Problem Sheet 10 with Solutions
GRA 6035 Mathematics

BI Norwegian Business School

Problems

Problem 1. Find $\dot{x}$.
(a) $x = \frac{1}{2}t - \frac{3}{2}t^2 + 5t^3$
(b) $x = (2t^2 - 1)(t^4 - 1)$
(c) $x = (\ln t)^2 - 5\ln t + 6$
(d) $x = \ln(3t)$
(e) $x = 5e^{-3t^2+t}$
(f) $x = 5t^2e^{-3t}$

Problem 2. Find the integrals.
(a) $\int t^3 dt$
(b) $\int_0^1 (t^3 + t^5 + \frac{1}{3})dt$
(c) $\int \frac{1}{t} dt$
(d)$\int te^{t^2} dt$
(e)$\int \ln t dt$

Problem 3. The following differential equations may be solved by integrating the right hand side. Find the general solution, and the particular solution satisfying $x(0) = 1$.
(a) $\dot{x} = 2t$.
(b) $\dot{x} = e^{2t}$
(c) $\dot{x} = (2t+1)e^{t^2+t}$
(d) $\dot{x} = \frac{2t+1}{t^2+t+1}$.

Problem 4. Show that $x(t) = Ce^{-t} + \frac{1}{2}e^t$ is a solution of the differential equation $\dot{x}(t) + x(t) = e^t$ for all values of the constant C.

Problem 5. Show that $x = Ct^2$ is a solution of $t\dot{x} = 2x$ for all choices of the constant C. Find the particular solution satisfying $x(1) = 2$.

Problem 6. Solve the equation $x^2\dot{x} = t + 1$. Find the integral curve through $(t, x) = (1, 1)$

Problem 7. Solve the following differential equations:
a. $\dot{x} = t^3 - 1$
b. $\dot{x} = te^t - t$
c. $e^x\dot{x} = t + 1$

Problem 8. Solve the following differential equations:
a. $t\dot{x} = x(1-t)$, $(t_0, x_0) = (1, \frac{1}{e})$
b. $(1+t^3)\dot{x} = t^2x$, $(t_0, x_0) = (0, 2)$
c. $x\dot{x} = t$, $(t_0, x_0) = (\sqrt{2}, 1)$
d. $e^{2t}\dot{x} - x^2 - 2x = 1$, $(t_0, x_0) = (0, 0)$

Problem 9. Final Exam in GRA6035 30/05/2011, 3c
Solve the initial value problem $(2t + y) - (4y - t)y' = 0,\ y(0) = 0$.

Problem 10. Final Exam in GRA6035 10/12/2010, 3c
Solve the initial value problem

$$\frac{t}{y^2}\,y' = \frac{1}{y} - 3t^2, \quad y(1) = \frac{1}{3}$$

Solutions

Solution 1. (a) $\dot{x} = \frac{1}{2} - 3t + 15t^2$
(b) $\dot{x} = 4t(t^4 - 1) + (2t^2 - 1)4t^3 = 12t^5 - 4t^3 - 4t$
(c) $\dot{x} = 2(\ln t)\frac{1}{t} - 5\frac{1}{t}$
(d) $\dot{x} = \frac{1}{t}$
(e) $\dot{x} = 5e^{-3t^2+t}(-6t+1)$
(f) $\dot{x} = 10te^{-3t} - 15t^2e^{-3t}$

Solution 2. (a) $\int t^3 dt = \frac{1}{4}t^4 + C$
(b) $\int_0^1 (t^3 + t^5 + \frac{1}{3})dt = \frac{3}{4}$
(c) $\int \frac{1}{t} dt = \ln|t| + C$
(d) To find the integral $\int te^{t^2} dt$ we substitute $u = t^2$. This gives $\frac{du}{dt} = 2t$ or $\frac{du}{2} = tdt$. We get

$$\int te^{t^2} dt = \int e^u \frac{du}{2} = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{t^2} + C$$

(e) We use integration by parts

$$\int uv' dt = uv - \int u'v dx.$$

We write $\int \ln t dt$ as $\int (\ln t) \cdot 1 dt$ and let $u = \ln t$ and $v' = 1$. Thus $u' = \frac{1}{t}$ and $v = t$, and

$$\begin{aligned}\int \ln t dt &= (\ln t)t - \int \frac{1}{t} t dt \\ &= t\ln t - \int 1 dt \\ &= t\ln t - t + C\end{aligned}$$

Solution 3. (a) $x = \int 2t dt = t^2 + C$. The general solution is $x = t^2 + C$. We get $x(0) = C = 1$, so $x = t^2 + 1$ is the particular solution satisfying $x(0) = 1$.
(b) $x = \frac{1}{2}e^{2t} + C$ is the general solution. We get $x(0) = \frac{1}{2}e^{2\cdot 0} + C = \frac{1}{2} + C = 1 \implies C = \frac{1}{2}$. Thus $x(t) = \frac{1}{2}e^{2t} + \frac{1}{2}$ is the particular solution.
(c) To find the integral $\int (2t+1)e^{t^2+t} dt$, we substitute $u = t^2 + t$. We get $\frac{du}{dt} = 2t + 1 \implies du = (2t+1)dt$, so

$$\int (2t+1)e^{t^2+t} dt = \int e^u du = e^u + C = e^{t^2+t} + C.$$

The general solution is $x = e^{t^2+t} + C$. This gives $x(0) = 1 + C = 1 \implies C = 0$. The particular solution is $x = e^{t^2+t}$.

(d) We substitute $u = t^2+t+1$ in $\int \frac{2t+1}{t^2+t+1}dt$ to find the general solution $x = \ln(t^2+t+1)+C$. We get $x(0) = \ln 1 + C = C = 1$. The particular solution is $x(t) = \ln(t^2+t+1)+1$.

Solution 4. $x(t) = Ce^{-t} + \frac{1}{2}e^t \implies \dot{x} = -Ce^{-t} + \frac{1}{2}e^t$. From this we get

$$\dot{x} + x = -Ce^{-t} + \frac{1}{2}e^t + Ce^{-t} + \frac{1}{2}e^t = e^t$$

so we see that $\dot{x} + x = e^t$ is satisfied when $x = Ce^{-t} + \frac{1}{2}e^t$.

Solution 5. $x = Ct^2 \implies \dot{x} = 2Ct$. We have

$$t\dot{x} = t \cdot 2Ct = 2Ct^2 = 2x.$$

Solution 6. The equation $x^2\dot{x} = t+1$ is separable:

$$x^2 \frac{dx}{dt} = t+1$$

gives

$$\begin{aligned} \int x^2 dx &= \int (t+1)dt \\ \frac{1}{3}x^3 &= \frac{1}{2}t^2 + t + C \\ x^3 &= \frac{3}{2}t^2 + 3t + 3C \end{aligned}$$

Taking third root and renaming the constant

$$x(t) = \sqrt[3]{\frac{3}{2}t^2 + 3t + K}$$

We want the particular solution with $x(1) = 1$. We have

$$\begin{aligned} x(1) &= \sqrt[3]{\frac{3}{2}1^2 + 3 + K} \\ &= \sqrt[3]{K + \frac{9}{2}} = 1 \implies K + \frac{9}{2} = 1 \end{aligned}$$

We get $K = -\frac{7}{2}$. Thus

$$x(t) = \sqrt[3]{\frac{3}{2}t^2 + 3t - \frac{7}{2}}$$

is the particular solution.

Solution 7. (a) $\dot{x} = t^3 - 1$ gives

$$x = \int (t^3 - 1)dt$$

We get

$$x = \frac{1}{4}t^4 - t + C.$$

(b) We must evaluate the integral $\int (te^t - t)dt$. To evaluate $\int te^t dt$ we use integration by parts

$$\int uv' dt = uv - \int u'v dt.$$

with $v' = e^t$ and $u = t$. We get $u' = 1$ and $v = e^t$. Thus

$$\int te^t dt = te^t - \int e^t dt = te^t - e^t + C$$

We get

$$x = \int (te^t - t)dt = te^t - e^t - \frac{1}{2}t^2 + C$$

(c) $e^x \dot{x} = t + 1$ is separated as

$$e^x dx = (t+1)dt \implies \int e^x dx = \int (t+1)dt$$

Thus we get

$$e^x = \frac{1}{2}t^2 + t + C.$$

Taking the natural logarithm on each side, we get

$$x(t) = \ln(\frac{1}{2}t^2 + t + C).$$

Solution 8. (a) $t\dot{x} = x(1-t)$ is separated as

$$\frac{dx}{x} = \frac{1-t}{t}dt \implies \int \frac{dx}{x} = \int \frac{1-t}{t}dt$$

Note that $\frac{1-t}{t} = \frac{1}{t} - 1$, so

$$\ln|x| = \ln|t| - t + C$$

From this we get

$$e^{\ln|x|} = e^{\ln|t| - t + C} = e^{\ln|t|}e^{-t}e^C \implies |x| = |t|e^{-t}e^C$$

From this we deduce that

$$x(t) = te^{-t}K$$

where K is a constant as the general solution. We will find the particular solution with $x(1) = \frac{1}{e}$. We get

$$x(1) = e^{-1}K = e^{-1} \implies K = 1.$$

The particular solution is

$$x(t) = te^{-t}.$$

(b) The equation $(1+t^3)\dot{x} = t^2 x$ is separated as

$$\frac{dx}{x} = \frac{t^2}{1+t^3}dt \implies \int \frac{dx}{x} = \int \frac{t^2}{1+t^3}dt$$

We get

$$\ln|x| = \frac{1}{3}\ln|1+t^3| + C = \ln|1+t^3|^{\frac{1}{3}} + C$$

This gives

$$e^{\ln|x|} = e^{\ln|1+t^3|^{\frac{1}{3}} + C}$$

This gives

$$|x| = |1+t^3|^{\frac{1}{3}} e^C$$

from which we deduce the general solution

$$x(t) = K(1+t^3)^{\frac{1}{3}}$$

where K is a constant. We which to find the particular solution with $x(0) = 2$. We get

$$x(0) = K = 2.$$

Thus the particular solution is

$$x(t) = 2(1+t^3)^{\frac{1}{3}}.$$

(c) $x\dot{x} = t$ is separated as

$$xdx = tdt \implies \int xdx = \int tdt$$

The general solution is

$$x^2 = t^2 + C$$

where x is define implicitly. We want the particular solution where $x(\sqrt{2}) = 1$. We get

$$1^2 = (\sqrt{2})^2 + C \implies 1 = 2 + C \implies C = -1$$

We have

$$x^2 = t^2 - 1 \implies x = \pm\sqrt{t^2 - 1}$$

since $x(\sqrt{2}) < 0$ we have

$$x(t) = \sqrt{t^2 - 1}$$

as the particular solution.

(d) $e^{2t}\frac{dx}{dt} - x^2 - 2x = 1$, is separated as follows:

$$e^{2t}\dot{x} - x^2 - 2x = 1 \implies e^{2t}\dot{x} = 1 + x^2 + 2x = (x+1)^2 \implies$$
$$\frac{dx}{(x+1)^2} = e^{-2t}dt \implies \int \frac{dx}{(x+1)^2} = \int e^{-2t}dt$$

To solve the integral

$$\int \frac{dx}{(x+1)^2}$$

we substitute $u = x+1$. We get $\frac{du}{dx} = 1 \implies dx = du$. Thus

$$\int \frac{dx}{(x+1)^2} = \int \frac{1}{u^2}du = \int u^{-2}du = \frac{1}{-1}u^{-2+1} + C = -u^{-1} + C = -\frac{1}{(x+1)} + C$$

Thus we get

$$-\frac{1}{(x+1)} = \frac{1}{-2}e^{-2t} + C = -\frac{1}{2}e^{-2t} + C \implies -x-1 = \frac{1}{-\frac{1}{2}e^{-2t} + C}$$

From this we get

$$x(t) = \frac{-1}{-\frac{1}{2}e^{-2t} + C} - 1$$

as the general solution. We want the particular solution with $x(0) = 0$. We get

$$x(0) = \frac{-1}{-\frac{1}{2}e^{0} + C} - 1 = 0$$

From this we get $C = -\frac{1}{2}$. Thus the particular solution is

$$x(t) = \frac{-1}{-\frac{1}{2}e^{-2t} - \frac{1}{2}} - 1$$
$$= \frac{1 - e^{-2t}}{1 + e^{-2t}}.$$

Solution 9. The differential equation can be written in the form

$$(2t + y) + (t - 4y)y' = 0$$

and we see that it is exact. Hence its solution can be written in the form $u(y,t) = C$, where $u(y,t)$ is a function that satisfies

$$\frac{\partial u}{\partial t} = 2t + y \quad \text{and} \quad \frac{\partial u}{\partial y} = t - 4y$$

One solution is $u(y,t) = t^2 + ty - 2y^2$, and the initial condition $y(0) = 0$ gives $C = 0$. Hence

$$t^2 + ty - 2y^2 = 0 \quad \Leftrightarrow \quad y = \frac{-t \pm 3t}{-4}$$

The solution to the initial value problem is therefore

$$y = -\frac{1}{2}t \text{ or } y = t$$

Solution 10. The differential equation can be written in the form

$$\left(3t^2 - \frac{1}{y}\right) + \frac{t}{y^2}y' = 0$$

and we see that it is exact. Hence it can be written of the form $u(y,t) = C$, where $u(y,t)$ is a function that satisfies

$$\frac{\partial u}{\partial t} = 3t^2 - \frac{1}{y} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{t}{y^2}$$

One solution is $u(y,t) = t^3 - t/y$, and this gives

$$t^3 - \frac{t}{y} = C \quad \Leftrightarrow \quad y = \frac{t}{t^3 - C}$$

The initial condition gives $1/(1-C) = 1/3$ or $C = -2$. The solution to the initial value problem is therefore

$$y = \frac{t}{t^3 + 2}$$