D_2 = -120(z-x)^2 - 49 < 0$ and therefore f is **not convex and not concave**.

Question 2.

(a) The homogeneous equation $y'' - 7y' + 12y = 0$ has characteristic equation $r^2 - 7r + 12 = 0$, and therefore roots $r = 3, 4$. Hence the homogeneous solution is $y_h(t) = C_1 e^{3t} + C_2 e^{4t}$. To find a particular solution of $y'' - 7y' + 12y = t - 3$, we try $y = At + B$. This gives $y' = A$ and $y'' = 0$, and substitution in the equation gives $-7A + 12(At + B) = t - 3$. Hence $A = 1/12$ and $B = -29/144$ is a solution, and $y_p(t) = \frac{1}{12}t - \frac{29}{144}$ is a particular solution. This gives general solution

$$y(t) = \mathbf{C_1 e^{3t} + C_2 e^{4t} + \frac{1}{12} t - \frac{29}{144}}$$

(b) We rewrite the differential equation as $3y^2 y' = 1 - te^t$. This differential equation is separable, and we integrate on both sides to solve it:

$$\int 3y^2 \, dy = \int 1 - te^t \, dt \quad \Rightarrow \quad y^3 = t - \int te^t \, dt = t - (te^t - e^t) + \mathcal{C} = t - te^t + e^t + \mathcal{C}$$

This gives

$$y = \sqrt[3]{t - te^t + e^t + \mathcal{C}}$$

(c) We rewrite the differential equation as $(t/y) \cdot y' + (\ln y - 1) = 0$, and try to find a function $u = u(y,t)$ such that $u'_t = \ln y - 1$ and $u'_y = t/y$ to find out if the equation is exact. We see that $u = t\ln y - t$ is a solution, so the differential equation is exact, with solution $t \ln y - t = \mathcal{C}$ or $\ln y - 1 = \mathcal{C}/t$. The solution is therefore

$$\ln y = \frac{\mathcal{C}}{t} + 1 \quad \Rightarrow \quad y = \mathbf{e}^{\mathcal{C}/\mathbf{t}+\mathbf{1}}$$

QUESTION 3.

(a) We compute the minor of order two in A consisting of the first two columns:

$$\begin{vmatrix} 5 & -5 \\ 2 & t-4 \end{vmatrix} = 5(t-4) + 10 = 5t - 10$$

We see that this minor is non-zero when $t \neq 2$, hence A has rank two (the maximal rank) when $t \neq 2$. When $t = 2$, we have

$$A = \begin{pmatrix} 5 & -5 & -5 \\ 2 & -2 & -2 \end{pmatrix}$$

and we see that A has rank one. This means that

$$\text{rk}(A) = \begin{cases} 2, & t \neq 2 \\ 1, & t = 2 \end{cases}$$

The three column vectors of A are not linearly independent for any values of t since the rank of A cannot be three.

(b) When $t \neq 2$, we have that $\text{rk}(A) = 2$, and and the first two columns of A are pivot columns. This impliest that $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions when $t \neq 2$ (with one degree of freedom, and we can choose the third variable to be free). When $t = 2$, we get the linear system

$$\begin{pmatrix} 5 & -5 & -5 \\ 2 & -2 & -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

and we see that this linear system is inconsistent (no solutions). We conclude that the linear system has infinitely many solutions (one degree of freedom) when $t \neq 2$, and no solutions when $t = 2$.

(c) We claim that $(A^T A)\mathbf{x} = \mathbf{0}$ has the same solutions as $A\mathbf{x} = \mathbf{0}$: If $A\mathbf{x} = \mathbf{0}$, then clearly $A^T A\mathbf{x} = A^T\mathbf{0} = \mathbf{0}$. Conversely, if $(A^T A)\mathbf{x} = \mathbf{0}$, then $\mathbf{x}^T(A^T A)\mathbf{x} = \mathbf{x}^T\mathbf{0} = \mathbf{0}$, and this implies that $(A\mathbf{x})^T(A\mathbf{x}) = \mathbf{0}$. But if an n-vector $\mathbf{y}$ satisfy $\mathbf{y}^T\mathbf{y} = \mathbf{0}$, then $y_1^2 + \cdots + y_n^2 = 0$ and therefore $y_1 = y_2 = \cdots = y_n = 0$; that is $\mathbf{y} = \mathbf{0}$. When we apply this to $\mathbf{y} = A\mathbf{x}$, we see that $A\mathbf{x} = \mathbf{0}$. This proves the claim. We conclude that the number of degrees of freedom of $(A^T A)\mathbf{x} = \mathbf{0}$ is the same as the number of degrees of freedom of $A\mathbf{x} = \mathbf{0}$, which is

$$3 - \text{rk}\, A = \begin{cases} 1, & t \neq 2 \\ 2, & t = 2 \end{cases}$$

Alternatively, we could solve this problem computing $A^T A$ explicitly.

QUESTION 4.

(a) The Lagrangian for this problem is given by $\mathcal{L} = x^2yz - \lambda(x^2 + 2y^2 - 2z^2)$, and the first order conditions are

$$\begin{aligned}
\mathcal{L}'_x &= 2xyz - 2x\lambda = 0 \\
\mathcal{L}'_y &= x^2z - 4y\lambda = 0 \\
\mathcal{L}'_z &= x^2y + 4z\lambda = 0
\end{aligned}$$

The complementary slackness conditions are given by $\lambda \geq 0$, and $\lambda = 0$ if $x^2 + 2y^2 - 2z^2 < 32$. Let us find all admissible points satisfying these conditions. We solve the first order conditions, and get $x = 0$ or $\lambda = yz$ from the first equation. If $x = 0$, then $y\lambda = z\lambda = 0$, so either $\lambda = 0$ or $\lambda \neq 0 \Rightarrow y = z = 0$. In the first case, the constraint gives $2y^2 - 2z^2 \leq 32 \Rightarrow y^2 - z^2 \leq 16$. This gives the solution

$$\boxed{x = 0,\ y^2 - z^2 \leq 16,\ \lambda = 0}$$

In the second case, $x = y = z = 0$, $\lambda \neq 0$. Since the constraint is not binding, this is not a solution. If $x \neq 0$, then $\lambda = yz$, and the last two first order conditions give

$$\begin{aligned}
x^2z - 4y \cdot yz = 0 &\Rightarrow z(x^2 - 4y^2) = 0 \\
x^2y + 4z \cdot yz = 0 &\Rightarrow y(x^2 + 4z^2) = 0
\end{aligned}$$

If $z = 0$, then $x^2 + 4z^2 \neq 0 \Rightarrow y = 0$ and $\lambda = yz = 0$ give a solution

$$\boxed{0 < x^2 \leq 32,\ y = z = 0,\ \lambda = 0}$$

If both $x \neq 0$ and $z \neq 0$, then $x^2 - 4y^2 = 0 \Rightarrow y \neq 0$, and $x^2 = 4y^2 = -4z^2$. This is not possible. Therefore, there are no more solutions.

(b) For any number a, we have that $x = \sqrt{32}, y = a, z = a$ is an admissible point for any value of a, since

$$x^2 + 2y^2 - 2z^2 = 32 + 2a^2 - 2a^2 = 32$$

The value of the function $f(x, y, z) = x^2yz$ at this point is $f(\sqrt{32}, a, a) = 32a^2$. When $a \to \infty$, we see that $f(\sqrt{32}, a, a) = 32a^2 \to \infty$, and this means that there is no maximum value. Hence the maximum problem has no solution.