PROGRAM: KUHN-TUCKER
BORDERED HESSIANS
DIFFERENTIAL EQUATIONS

EARLIER EXAMS + SOLUTIONS:

IT'S LEARNING (GRA 6035)

COURSE WEBPAGE

GRA 6035 MATHEMATICS 2010/2011
(LAST YEARS WEBPAGE)

THERE YOU WILL FIND:

10.12.2010 LAST YEARS EXAM

22.12.2010 MOCK EXAM
(QUITE HARD)

30.05.2011

KUHN-TUCKER

max $f(x_1, \dots x_n)$ SUB. to $\begin{cases} g_1(x_1, \dots x_n) \leq b_1 \\ \vdots \\ g_m(x_1 \dots x_n) \leq b_m \end{cases}$

THE VECTORS SATISFIING THESE CONSTRAINTS ARE CALLED ADMISSIBLE.

① $\mathcal{L} = f(x_1, \dots x_n) - \lambda_1 (g_1(x_1 \dots x_n) - b_1) - \lambda_2 \dots$

JUST THE SUM OVER THE CONSTRAINTS

THIS GIVES THE FOLLOWING FIRST ORDER CONDITIONS

② $\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial f}{\partial x_i} - \sum_{j=1}^{m} \lambda_j \frac{\partial g}{\partial x_i} = 0$

NEW THING:

③ $\lambda_j \geq 0$ WITH $\lambda_i = 0 \Rightarrow g_j(x) < b_j$

THIS IS ~~USUSA~~ USUALLY WRITTEN

$\lambda_j (g_j(x) - b_j) = 0$

(ONLY ONE ZERO AT THE TIME.

COMPLEMENTARY SLACKNESS CONDITION

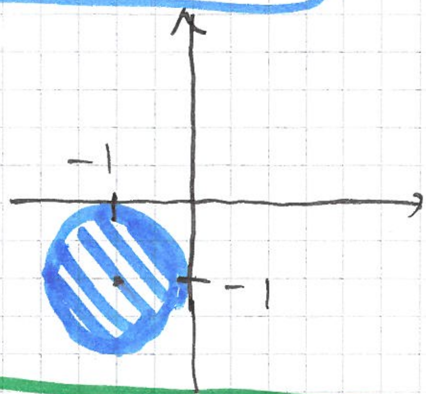
EX. ~~PROB~~ EXERCISE 4 EX. 30.05.11

~~NUMBER~~ 4 b)

$$f(x,y) = xy e^{x+y}$$

SUB

$$(x+1)^2 + (y+1)^2 \leq 1$$



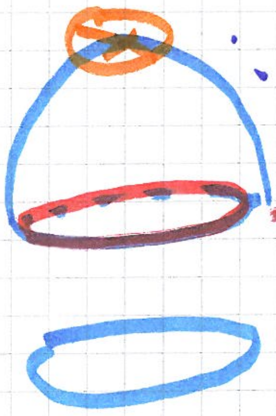
$$\mathcal{L} = xye^{x+y} - \lambda[(x+1)^2 + (y+1)^2 - 1]$$

I. $\mathcal{L}_x = (x+1)ye^{x+y} - 2\lambda(x+1) = 0$

II $\mathcal{L}_y = (y+1)xe^{x+y} - 2\lambda(y+1) = 0$

III $\boxed{\lambda \geq 0}$ $\lambda[(x+1)^2 + (y+1)^2 - 1] = 0$

$$\frac{\partial f}{\partial x} = ye^{x+y} + xye^{x+y} \cdot 1 = (x+1)ye^{x+y}$$



BINDING CONSTRAINT

① $\lambda = 0$ $\Rightarrow$ WE ARE LOOKING FOR ⊗

② $\lambda > 0$ CONSTRAINT TURNED ON

$\Rightarrow$ WE ARE ON THE BOUNDARY.

THE CANDIDATES: I. $(x+1)[ye^{x+y} - 2\lambda] = 0$

II $(y+1)[xe^{x+y} - 2\lambda] = 0$

$$\boxed{x = -1 \quad y = -1 \quad \lambda = 0}$$

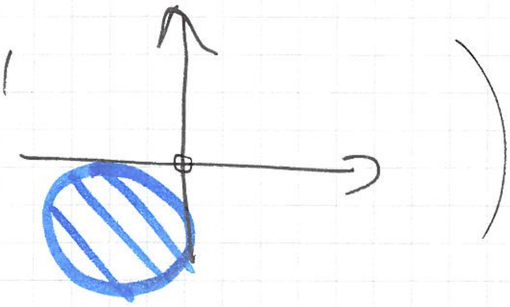
THIS IS THE ONLY ONE. WHY? NOTE $\to$ X OR Y ~~...~~ $x+1 \neq 0$

WE GET $2\lambda = ye^{x+y}$ AND SINCE

$y < 0$

WHY? NOTE THAT X OR Y $\neq 0$.

($(x+1)^2 + (y+1)^2 \leq 1$)



ASSUME $y \neq 0$ ($x+1 \neq 0$)

WE GET $2\lambda = y e^{x+y}$

AND [I. $(x+1)(y e^{x+y} - 2\lambda) = 0$]

SINCE $y < 0$ WE GET A CONTRADICTION

TO III $\lambda \geq 0$.

IN SHORT: ONLY ~~NO~~ ONE CANDIDATE

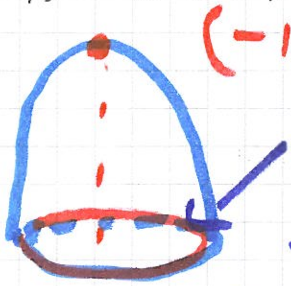
$(-1, -1, \lambda = 0)$

(YESTERDAY WE PROVED THAT f WAS ~~CONV~~ CONCAVE. HENCE

$(-1,-1)$ SOLVES THE PROBLEM.

[$(-1,-1)$ GIVES THE MAX!]

(-1, -1, SOMETHING)



THE $\lambda \geq 0$ CONDITION IS SMART ENOUGH TO SEE THAT BORDER POINTS HERE CAN NOT BE MAX!!

b. FIND MAX/MIN VALUES OF f

$$\mathcal{L} = xy e^{x+y} - \lambda \left((x+1)^2 + (y+1)^2 - 1 \right)$$

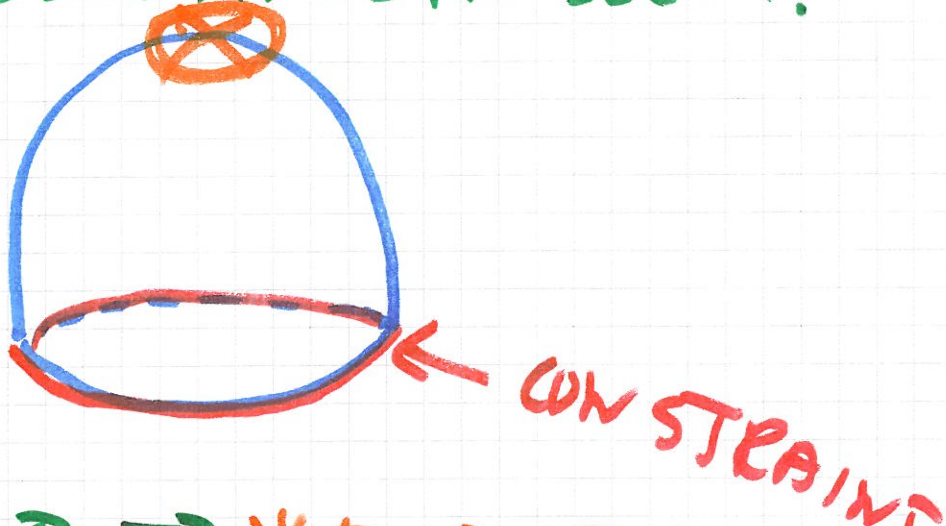
I. $\mathcal{L}'_x = (x+1) y e^{x+y} - 2\lambda (x+1) = 0$

II. $\mathcal{L}'_y = (y+1) \cancel{x} e^{x+y} - 2\lambda (y+1) = 0$

III. $\lambda \geq 0 \quad \lambda \left((x+1)^2 + (y+1)^2 - 1 \right) = 0$

SLACKNESS CONDITION
NOT BOTH ZERO.

NOTE: THAT $\lambda = 0$ THEN CORRESPONDS TO THE UN-CONSTRAINED PROBLEM.



① $\lambda = 0 \Rightarrow$ WE ARE LOOKING FOR $\otimes$

② $\lambda > 0$ CONSTRAINT ON $\Rightarrow$ WE ARE ON THE BOUNDARY

A BIRDS VIEW TO THE SOLUTION OF ~~$~~THE PROBLEM:

$$\max\ xye^{x+y} \quad \text{sub} \quad (x+1)^2+(y+1)^2 \leq 1)$$

I. $(x+1)y\, e^{x+y} \cancel{- 2\lambda(x+1)} = 0$

II $(y+1)\, x\, e^{x+y} \cancel{- 2\lambda(y+1)} = 0$

III $\lambda \geq 0 \qquad \lambda\left((x+1)^2+(y+1)^2-1\right)=0$

$\lambda = 0$ BACK TO NORMAL

NO CONSTRAINTS

$x = -1$

$y = -1$

$\lambda \neq 0$ CONSTRAINT TURNED ON:

$$(x+1)^2+(y+1)^2=1$$

SO IT IS IN SOME SENSE ORDINARY LAGRANGE + AN ADDED BONUS: $\lambda > 0$, WHICH ALLOWS US TO ~~E~~ GET RID OF IN THIS CASE ALL CANDIDATES.

RECALL: THIS CORRESPONDS TO WHAT WE DID IN THE FIRST YEAR COURSE:

1. SEARCH FOR INTERIOR POINTS: $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$
2. ON THE BOUNDARY: USE THE BOUNDRY CONDITION AND FIND POINTS THERE.

IS THE KUHN-TUCKER CONDITIONS REALLY WHAT WE WANT?

KUHN-TUCKER: (2) $\frac{\partial L}{\partial x_i} = \frac{\partial f}{\partial x_i} - \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i}$

(3) $\lambda_j \geq 0 \quad \lambda_j (g_j(x_1 \ldots x_n) - b_j) = 0$

DOES IT SOLVE THE MAX f PROBLEM?

~~YES~~... OR BETTER ALMOST...

FIRST RESULT: KUHN ~~NEC~~-TUCKER NECESSARY CONDITIONS.

SUPPOSE $x^* = (x_1^*, \dots, x_n^*)$

SOLVES:

$$\max f(x) \quad \text{SUB. TO} \quad \begin{cases} g_1(x) \le b_1 \\ \vdots \\ g_m(x) \le b_m \end{cases}$$

(AND f DIFFERENTIABLE ON S
x^* AN INTERIOR POINT OF S)

ASSUME FURTHERMORE THAT THE FOLLOWING **CONSTRAINT QUALIFICATION** IS SATISFIED:

CQ: $\nabla g_j(x^*)$ $1 \le j \le m$
CORRESPONDING TO ACTIVE CONSTRAINTS AT x^* ARE LINEARLY INDEPENDENT,

THEN THERE EXISTS $\lambda_1 \dots \lambda_m$ SUCH THAT THE KUHN TUCKER CONDITIONS HOLD AT $x = x^*$.

THE IMPORTANT THING IN COMPUTATIONS:
IS THE FOLLOWING:

THE KUHN-TUCKER CONDITIONS ARE SUFFICIENT IF THE LAGRANGIAN IS CONCAVE.

SO IN PRACTICE, THE WORKING HORSE IS:

KUHN TUCKER
+ CONCAVE

BORDERED HESSIANS

COMMENT ON ; REVISITING

LOCAL MAX (MIN) $f(x,y,z) = x^2 + y^2 + z^2$

$$g_1(x,y,z) = x + 2y + z = 30$$
$$g_2(x,y,z) = 2x - y - 3z = 10$$

GIVEN $\underline{P(10,10,0)}$ WHAT CAN WE SAY?

CRITICAL POINT !!

RECALL: $\underline{(-1)^m B_r(x^*) > 0}$ $\underline{r = m+1 \cdots n}$ LOC MIN

$(-1)^r B_r(x^*) > 0$ $r = m+1 \cdots n$ LOC MAX

NOTE THAT r IS NOT THE DIMESION OF THE DETERINANT.

IN THIS CASE: $\underline{m = 2}$ AND $n = 3$

SO $\underline{r = 3}$ IS THE ONLY ONE WE NEED TO CHECK.

$$B_3(P) = \begin{vmatrix} 0 & 0 & \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial z} \\ 0 & 0 & \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z} \\ & & \mathcal{L}''_{xx} & \mathcal{L}''_{xy} & \mathcal{L}''_{xz} \\ & & \mathcal{L}''_{yx} & \mathcal{L}''_{yy} & \mathcal{L}''_{yz} \\ & & \mathcal{L}''_{zx} & \mathcal{L}''_{zy} & \mathcal{L}''_{zz} \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & -1 & -3 \\ 1 & 2 & 2 & 0 & 0 \\ 2 & -1 & 0 & 2 & 0 \\ 1 & -3 & 0 & 0 & 2 \end{vmatrix}$$

$$= \cdots = 150$$

$(-1)^2 B_3(P) = 150 > 0$ LOC. MIN.

DIFFERENTIAL EQUATIONS.

THE UNKNOWN, THE OBJECT OF INTEREST, IS A FUNCTION, NOT A NUMBER.

$$f(x) = ax^2 + bx + c = 0$$

EQUAL AS WHAT?

WHAT DOES THAT MEAN:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

OR

$$a = b = c = 0$$

SO $a = b = c = 0$ IS EQUALITY AS FUNCTIONS

$f(x) =$ ZERO FUNCTION MEANS THAT $f(x) = 0$ FOR ALL x WHICH GIVES US THE CONCLUSION

$$a = b = c = 0$$

IN THIS CHAPTER OUR EQUATIONS, ARE EQUATIONS DEFINING FUNCTIONS, NOT GIVING NUMBERS....

EXAMPLE

$$X = X(t)$$

$$X'(t) = a\,X(t) \qquad a \text{ NUMBER.}$$

THIS IS AN EXAMPLE OF A DIFFERENTIAL EQUATION.

WHAT DOES A SOLUTION LOOK LIKE?

CANDIDATE: $\underline{X(t) = e^{at}}$

TEST: $x'(t) = a\,\underline{e^{at}}$

GOOD! $x'(t) = a\,\underline{x(t)}$

SO $x(t) = e^{at}$ IS A SOLUTION OF THE DIFFERENTIAL EQUATION

$$x'(t) = a\,x(t)$$

TO MUCH NOTATION, WE WANT SOMETHING SHORTER:

$$\dot{x} = a x$$

(SHORT FOR $X'(t) = a\,x(t)$)

SOME INITIAL INSIGHTS ABOUT DIFFERENTIAL EQUATIONS.

① $\dot{x} = x + t$ $\quad$ ($x = x(t)$
$\dot{x} = x'(t)$)

IS $x(t) = Ce^t - t - 1$ (C CONSTANT)

A SOLUTION OF ①?

WE ANSWER THAT BY PLUGGING IN?

$$x(t) = Ce^t - t - 1$$

$$x'(t) = Ce^t - 1$$

TEST: LEFT HAND SIDE: $Ce^t - 1$

RIGHT HAND SIDE: $Ce^t - \cancel{t} - 1 + \cancel{t}$

LS = RS  IT'S OK.

SOME NOTATION AND NOTIONS:

IT IS ~~COSTU~~ COSTUMARY TO WRITE

A. ① IN THIS WAY:

$$\dot{x} - x = t$$

[THE $\dot{x}$, x ON THE LEFT SIDE]

B. ONE SOLUTION OF ①, SAY $e^t - t - 1$ (C=1)

IS CALLED A **PARTICULAR SOLUTION**.

[ANOTHER PARTICULAR SOLUTION IS $-t-1$ (C=0)]

C. IMPORTANT FACT:

THE DIFFERENCE OF TWO PARTICULAR SOLUTIONS:

$$x_1(t) = e^t - t - 1 \qquad (c=1)$$

$$x_2(t) = \quad - t - 1 \qquad (c=0)$$

$$x_h(t) = x_1(t) - x_2(t) = e^t - t - 1 - (-t-1)$$

$$= e^t$$

IS A SOLUTION OF

$$\dot{x} - x = 0$$

$$\boxed{\begin{array}{c} \dot{x} - x = t \\ \dot{x} - x = 0 \end{array}}$$

FORGETTING ABOUT THE t ON THE RIGHT HAND SIDE

WHY IS THIS TRUE?

CHECK IN THE EXAMPLE WE ARE CONSIDERING.

$\cancel{x_h = e}$ $\quad x_h(t) = e^t \quad x_h'(t) = e^t$

$$\dot{x} - x \stackrel{?}{=} 0$$

CHECK: $e^t - e^t = 0$ ☺ OK!

WHY IS THIS INTERESTING:

CONSIDER THE MORE GENERAL PROBLEM

$$\dot{x} + g(t)x = f(t)$$

THEN THIS PROBLEM CAN BE BROKEN DOWN TO TWO SEPERATE PROBLEMS:

1. FINDING <u>ONE</u> SOLUTION, ONE PARTICULAR SOLUTION TO $\dot{x} + g(t)x = f(t)$

2. FINDING ALL SOLUTIONS TO $\dot{x} + g(t)x = 0$.