~~[illegible]~~ 14.10.11

HELLO!

"PROBLEM TYPE"	CANDIDATES	DIAGNOSTICS
NO CONSTRAINTS	$\frac{\partial f}{\partial x_i} = 0$	H (HESSIAN)
BINDING = CONSTRAINTS	$\frac{\partial \mathcal{L}}{\partial x_i} = 0$	bH (BORDERED HESSIANS)
NOT BINDING CONSTRAINTS $\leq$	$\frac{\partial \mathcal{L}}{\partial x_i}$ KUHN TUCKER CONDIT.	HARDER... USUALLY $\mathcal{L}$ CONCAVE

CONSIDER THE FOLLOWING:

FIX $b = \bar{b}$

LET $\bar{x}$ BE THE POINT THAT GIVE THE MAX VALUE:

$$f(\bar{x}) = f^*(\bar{b})$$

$$\Rightarrow f(x) \le f^*(g(x))$$

$$g(x) = b$$

AND DEFINE

$$\boxed{\varphi(x) = f(x) - f^*(g(x))}$$

THE POINT $\varphi(x)$ IS MAX IN $\bar{x}$.

[WHY? LESS THAN OR EQUAL TO ZERO AND ZERO IN $\bar{x}$.

$$\Rightarrow \frac{\partial \varphi}{\partial x_i}(\bar{x}) = \frac{\partial f}{\partial x}(\bar{x}) \sum_{j=1}^{m} \frac{\partial f^*(\bar{b})}{\partial b_j} \frac{\partial g}{\partial x_j}(x) \Big|_{b=g(x)}$$

$$\frac{\partial f^*(\bar{b})}{\partial b_j} = \lambda_j$$

$$\boxed{\frac{\partial g}{\partial x_j}(\bar{x})} \qquad \frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial f}{\partial x} - \sum_{j=1}^{m} \lambda_j \frac{\partial g}{\partial x_i}$$

INTERPRETATION OF THE LAGRANGE MULTIPLICATOR

CONSIDER

$\max f(x)$ subject to

$g_j(x) = b_j \quad j = 1, \ldots, m$

$(x_1, \ldots, x_n)$

ASSUME THAT

$(b_1, \ldots, b_n)$

$$f^*(b) = \max \{ f(x) : g_j(x) = b_j \}$$

[THIS IS JUST THE SOLUTION TO THE MAX PROBLEM <u>VIEWED</u> AS A FUNCTION OF THE (BINDING) CONSTRAINTS]

FURTHERMORE, ASSUME THAT THIS FUNCTION IS DIFFERENTIABLE IN THE b_j. (THIS IS JUST A TECHNICAL ASSUMPTION, FOR US ALLWAYS THE CASE.)

WOW!

$$\lambda_i = \frac{\partial f^*(b)}{\partial b_i}$$

[THE λ's ARE JUST THE PARTIAL DERIVATIVES OF THE VALUE FUNCTION $f^*(b)$!]

WHY DO WE CARE?

ECONOMIC INTERPRETATION.

QUASI EXAMPLE

f PROFITT

$g(x) \leq b$: BOUND ON INPUT FACTORS (SAY METALS, IRON, COPPER ..)

IRON IS NUMBER 1

THEN $\frac{\partial f^*}{\partial b_1}$ ~~IS THE INCREASE~~ IS THE INCREASED PROFITT ATTRIBUTED TO AN INCREASE OF ONE UNIT IRON.

MORE INTERESTINGLY, IT IS ALSO THE ~~MAX~~ LIMIT PRICE FOR IRON THAT MAKES ~~AN~~ INCREASE IN INPUT OF IRON PROFITABLE.

IN SOME SENSE ~~THIS~~ IS $\frac{\partial f^*}{\partial b_1}$ THE INTERNAL PRICE OF IRON FOR THE FIRM.
(TYPICALLY NOT MARKET PRICE)
THIS IS WHY $\frac{\partial f^*}{\partial b_1}$ (or λ_1) IS CALLED THE SHADOW PRICE.

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial f}{\partial x_i} - \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i}$$

$$\lambda_j = \frac{\partial f^*}{\partial b_j}$$

HELLO! ?

A REMARK ABOUT THE VALUE FUNCTION $f^*(b) = \cancel{\max\{f(x) \text{ sub}_j\ g_j(x)}$

$= \max\{f(x) \text{ SUB. } g_j(x) = b_j \quad j=1,\dots m\}$

BUT REALIZE THAT

THIS VALUE FUNCTION IS "BORN" NATURALLY WHEN WE "DO $\max f(x) \quad g(x) = a$

THEN WE TYPICALLY GET

$x = \frac{a^2}{2}$ AND THEN

THE VALUE FUNCTION IS $f\left(\frac{a^2}{2}\right)$

THE ENVELOPE THEOREM

IN THE PROBLEM

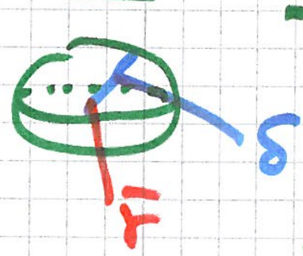
$$\max_{x \in S} f(x,r) \qquad S \subseteq \mathbb{R}^n$$

$r = (r_1 \dots r_n)$

ASSUME THAT THE SOLUTIONS

$$x^*(r) \in \mathrm{INT}(S)$$

FOR EVERY r in $\underline{B(\bar{r},\delta)}$

[$B(\bar{r},\delta)$ IS A BALL 

]

ASSUME

$$r \longrightarrow f(x^*(\bar{r}), r)$$

$$r \longrightarrow f^*(r)$$

[CORRESPONDS TO THE VALUE FUNCTION IN OUR DISCUSSION OF THE λ's IN LAGRANGIANS]

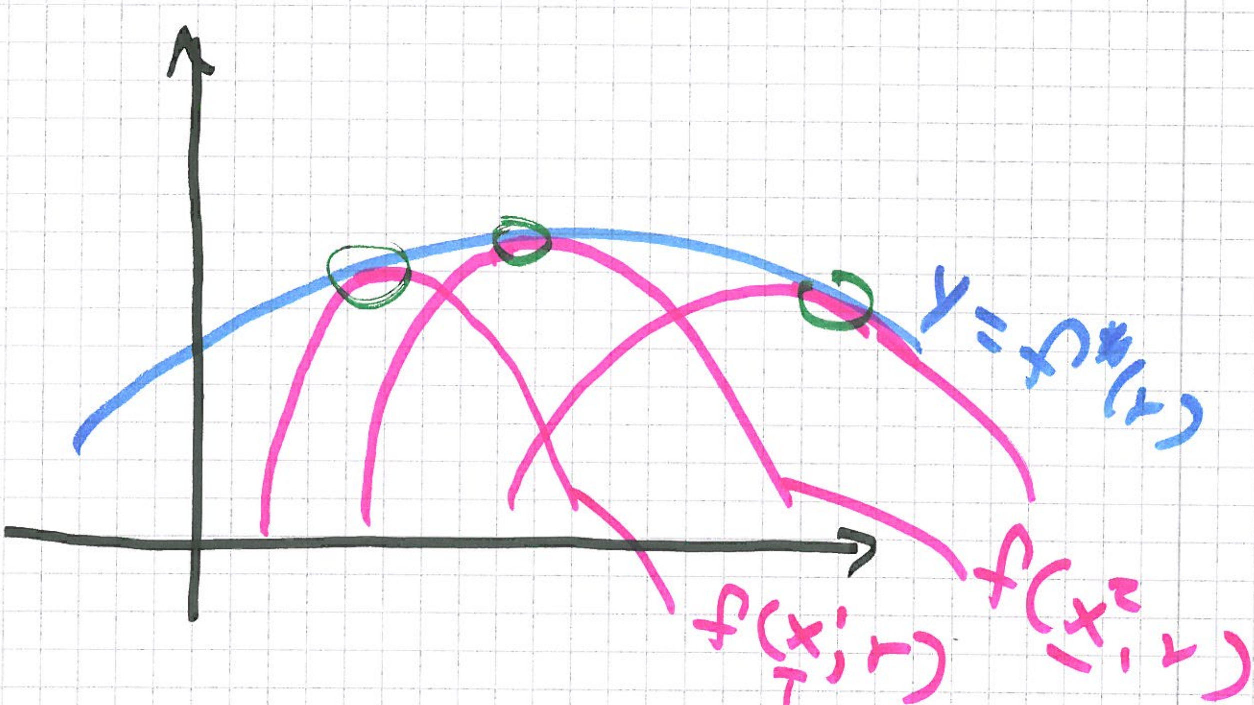
ARE DIFFERENTIABLE.

THEN

$$\frac{\partial f^*(\bar{r})}{\partial r_j} = \frac{\partial f(x,r)}{\partial r_j} \quad \begin{pmatrix} x = x^*(r) \\ r = \bar{r} \end{pmatrix}$$

THIS IS THE ENVELOPE THEOREM.

WHAT DOES THIS MEAN?



THE ENVELOPE TEOREM TELLS US THAT $f^*(r)$ CREATES A BORDER OF THIS TYPE)

WHY IS THIS TRUE?

SAME PROOF AS IN THE λ CASE:

DO $\varphi(r) = f(x^*(\bar{r}), r) - f^*(r)$

SMALL EXAMPLE OF USE:

CONSIDER

$$\max f(x; a) = -x^2 + 2ax + 4a^2$$

WHAT IS THE EFFECT OF AN INCREASE IN a?

$2x^* + 8a^{\cancel{2}}$

SOLUTION:

$$f'(x) = -2x + 2a = 0$$

$$x = a$$

$$x^*(a) = a$$

$$f^*(a) = -a^2 + 2a \cdot a + 4a^2 = 5a^2$$

$$\frac{\partial f^*}{\partial a}(a) = \underline{10a}$$

DIFF THIS

$x^* = a$

$$\frac{\partial f}{\partial a}(x^*(a), a) = 2x^* + 8a = \underline{10a}$$

THIS IS A CHECK OF ENVELOPE THEOREM

BORDERED HESSIANS

MOTIVATION / INTUITION:

WHAT ABOUT "LOCAL SECOND ORDER CONDITIONS" WITH CONSTRAINTS?

[WE HAVE DIAGNOSTICS IN THE CASE WITHOUT CONSTRAINTS... NOW WE WANT DIAGNOSTICS WITH CONSTRAINTS!]

FIRST IDEA:

DO HESSIAN FOR $\mathcal{L}$ IN CONTRAST TO f

$$H = \begin{bmatrix} \mathcal{L}''_{11} & \mathcal{L}''_{12} & \cdots \\ \vdots & & \\ \vdots & & \end{bmatrix}$$

IT IS A GOOD IDEA, BUT
TO STRONG.

WHY?

WE DONT NEED CONVEXITY
(OR CONCAVITY) EVERYWHERE
WE NEED IT IN THE
DIRECTIONS DEFINED BY
THE CONSTRAINTS).

LESS CRYPTIC, BUT HORRIBLE

CONSIDER

$$\max_{(\min)} f(x) \quad \text{SUB.} \quad g_j(x) = b_j \quad j = 1, \dots m$$

$$\mathcal{L} = f(x) - \sum_{j=1}^{m} \lambda_j (g_j(x) - b_j)$$

DEFINE THE FOLLOWING MONSTER:

$$bH = \begin{vmatrix} 0 & \cdots & 0 & \frac{\partial g_1(x^*)}{\partial x_1} & \cdots & \frac{\partial g_1(x^*)}{\partial x_r} \\ \vdots & & \vdots & \vdots & & \\ 0 & \cdots & 0 & \frac{\partial g_m(x^*)}{\partial x_1} & \cdots & \frac{\partial g(x^*)}{\partial x_r} \\ & & & \mathcal{L}''_{11}(x^*) & \cdots & \mathcal{L}''_{1r}(x^*) \\ & & & \vdots & & \vdots \\ & & & \mathcal{L}''_{r1}(x^*) & & \mathcal{L}''_{rr}(x^*) \end{vmatrix}$$

THIS IS THE DEFINITION OF THE BORDER HESSIAN

THE RESULT IS:

f SUB $g_1 \dots g_m$ $S \subseteq \mathbb{R}^m$

$x^* \in INT(S)$

SATISFYING THE NEC. CONDITIONS (Th.3.3)

$$\begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \dots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & \vdots \end{bmatrix} \quad \text{rank } m$$

THEN

a. $(-1)^{\underline{m}} B_r(x^*) > 0 \quad r = m+1, \dots n$
THEN x^* SOLVES LOC. MIN

b. $(-1)^{\underline{r}} B_r(x) > 0 \quad r = m+1, \dots n$
THEN x^* SOLVES SOLVES LOC. MAX.

NOTE INDEPENDENT OF r. THAT IS ALL NEG. OR POS.

EXAMPLE

LOC. $\substack{\text{max}\\ \text{min}}$ $f(x,y,z) = x^2+y^2+z^2$

$g_1(x,y,z) = x+2y+z = 30$

$g_2(x,y,z) = 2x-y-3z = 10$

SOLUTION

$$\boxed{\mathcal{L} = x^2+y^2+z^2 - \lambda_1(x+2y+z-30) - \lambda_2(2x-y-3z-10)}$$

WE WILL NOT SPEND TIME ON FINDING THE STAT. POINT JUST STATE IT

$P(10, 10, 0)$.

DOUBLE DERIVATIVES

THE POINT IS DIAGNOSTICS:

$m=2$

$n=3$

$$bH = \begin{bmatrix} 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & -1 & -3 \\ 1 & 2 & 2 & 0 & 0 \\ 2 & -1 & 0 & 2 & 0 \\ +1 & -3 & 0 & 0 & 2 \end{bmatrix}$$

$\det(bH) = 150$

COMPUTATION

NOTE: $m+1$ TO n IN THIS $m=2$ $n=3$.

KUHN – TUCKER

CONCERNS THE MOST GENERAL AND INTERESTING CASE WHERE CONSTRAINTS ARE JUST AN UPPER LIMIT.

$$\max f(x_1, \ldots x_m) \quad \text{SUBJ.} \quad \begin{cases} g_1(x_1 \ldots x_n) \leq b_1 \\ \vdots \\ g_m(x_1 \ldots x_n) \leq b \end{cases}$$

$(x_1 \ldots x_m)$ THAT SATISFY THESE CONSTRAINTS ARE CALLED **ADMISSIBLE.**

AS SEEN MANY TIMES BEFORE

① $\mathcal{L} = f(\overset{(x_1 \dots x_n)}{x}) - \lambda_1 (g_1(x) - b_1) \dots$

LEADS TO 1. ORDER COND.

② $\dfrac{\partial \mathcal{L}}{\partial x_i} = \dfrac{\partial f}{\partial x_i} - \sum_{j=1}^{m} \lambda_j \dfrac{\partial g_j}{x_i} = 0$

NEW THING: COMPLEMENTARY SLACKNESS CONDITION

③ $\lambda_j \geq 0$ WITH $\lambda_i = 0 \Rightarrow g_j(x) < b_j$

~~(INTERPRETATION: IF THE CONSTRAINT IS NOT BINDING, THEN~~

IF ~~THE~~ $\lambda_i = 0$ THEN THE CONSTRAINT IS NOT BINDING.

②+③ IS CALLED KUHN TUCKER CONDITIONS.

AN ALTERNATIVE WAY TO WRITE (3):

$$\lambda_j \geq 0 \qquad \lambda_j \left(g_j(x) - b_j \right) = 0$$

$$j = 1, 2, \dots, m$$

ONLY ONE OF THESE CAN BE ZERO AT THE TIME.

ILLUSTRATION ON HOW TO USE THE KUHN-TUCKER CONDITIONS.

NEXT LECTURE ... THURSDAY

- OTHER THINGS:
  - BORDERED HESSIAN
  - KUHN TUCKER & DIAGNOSTICS.