LECTURE 3

GRA 6035

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REVIEW OF LECTURE 2:

- Matrices and matrix operations, matrix laws
  - addition / subtraction, scalar multiplication, multiplication, transpose, inverse, determinants
- Rank and minors:

Reformulation of rank: $\operatorname{rk} A = \begin{cases} \text{maximal order of a non-zero} \\ \text{minor in } A \end{cases}$

Ex:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$$

2×2

Minor of order 2: $\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1 \cdot 1 - 3 \cdot 2 = -5 \neq 0$

$\Rightarrow \operatorname{rk} A = 2$

Fact:

If A is an $n \times n$-matrix, then $\begin{cases} \operatorname{rk} A = n & \text{if } |A| \neq 0 \\ \operatorname{rk} A < n & \text{if } |A| = 0 \end{cases}$

Ex:

$$A = \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 3 & 1 & 2 \end{array}\right)$$

2×3

$\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5 \neq 0$

$\Rightarrow \operatorname{rk} A = 2$

Interpretation of matrix as augmented matrix:

$\left.\begin{array}{l} \operatorname{rk} \text{augmented matrix} = 2 \\ \operatorname{rk} \text{coefficient matrix} = 2 \end{array}\right\} \Rightarrow$ One unique solution.

Ex:

$$A = \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 3 & -1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

3×4

$\text{rk } A = ?$

Minor of order 3: $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & -1 \\ 0 & 0 & 0 \end{vmatrix} = 0$

All 3-minors are 0.

Minor of order 2: $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1 \neq 0$

$\Rightarrow \text{Rk } A = \underline{\underline{2}}.$

Interpretation as an augmented matrix:

$\left.\begin{array}{l} \text{rk augmented matrix} = 2 \\ \text{rk coefficient matrix} = 2 \end{array}\right\}$

$\Rightarrow$ There are infinitely many solutions, one degree of freedom: $\underset{\substack{\uparrow \\ n = \\ \# \text{variables}}}{3} - \underset{\substack{\uparrow \\ \text{rank}}}{2} = \underline{1}$

* Equation 3 is superflous (unnecessary) (follows from the other equations)

* Variable 3 = z is free

$$\begin{array}{l} x + 2y + 3z = 4 \\ 2x + 3y - z = 5 \end{array} \Rightarrow \begin{array}{l} x + 2y = 4 - 3z \\ 2x + 3y = 5 + z \end{array} \Rightarrow \text{Solve for } x \text{ and } y.$$

$-2 \cdot \mathrm{I} + \mathrm{II}$: $\quad -y = -3 + 7z$

$\Rightarrow y = \underline{3 - 7z}$

$x + 2y = x + 2(3 - 7z) = 4 - 3z$

$x = 4 - 3z - 6 + 14z$

$= \underline{-2 + 11z}$

Solution: $\begin{cases} x = -2 + 11z \\ y = 3 - 7z \\ z = \text{free} \end{cases}$

Fact: $\operatorname{rk} A = \operatorname{rk} A^T$

Plan for LECTURE 3:

① Vectors
② Linear independence
③ Rank and linear independence

$\Big\}$ [FMEA] 1.1 – 1.3

① Vectors

A column vector is an $m \times 1$-matrix; i.e. a matrix with one column. It is also called an m-vector. The collection of all m-vectors is written $\mathbb{R}^m$.

Ex:

$$\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \underline{w} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

2-vector 3-vector

Note: We write
small letter for numbers – a
capital letter for matrices – A
bold face / underlined letters for vectors – $\underline{a}$

We compute with vectors using matrix algebra – since vectors are special cases of matrices:

addition, subtraction, scalar mult. are defined for vectors

Ex:

$$2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 3 \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 6 \\ 9 \end{pmatrix} = \begin{pmatrix} -4 \\ -5 \end{pmatrix}$$

Vectors can be interpreted geometrically. An m-vector $\underline{v}$ in $\mathbb{R}^m$ has the form

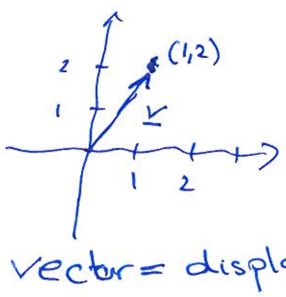
$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix}$$

and we interpret this vector as the directed line segment from the origin $(0, 0, .., 0)$ to the point $(v_1, v_2, .., v_m)$ in an m-dimensional coordinate system:

Ex:

$\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

2-vector



vector = displacement

Addition and scalar multiplication of vectors can be interpreted geometrically

Parallelogram rule for addition of two vectors:

If $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^2$ are represented as points in the plane, then $\mathbf{u}+\mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are $\mathbf{0}$, $\mathbf{u}$ and $\mathbf{v}$. (Note that $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.)

EXAMPLE: Let $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Graphs of $\mathbf{u}, \mathbf{v}$ and $\mathbf{u}+\mathbf{v}$ are given below:

$\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 3 \\ 4 \end{pmatrix}}}$

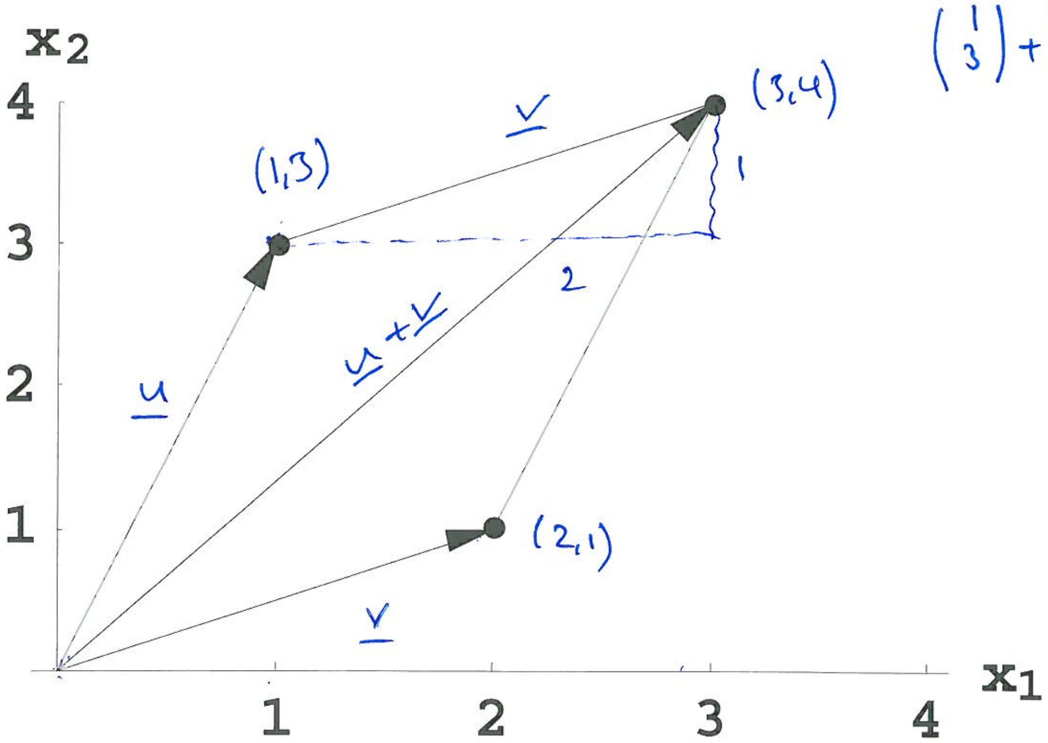
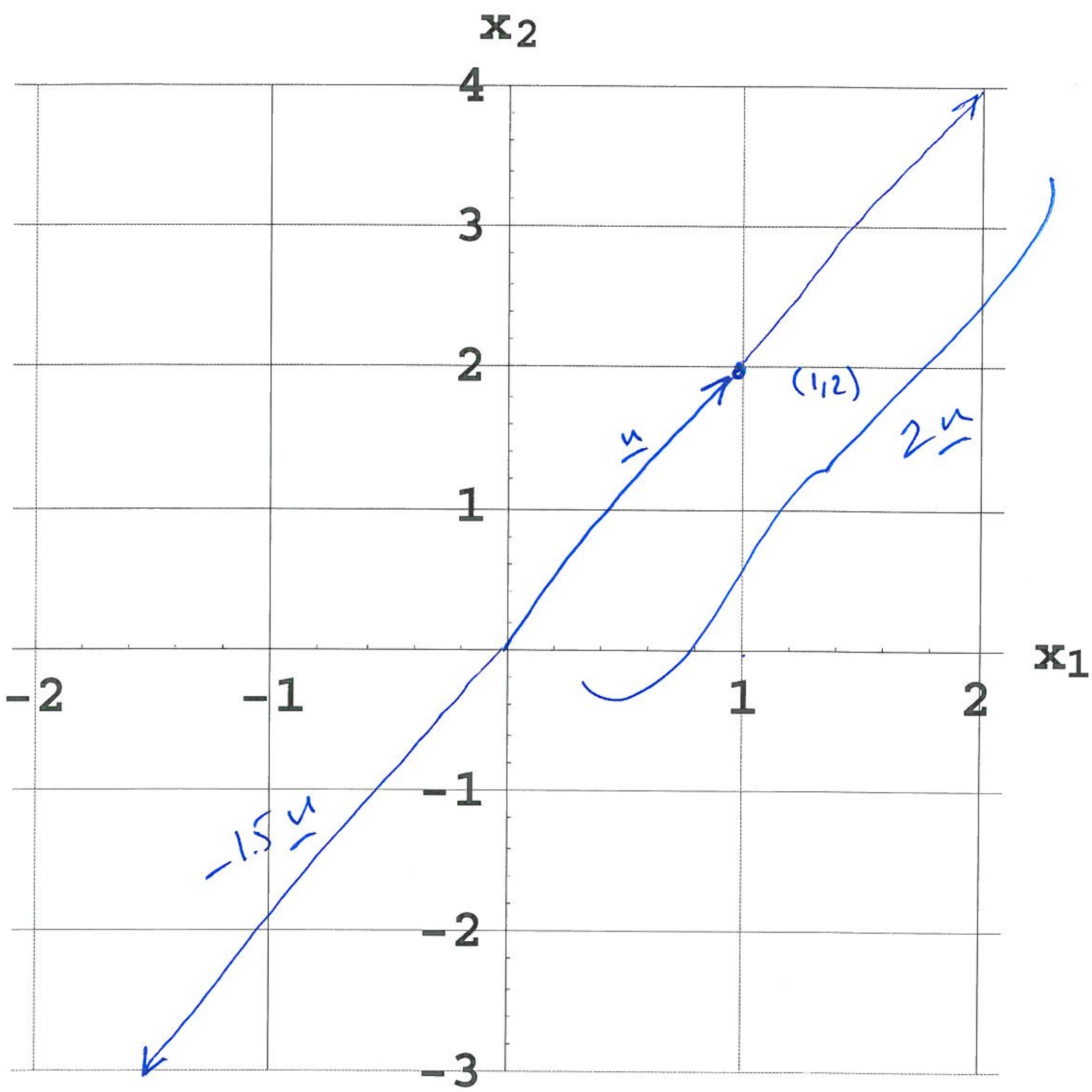


Illustration of the Parallelogram Rule

EXAMPLE: Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Express $\mathbf{u}$, $2\mathbf{u}$, and $\frac{-3}{2}\mathbf{u}$ on a graph.

$2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$



$-1.5 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1.5 \\ -3 \end{pmatrix}$

Line and line segments

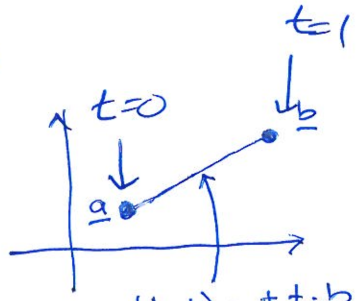
Let $\underline{a}$ and $\underline{b}$ be two points in m-dimensional space $\mathbb{R}^m$.
The **line segment** from $\underline{a}$ to $\underline{b}$ is the collection of points

$$[\underline{a},\underline{b}] = \{ (1-t)\cdot\underline{a} + t\cdot\underline{b} \mid t\in[0,1] \}$$

= { all the points that can be written on the form

$(1-t)\,\underline{a} + t\cdot\underline{b}$

with $t\in[0,1]$

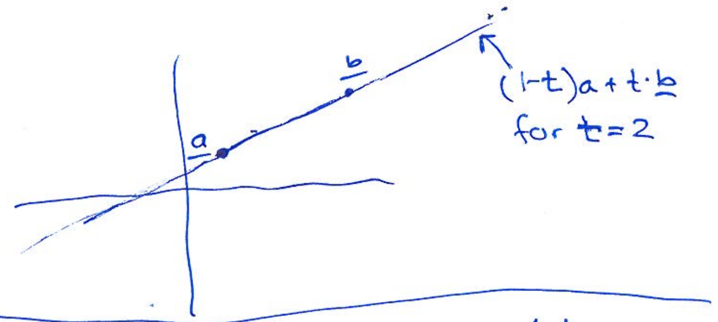


$$(1-t)\underline{a} + t\cdot\underline{b} = \underline{a} + t\cdot(\underline{b}-\underline{a})$$

The **line** through $\underline{a}$ and $\underline{b}$ are all the points that can be written

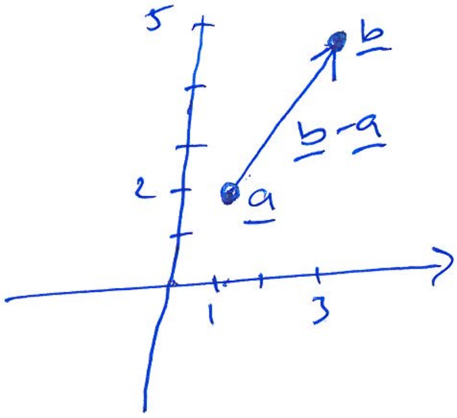
$$(1-t)\cdot\underline{a} + t\cdot\underline{b}$$

for any ~~n~~number t.



Ex: $\underline{a} = (1,2)$

$\underline{b} = (3,5)$



$\underline{a} = \begin{pmatrix}1\\2\end{pmatrix}$

$\underline{b} = \begin{pmatrix}3\\5\end{pmatrix}$

$\underline{b}-\underline{a} = \begin{pmatrix}3\\5\end{pmatrix} - \begin{pmatrix}1\\2\end{pmatrix} = \underline{\begin{pmatrix}2\\3\end{pmatrix}}$

$$[\underline{a},\underline{b}] = \{ \underline{a} + t\cdot(\underline{b}-\underline{a}) \mid \text{for } t\in[0,1] \}$$

$$\begin{pmatrix} t=0: & \underline{a} \\ t=1: & \underline{b} \end{pmatrix}$$

$$= \{ \underline{a}\cdot(1-t) + \underline{b}\cdot t \mid t\in[0,1] \}$$

$$\begin{pmatrix}1\\2\end{pmatrix}\cdot(1-t) + \begin{pmatrix}3\\5\end{pmatrix}\cdot t = \begin{pmatrix}1-t+3t\\2(1-t)+5t\end{pmatrix} = \underline{\underline{\begin{pmatrix}2t+1\\3t+2\end{pmatrix}}}, \quad t\in[0,1]$$

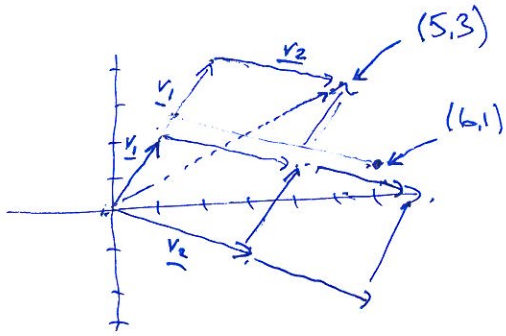
Linear combinations of vectors:

Let $\underline{v}_1, \underline{v}_2, \underline{v}_3, \dots, \underline{v}_n$ be a collection of m-vectors.
A linear combination of these vectors is an expression

$$c_1 \cdot \underline{v}_1 + c_2 \cdot \underline{v}_2 + \dots + c_n \cdot \underline{v}_n$$

where $c_1, c_2, \dots, c_n$ are numbers. A vector $\underline{w}$ is a linear combination of $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$ if there are numbers $c_1, c_2, \dots, c_n$ such that

$$c_1 \underline{v}_1 + c_2 \underline{v}_2 + \dots + c_n \underline{v}_n = \underline{w}$$



Ex: $\underline{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

a) $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ is a linear combination of $\{\underline{v}_1, \underline{v}_2\}$ since

$$2\underline{v}_1 + \underline{v}_2 = \underset{c_1}{2} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

$c_2 = 1$

b) Is $\begin{pmatrix} 6 \\ 1 \end{pmatrix}$ a linear combination of $\{\underline{v}_1, \underline{v}_2\}$?

$$\underbrace{\begin{pmatrix} 6 \\ 1 \end{pmatrix} = x_1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix}}_{\text{vecter equation}}$$

$$\begin{pmatrix} 6 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_1 \end{pmatrix} + \begin{pmatrix} 3x_2 \\ -x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 3x_2 \\ 2x_1 - x_2 \end{pmatrix}$$

$$\begin{array}{l} 6 = x_1 + 3x_2 \\ 1 = 2x_1 - x_2 \end{array} \Rightarrow \boxed{\begin{array}{l} x_1 + 3x_2 = 6 \\ 2x_1 - x_2 = 1 \end{array}}$$

linear system

$$\left.\begin{aligned} x_1 + 3x_2 &= 6 \\ 2x_1 - x_2 &= 1 \end{aligned}\right\}$$

$$\left(\begin{array}{cc|c} 1 & 3 & 6 \\ 2 & -1 & 1 \end{array}\right) \xrightarrow{-2} \left(\begin{array}{cc|c} 1 & 3 & 6 \\ 0 & -7 & -11 \end{array}\right)$$

$$x_1 + 3x_2 = 6$$
$$-7x_2 = -11 \qquad \underline{x_2 = 11/7}$$

$$x_1 + 3 \cdot (11/7) = 6$$
$$\Rightarrow x_1 = 6 - 33/7$$
$$= \underline{9/7}$$

<u>Solution:</u> $x_1 = 9/7$, $x_2 = 11/7$

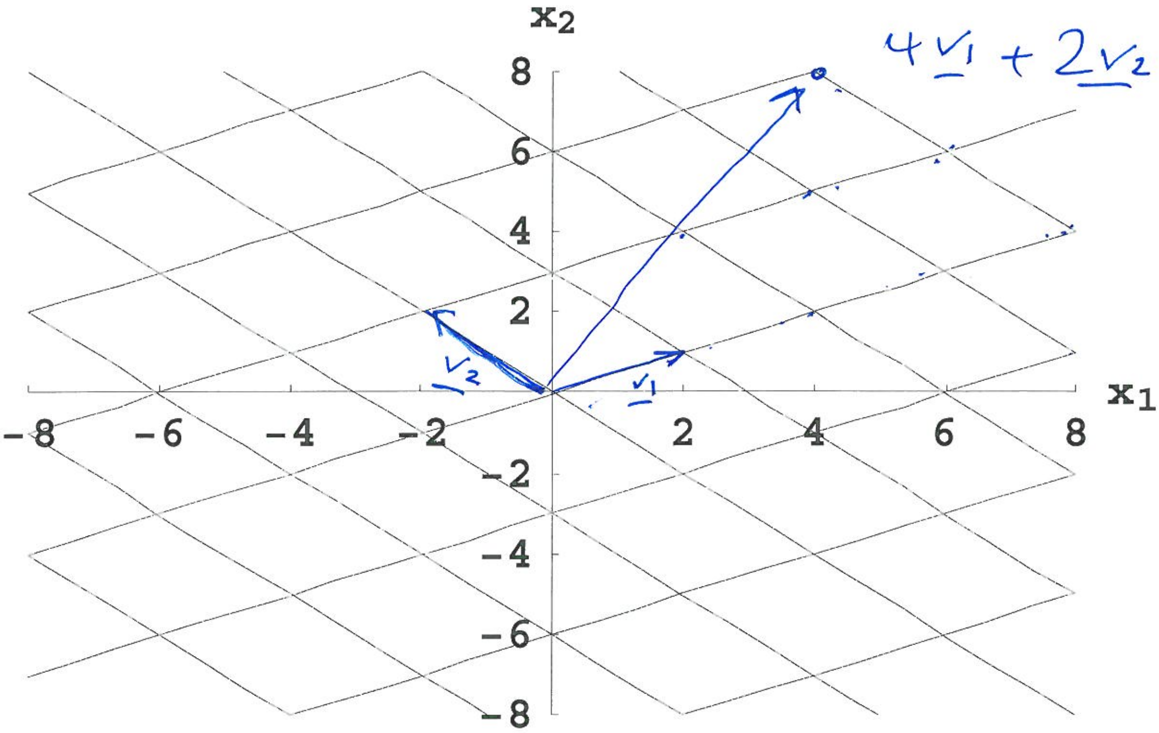
$$\Rightarrow \begin{pmatrix} 6 \\ 1 \end{pmatrix} = \frac{9}{7} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{11}{7} \cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

So $\begin{pmatrix} 6 \\ 1 \end{pmatrix}$ is a linear combination of $\underline{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\underline{v}_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

vector equation $\longleftrightarrow$ linear system $\longleftrightarrow$ matrix equation $A \cdot \underline{x} = \underline{b}$

EXAMPLE: Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Express each of the following as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{a} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$



All 2-vectors are linear combination of $\underline{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\underline{v}_2 = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$.

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = x_1 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \cdot \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$ has a solution for all b_1, b_2

$$2x_1 - 2x_2 = b_1$$
$$x_1 + 2x_2 = b_2$$

$$\begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix} = 2 \cdot 2 - 1 \cdot (-2) = 4 + 2 = 6 \neq 0.$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

② Linear independence

Let $\underline{v}_1, \underline{v}_2, \underline{v}_3, \dots, \underline{v}_n$ be m-vectors.
We say that $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$ is

(a) linearly dependant

if at least one of the vectors can be expressed as a linear combination of the others

(b) linearly independent

if none of the vectors can be written as a linear combination of the others.

Ex: $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \underline{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \underline{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$ Are $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ linearly independant?

lin. dep. $\left\{ \begin{array}{lcl} \underline{v}_1 = x \cdot \underline{v}_2 + y \cdot \underline{v}_3 & \Leftrightarrow & 1 \cdot \underline{v}_1 - x \cdot \underline{v}_2 - y \underline{v}_3 = \underline{0} \\ \text{or} & & \text{or} \\ \underline{v}_2 = x \cdot \underline{v}_1 + y \cdot \underline{v}_3 & \Leftrightarrow & -x\underline{v}_1 + \underline{v}_2 - y \underline{v}_3 = \underline{0} \\ \text{or} & & \text{or} \\ \underline{v}_3 = x \cdot \underline{v}_1 + y \cdot \underline{v}_2 & \Leftrightarrow & -x\underline{v}_1 - y\underline{v}_2 + \underline{v}_3 = \underline{0} \end{array} \right.$

$\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ are linearly dependant if and only if there is a solution to the vector equation

$$\boxed{x \cdot \underline{v}_1 + y \underline{v}_2 + z \underline{v}_3 = \underline{0}}$$

different from $(x, y, z) = (0, 0, 0)$.

In the example: $x_1 \cdot \underline{v}_1 + x_2 \underline{v}_2 + x_3 \underline{v}_3 = \underline{0}$

$$x_1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + x_3 \cdot \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 - x_2 + x_3 &= 0 \\ x_1 + 2x_2 + 4x_3 &= 0 \end{aligned}\right\} \iff \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

homogeneous linear system (See Ch. 3.3 of [LS6E])

Possibilities for solutions:

unique solution	$(x,y,z) = (0,0,0)$ trivial solution	linearly independent
infinitely many solutions	$(x,y,z) = (0,0,0)$ and many others	linearly dependent

Solution:

(no free var's)

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{vmatrix} = -6 \neq 0 \implies \text{unique solution } (x,y,z) = (0,0,0)$$

$\implies \{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ linearly independent

In general, if $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$ are m-vectors, then we consider the vecter equation

$$\boxed{x_1 \cdot \underline{v}_1 + x_2 \underline{v}_2 + \dots + x_n \cdot \underline{v}_n = \underline{0}}$$

<u>Two cases:</u>

only one solution (the trivial solution $(x_1, \dots, x_n) = (0, \dots, 0)$)	$\{\underline{v}_1, \dots, \underline{v}_n\}$ linearly independat
many solutions (non-trivial solutions)	$\{\underline{v}_1, \dots, \underline{v}_n\}$ linearly dependant

<u>The special case $m=n$:</u>

$$A = \left(\underline{v}_1 \,\Big|\, \underline{v}_2 \,\Big|\, \underline{v}_3 \,\Big|\, \dots \,\Big|\, \underline{v}_n \right)$$

$n \times n$-matrix

$\lvert A \rvert \neq 0$:	linearly independent
$\lvert A \rvert = 0$:	linearly dependant

If $m \neq n$, the matrix A is <u>not</u> square, and we cannot use $\lvert A \rvert$ to distinguish the two cases.

We must use <u>$rk(A)$</u> instead!

③ Rank and linearly independant vectors

Reformulation of rank:

The rank of a matrix is equal to the maximal number of linearly independent column vectors in the matrix.

A $m\times n$-matrix (n columns)

- $rk\,A = n$: All n column vecters are linearly independant
- $rk\,A < n$: Not all n column vecter are linearly independant

Ex:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 7 & 8 \\ 1 & -1 & 2 & 4 \end{pmatrix} \leftarrow \left\{ \underline{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \underline{v}_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \underline{v}_3 = \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix}, \underline{v}_4 = \begin{pmatrix} 4 \\ 8 \\ 4 \end{pmatrix} \right.$$

3x4-matrix

$rk\,A = 4$? $\to$ Linearly independent

$rk\,A < 4$? $\to$ Linearly dependant

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 7 \\ 1 & -1 & 2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 7 \\ -1 & 2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 3 \\ 1 & 7 \end{vmatrix} = 9 + 11 = 20 \neq 0$$

$\Rightarrow rk\,A = 3 \Rightarrow x_4$ free $\Rightarrow$ the vector $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\}$ are lin. dependant

But: $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ are linearly independant.

Conclusion:

The rank $rk(A)$ of an $m \times n$-matrix does not just tell us if the column vectors in A,

$$A = \left(\underline{v_1} \middle| \underline{v_2} \middle| \cdots \middle| \underline{v_n} \right)$$

are linearly independent. If $rk\, A < n$, then the pivot columns tells us how to pick a maximal subset of $\{\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}\}$ of linearly independent vectors:

$$\underline{v_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \underline{v_2} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \underline{v_3} = \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix}, \underline{v_4} = \begin{pmatrix} 4 \\ 8 \\ 4 \end{pmatrix} \Rightarrow A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 7 & 8 \\ 1 & -1 & 2 & 4 \end{pmatrix}$$

$rk\, A = 3$ since

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 7 \\ 1 & -1 & 2 \end{vmatrix} = 20 \neq 0$$

$\Rightarrow \{\underline{v_1}, \underline{v_2}, \underline{v_3}, \underline{v_4}\}$ linearly dependent

$\Downarrow$

pivot columns = column 1, 2, 3

$\Rightarrow \{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ linearly independent