LECTURE 6

GRA 6035

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REVIEW LECTURE 5:

- QUADRATIC FORMS
- DEFINITENESS
- QUADRATIC FORMS WITH LINEAR CONSTRAINTS, BORDERED HESSIANS

pos. definite

negative definite

Ex: $Q(x_1, x_2) = 12x_1^2 - 6x_1x_2 + 3x_2^2$ subject to $3x_1 - 5x_2 = 0$

$$H = \left(\begin{array}{c|cc} 0 & 3 & -5 \\ \hline 3 & 12 & -3 \\ -5 & -3 & 3 \end{array}\right)$$

bordered Hessian matrix

Last $n-m = 2-1 = 1$ leading principal minors:

$$|H| = -3\big(3\cdot 3 - (-3)(-5)\big) + (-5)\cdot\big(3(-3) - 12(-5)\big)$$

$$= (-3)(-6) + (-5)(51) = \underline{18 - 255 < 0}$$

Same sign as $(-1)^n = -1$.

$\Rightarrow$ positiv definit

$\underline{x} = \underline{0}$ is global minimum

In the case with

(a) $n = 2$ variables

(b) $m = 1$ constraints

Same sign as $(-1)^m = -1$

$|H| < 0 \Rightarrow$ positive definite

$|H| > 0 \Rightarrow$ negative definite

Same sign as $(-1)^n = 1$

Conclusion:

$Q(x_1, x_2) = 12x_1^2 - 6x_1x_2 + 3x_2^2$ subject to $3x_1 - 5x_2 = 0$ is positive definite

$\Rightarrow$ global minimum for the constrained optimalization problem.

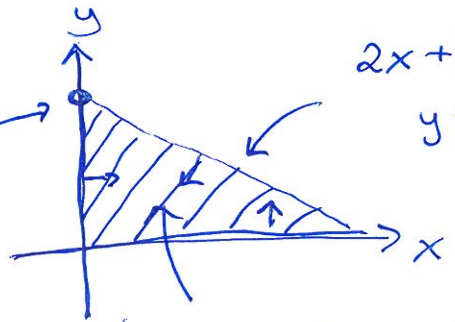
Plan for Lecture 6:

① Sets and topology
② Convex and concave functions

} [FMEA] Ch. 2.2–2.3

① Sets and topology

Ex: Max $f(x,y) = x^2 + 3y^2$ subject to $\begin{cases} x \geq 0,\ y \geq 0, \\ 2x + 3y \leq 30 \end{cases}$



$2x + 3y = 30$

$y = 10 - \frac{2}{3}x$

Solution to the maximization problem: $(x,y) = (0,10)$; $f = 300$.

allowed inputs for f = domain of definition for f

$= D_f$

Notation:

$$D_f = \{ (x,y) : x \geq 0, y \geq 0, 2x + 3y \leq 30 \}$$

= the set of all points (x,y) such that $x \geq 0$, $y \geq 0$, $2x + 3y \leq 30$

If the domain of definition were all points (x,y), then we write $D_f = \mathbb{R}^2$.

$\mathbb{R}^2$ ↑ all points (x,y)

$\mathbb{R}^3$ = all pts (x,y,z)

- Open sets, closed sets
- Bounded sets
- Convex sets

<u>Defn:</u> If $\underline{a} = (a_1, a_2, \ldots, a_n)$ is a point and $r > 0$, then

$$B(\underline{a}, r) = \{ \underline{x} : \text{the distence from } \underline{x} \text{ to } \underline{a} \text{ is less then } r \}$$

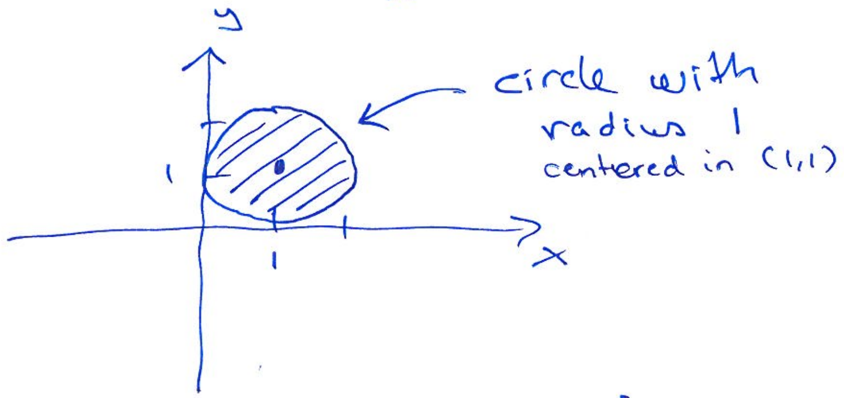
open ball

around $\underline{a}$ with radius r

$$= \{ \underline{x} : d(\underline{x}, \underline{a}) < r \}$$

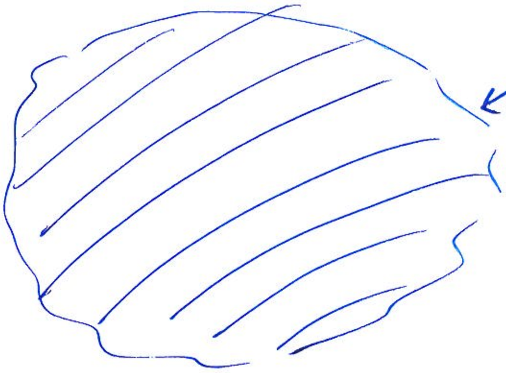
d = distance

<u>Ex:</u> $\left.\begin{array}{l} \underline{a} = (1,1) \\ r = 1 \end{array}\right\}$ $\quad B(\underline{a}, r) = \{ \underset{\underset{\underline{x}}{\shortparallel}}{(x,y)} : d(\underline{x}, \underline{a}) < 1 \}$



$B(\underline{a}, r)$ is the dish inside the circle.

Open and closed sets



boundary: the curve that is on the boundary (between points in the set and outside the set).

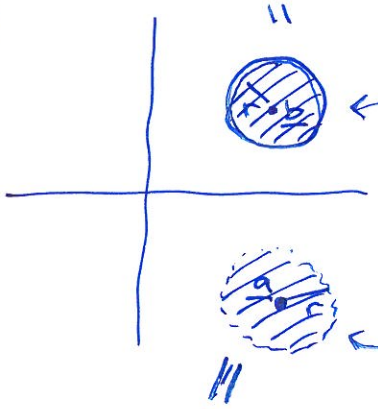
S : a set (points in S marked)

open set: no boundary points are in the set

closed set: all boundary points are in the set.

$\overline{B}(\underline{b}, r) = \{\underline{x} : d(\underline{x}, \underline{b}) \le r\}$ closed

Ex:



dish including the circle = closed

dish excluding the circle = open

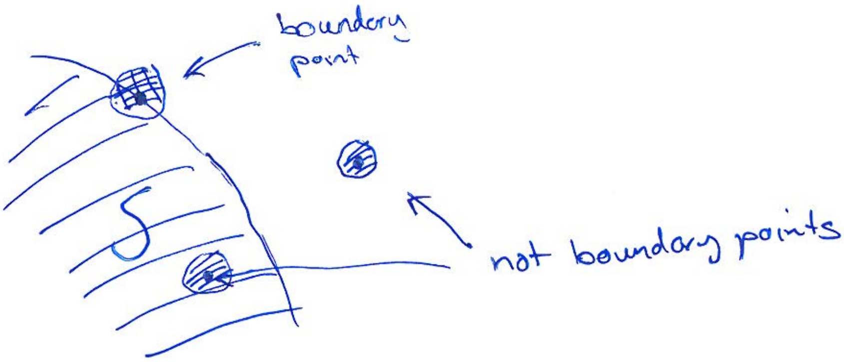
$B(\underline{a}, r) = \{\underline{x} : d(\underline{x}, \underline{a}) < r\}$

is an open set

An <u>open set</u> is a set that does <u>not</u> include any boundary points. A <u>closed set</u> is a set that includes all boundary points.

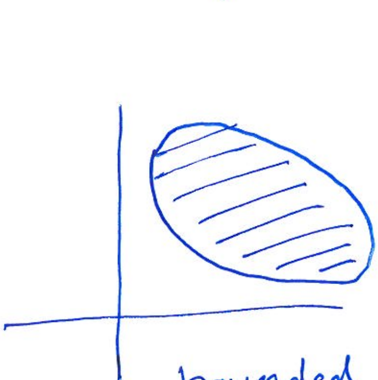
The sets $\mathbb{R}^n$ is considered both open and closed, $n = 1, 2, 3, \dots$

A point $\underline{x}$ is a <u>boundary point</u> of a set S if any open ball centered in $\underline{x}$ includes both points in S and points outside S.

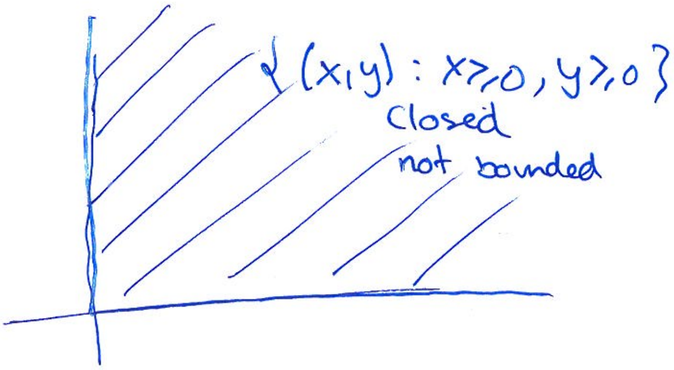


<u>Defn:</u>

A <u>bounded set</u> is a set that is contained in an open ball with a large enough radius.



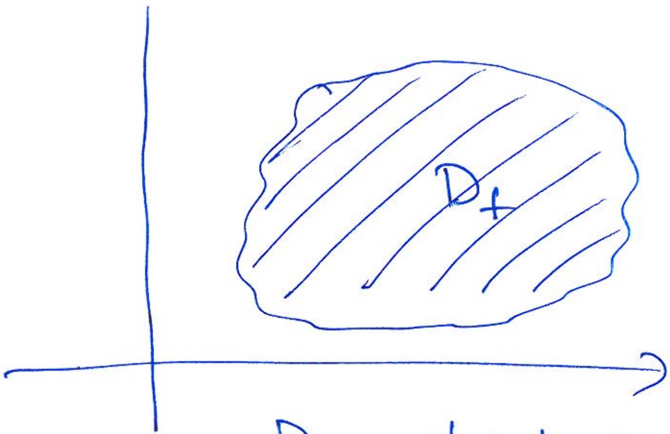
bounded



not bounded
("goes to infinity")

THEOREM: (EXTREME VALUE THEOREM)

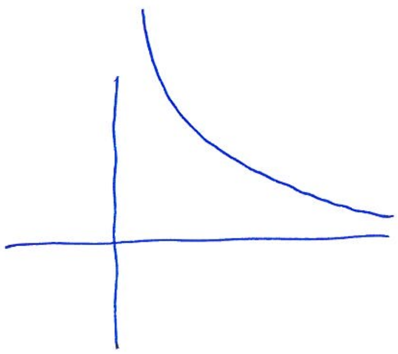
If f is a continuous function defined on a closed and bounded set, then f has a global maximum and a global minimum.



$$\left.\begin{array}{l} D_f \text{ closed and bounded} \\ f \text{ continuous} \end{array}\right\} \Rightarrow \begin{array}{l} \text{global} \\ \text{min/max} \\ \text{exists.} \end{array}$$

Ex: $f(x) = 1/x$, $x > 0$ $\qquad D_f = (0, \infty)$

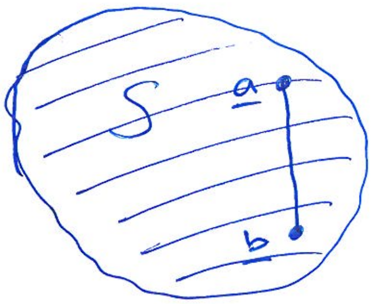
open, not closet set.



no global max since

$$\lim_{x \to 0^+} f(x) = \infty$$

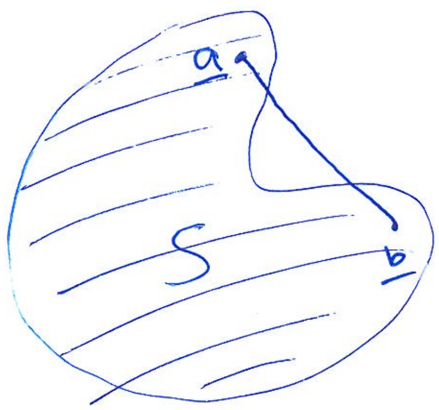
Convex sets:



a convex set

Defn: A set S is convex if the following condition holds:

If $\underline{a}$ and $\underline{b}$ are points in S, then the line segment $[\underline{a}, \underline{b}]$ is contained in S.



not convex

$\underline{a}, \underline{b}$ is in S

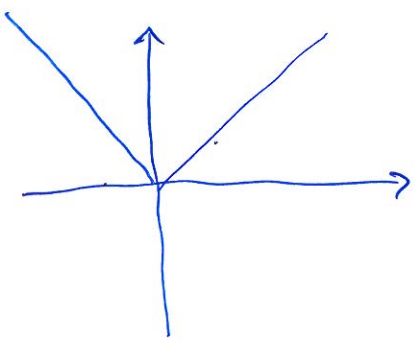
$[\underline{a}, \underline{b}]$ is not in S

② Convex and concave functions

A function $f(\underline{x}) = f(x_1, x_2, x_3, \ldots, x_n)$ is called C^2 if f, and all the partial derivatives of f of order one or two are continuous.

All functions in this course are C^2,
(except this example)

Ex: $f(x) = |x|$



$$f'(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

$f'(x)$ not defined in $x = 0$.

Ex:

$$f(x,y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

This function is not C^2.

Thm: (Clairaut's thm)

If f is a C^2 function, then

$$f''_{ij} = f''_{ji} \qquad \left(f''_{x_i x_j} = f''_{x_j x_i} \right)$$

The Hessian matrix:

Ex: $f(x,y) = x^2 + 6xy + y^2 \longrightarrow A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$

$f'_x = 2x + 6y$

$f'_y = 6x + 2y$

$f''_{xx} = 2$

$f''_{xy} = 6 = f''_{yx}$

$f''_{yy} = 2$

$$H(f) = f'' = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{pmatrix} = \underline{\underline{\begin{pmatrix} 2 & 6 \\ 6 & 2 \end{pmatrix}}}$$

If f is a quadratic form, with symmetric matrix A, then $H(f) = 2A$.

Ex: $f(x,y) = e^{x+y}$

$f'_x = e^{x+y} \cdot 1 = e^{x+y}$

$f'_y = e^{x+y} \cdot 1 = e^{x+y}$

$$\left.\begin{aligned} f''_{xx} &= e^{x+y} \\ f''_{xy} &= e^{x+y} \\ f''_{yy} &= e^{x+y} \end{aligned}\right\} \quad H(f) = \underline{\underline{\begin{pmatrix} e^{x+y} & e^{x+y} \\ e^{x+y} & e^{x+y} \end{pmatrix}}}$$

Defn: The Hessian matrix of $f(x_1, \ldots, x_n)$ is the matrix

$$H(f) = \begin{pmatrix} f''_{11} & f''_{12} & f''_{13} & \cdots & f''_{1n} \\ f''_{21} & f''_{22} & f''_{23} & \cdots & f''_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f''_{n1} & f''_{n2} & f''_{n3} & \cdots & f''_{nn} \end{pmatrix}$$

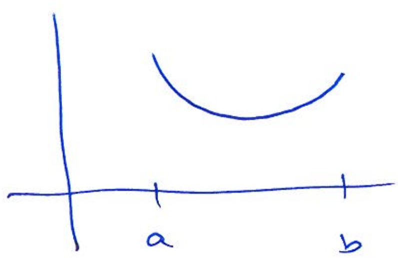
If f is C^2, then $H(f)$ is symmetric.

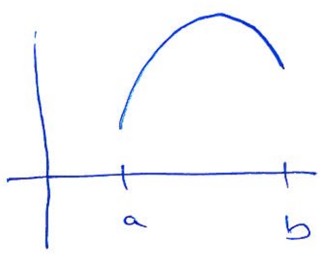
Recall: Convex/concave functions in one variable

a convex set

If $f(x)$ is defined on $[a,b]$, then

f convex	f is concave
$\Updownarrow$	$\Updownarrow$
$f''(x) \geq 0$ for all $x \in [a,b]$	$f''(x) \leq 0$ for all $x \in [a,b]$

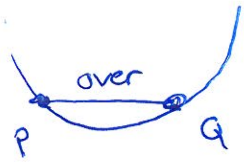




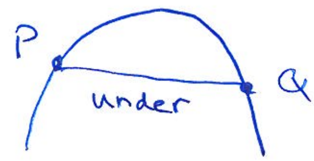
Defn:

Let $f(x_1, x_2, \dots x_n)$ be a function defined on a convex set, D_f.

f is <u>convex</u> $\Leftrightarrow$ If P, Q are points on the graph of f, then the line segment $[P, Q]$ lies <u>over</u> or on the graph of f.



f is <u>concave</u> $\Leftrightarrow$ If P, Q are points on the graph of f, then the line segment $[P, Q]$ lies <u>under</u> or on the graph of f.



f is <u>strictly convex</u> / <u>strictly concave</u> if the line segment $[P, Q]$ lies strictly over/under the graph (except at P, Q).

Result: Assume that f is defined on an open convex set D_f. Then we have

f convex $\Longleftrightarrow$ $H(f)$ is positive semidefinite

f concave $\Longleftrightarrow$ $H(f)$ is negative semidefinite

f strictly convex $\Longleftarrow$ $H(f)$ is positive definite

f strictly concave $\Longleftarrow$ $H(f)$ is negative definite

Ex: $f(x,y) = xe^{y}$, $D_f = \mathbb{R}^2$

Compute $H(f)$:

$f'_x = e^y$ $\qquad$ $f''_{xx} = 0$

$f'_y = x \cdot e^y$ $\qquad$ $f''_{xy} = e^y$

$f''_{yy} = xe^y$

$$H(f) = \begin{pmatrix} 0 & e^y \\ e^y & xe^y \end{pmatrix}$$

$D_1 = 0$

$D_2 = -(e^y)^2 < 0 \Rightarrow H(f)$ is indefinit

f is neither convex nor concave

Remember: Pos. semidefinite $\qquad$ Neg. semidefinite

$\Delta_1 \geq 0$ $\qquad$ $\Delta_1 \leq 0$

$\Delta_2 \geq 0$ $\qquad$ $\Delta_2 \geq 0$

Ex: $f(x,y) = x^2 - 6xy + 12y^2$

$$H(f) = 2 \cdot \begin{pmatrix} 1 & -3 \\ -3 & 12 \end{pmatrix} = \begin{pmatrix} 2 & -6 \\ -6 & 24 \end{pmatrix}$$

$$\left.\begin{aligned} D_1 &= 2 \\ D_2 &= 2 \cdot 24 - (-6)^2 = 48 - 36 = 12 \end{aligned}\right\} \quad H(f) \text{ is positive definite}$$

$\Downarrow$

f is convex

(strictly convex)

Result: If Q is a quadratic form, then

Q convex $\Leftrightarrow$ Q is positive semidefinite

Q concave $\Leftrightarrow$ Q is negative semidefinite

Ex: $f(x,y) = 12x - 13y + 4$

$f'_x = 12$ $\quad f''_{xx} = 0$ $\quad H(f) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$f'_y = -13$ $\quad f''_{xy} = 0$

$f''_{yy} = 0$

~~positive definite~~

positive semidefinite

negative semidefinite

f is convex and concave

Ex: $f(x,y) = e^{x+y}$

$$H(f) = \begin{pmatrix} e^{x+y} & e^{x+y} \\ e^{x+y} & e^{x+y} \end{pmatrix}$$

$D_1 = e^{x+y} > 0$ $\quad\Delta_1 = e^{x+y}, e^{x+y} > 0$

$D_2 = 0$ $\quad\Delta_2 = 0$

$H(f)$ is positive semidefinite $\Rightarrow$ f is convex