Problem Sheet 11 with Solutions
GRA 6035 Mathematics

BI Norwegian Business School

Problems

1. Find the general solution of the following differential equations:

a) $\ddot{x} = t$
b) $\ddot{x} = e^t + t^2$

2. Solve the initial value problem $\ddot{x} = t^2 - t$, $x(0) = 1$, $\dot{x}(0) = 2$.

3. Solve the problem $\ddot{x} = \dot{x} + t$, $x(0) = 1$, $x(1) = 2$.

4. Find the general solutions of the following differential equations:

a) $\ddot{x} - 3\dot{x} = 0$
b) $\ddot{x} + 4\dot{x} + 8x = 0$
c) $3\ddot{x} + 8\dot{x} = 0$
d) $4\ddot{x} + 4\dot{x} + x = 0$
e) $\ddot{x} + \dot{x} - 6x = 8$
f) $\ddot{x} + 3\dot{x} + 2x = e^{5t}$

5. Find the general solutions

a) $\ddot{x} - x = e^{-t}$
b) $3\ddot{x} - 30\dot{x} + 75x = 2t + 1$

6. Solve

a) $\ddot{x} + 2\dot{x} + x = t^2$, $x(0) = 0$, $\dot{x}(0) = 1$
b) $\ddot{x} + 4x = 4t + 1$, $x(\frac{\pi}{2}) = 0$, $\dot{x}(\frac{\pi}{2}) = 0$

7. Consider the equation $\ddot{x} + a\dot{x} + bx = 0$ when $\frac{1}{4}a^2 - b$, so that that the characteristic equation has a double root $r = -\frac{a}{2}$. Let $x(t) = u(t)e^{rt}$ and prove that this function is a solution if and only if $\ddot{x} = 0$. Conclude that the general solution is $x = (A + Bt)e^{rt}$.

8. Find the general solutions of the following equations for $t > 0$:

a) $t^2\ddot{x} + 5t\dot{x} + 3x = 0$
b) $t^2\ddot{x} - 3t\dot{x} + 3x = t^2$

9. Solve the differential equation $\ddot{x} + 2a\dot{x} - 3a^2x = 100e^{bt}$ for all values of the constants a and b.

10. Final Exam in GRA6035 30/05/2011, 3b
Find the general solution of the differential equation $y'' + 2y' - 35y = 11e^t - 5$.

11. Final Exam in GRA6035 10/12/2010, 3b
Find the general solution of the differential equation $y'' + y' - 6y = te^t$.

Solutions

1 We solve the differential equation by direct integration:

a) $\ddot{x} = t \implies \dot{x} = \frac{1}{2}t^2 + C_1 \implies x = \frac{1}{6}t^3 + C_1 t + C_2$
b) $\ddot{x} = e^t + t^2 \implies \dot{x} = e^t + \frac{1}{3}t^3 + C_1 \implies x = e^t + \frac{1}{12}t^4 + C_1 t + C_2$

2 We have $\ddot{x} = t^2 - t \implies \dot{x} = \frac{1}{3}t^3 - \frac{1}{2}t^2 + C_1 \implies x = \frac{1}{12}t^4 - \frac{1}{6}t^3 + C_1 t + C_2$. The initial condition $x(0) = 1$ gives $\frac{1}{12}0^4 - \frac{1}{6}0^3 + C_1 0 + C_2 = C_2 = 1$, and $\dot{x}(0) = 2$ gives $\frac{1}{3}0^3 - \frac{1}{2}0 + C_1 = C_1 = 2$. The particular solution is therefore

$$x(t) = \frac{1}{12}t^4 - \frac{1}{6}t^3 + 2t + 1$$

3 Substitute $u = \dot{x}$. Then $\ddot{x} = \dot{x} + t \Leftrightarrow \dot{u} = u + t \Leftrightarrow \dot{u} - u = t$. The integrating factor is e^{-t}, and we get

$$ue^{-t} = \int te^{-t}dt = -e^{-t} - te^{-t} + C_1$$

From this we obtain $u = (-e^{-t} - te^{-t} + C_1)e^t = C_1 e^t - t - 1$ and we integrate to find x from $u = \dot{x}$, and get $x = \int (C_1 e^t - t - 1)dt = C_1 e^t - t - \frac{1}{2}t^2 + C_2$. The initial condition $x(0) = 1$ gives $C_1 + C_2 = 1 \implies C_2 = 1 - C_1$, and the condition $x(1) = 2$ gives $C_1 e - 1 - \frac{1}{2} + C_2 = C_1 e - 3/2 + 1 - C_1 = 2$. This gives $C_1(e-1) = 5/2$, or

$$C_1 = \frac{5}{2(e-1)}, \quad C_2 = 1 - \frac{5}{2(e-1)} = \frac{2e-7}{2(e-1)}$$

The particular solution is therefore

$$x(t) = \frac{5}{2(e-1)} \cdot e^t - t - \frac{1}{2}t^2 + \frac{2e-7}{2(e-1)}$$

4

a) The characteristic equation is $r^2 - 3r = 0 \implies r = 0, 3 \implies x(t) = C_1 + C_2 e^{3t}$.
b) Characteristic equation is $r^2 + 4r + 8 = 0$. This has no real solutions. Thus we put $\alpha = -\frac{1}{2}a = -\frac{1}{2}4 = -2, \beta = \sqrt{b - \frac{1}{4}a^2} = \sqrt{8 - \frac{1}{4}4^2} = 2$. From this the general solution is $x(t) = e^{\alpha t}(A\cos\beta t + B\sin\beta t) = e^{-2t}(A\cos 2t + B\sin 2t)$.
c) $3\ddot{x} + 8\dot{x} = 0 \Longleftrightarrow \ddot{x} + \frac{8}{3}\dot{x} = 0$. The characteristic equation is $r^2 + \frac{8}{3}r = 0 \implies r = 0$ or $r = -\frac{8}{3}$. The general solution is $x(t) = C_1 e^{0t} + C_2 e^{-\frac{8}{3}t} = C_1 + C_2 e^{-\frac{8}{3}t}$.
d) $4\ddot{x} + 4\dot{x} + x = 0$ has characteristic equation $4r^2 + 4r + 1 = 0$. There is one solution $r = -\frac{1}{2}$. The general solution is $x(t) = (C_1 + C_2 t)e^{-\frac{1}{2}t}$.
e) First we solve the homogenous equation $\ddot{x} + \dot{x} - 6x = 8$. The characteristic equation is $r^2 + r - 6 = 0$. It has the solutions $r = -3$ and $r = 2$. The general solution of the homogenous equation is thus

$$x_h(t) = C_1 e^{-3t} + C_2 e^{2t}$$

In order to find the general solution of the non-homogenous equation $\ddot{x}+\dot{x}-6x = 8$, we need to find a particular solution and we guess on a solution of the form $x_p(t) = A$ for some constant A. Putting this into the equation gives $A = -\frac{8}{6} = -\frac{4}{3}$. Thus the general solution is

$$x(t) = -\frac{4}{3} + C_1 e^{-3t} + C_2 e^{2t}$$

f) We first solve the homogenous equation $\ddot{x}+3\dot{x}+2x = 0$. The characteristic equation is $r^2+3r+2 = 0$. The solutions are $r = -1$ and $r = -2$. The general solution of the homogenous equation is thus

$$x_h(t) = C_1 e^{-t} + C_2 e^{-2t}$$

To find a solution of the non-homogenous equation $\ddot{x}+3\dot{x}+2x = e^{5t}$, we guess on a solution of the form $x_p(t) = Ae^{5t}$. We have that

$$\dot{x}_p = 5Ae^{5t} \text{ and } \ddot{x}_p = 25Ae^{5t}$$

Putting this into the equation we obtain

$$25Ae^{5t} + 3\cdot 5Ae^{5t} + 2Ae^{5t} = e^{5t}$$

From this we get $42Ae^{5t} = e^{5t}$ and we must have $A = \frac{1}{42}$. Thus the solution is

$$x(t) = \frac{1}{42}e^{5t} + C_1 e^{-t} + C_2 e^{-2t}$$

5

a) We first solve $\ddot{x}-x = 0$. The characteristic equation is $r^2-1 = 0$. We get $x_h = C_1 e^{-t} + C_2 e^{t}$. To find a solution of $\ddot{x}-x = e^{-t}$, we guess on solution of the form $x_p = Ae^{-t}$. We have $\dot{x}_p = -Ae^{-t}$ and $\ddot{x}_p = Ae^{-t}$. Putting this into the left hand side of the equation, we get

$$Ae^{-t} - (Ae^{-t}) = 0$$

So this does not work. The reason is that e^{-t} is a solution of the homogenous equation. We try something else: $x_p = Ate^{-t}$. This gives

$$\begin{aligned}
\dot{x}_p &= A(e^{-t} - te^{-t}) \\
\ddot{x}_p &= A(-e^{-t} - (e^{-t} - te^{-t})) \\
&= Ae^{-t}(t-2)
\end{aligned}$$

Putting this into the left hand side of the equation, we obtain

$$\begin{aligned}\ddot{x}_p - x_p &= Ae^{-t}(t-2) - Ate^{-t}\\ &= -2Ae^{-t}\end{aligned}$$

We get a solution for $A = -\frac{1}{2}$. Thus the general solution is

$$x(t) = -\frac{1}{2}te^{-t} + C_1e^{-t} + C_2e^{t}$$

b) The equation is equivalent to

$$\ddot{x} - 10\dot{x} + 25x = \frac{2}{3}t + \frac{1}{3}$$

We first solve the homogenous equation for which the characteristic equation is

$$r^2 - 10r + 25 = 0$$

This has one solution $r = 5$. The general homogenous solution is thus

$$x_h = (C_1 + C_2t)e^{5t}$$

To find a particular solution, we try

$$x_p = At + B$$

We have $\dot{x}_p = A$ and $\ddot{x}_p = 0$. Putting this into the equation, we obtain

$$0 - 10A + 25(At + B) = \frac{2}{3}t + \frac{1}{3}$$

We obtain $25A = \frac{2}{3}$ and $-10A + 25B = \frac{1}{3}$. From this we get $A = \frac{2}{75}$ and $-\frac{20}{75} + 25B = \frac{25}{75} \implies B = \frac{45}{25\cdot 75} = \frac{3}{125}$. Thus

$$x(t) = \frac{2}{75}t + \frac{3}{125} + (C_1 + C_2t)e^{5t}$$

6

a) We first solve the homogenous equation $\ddot{x} + 2\dot{x} + x = 0$. The characteristic equation is $r^2 + 2r + 1 = 0$ which has the one solution, $r = -1$. We get

$$x_h(t) = (C_1 + C_2t)e^{-t}.$$

To find a particular solution we try with $x_p = At^2 + Bt + C$. We get $\dot{x}_p = 2At + B$ and $\ddot{x}_p = 2A$. Substituting this into the left hand side of the equation, we get

$$\begin{aligned}&2A + 2(2At + B) + (At^2 + Bt + C)\\ &= 2A + 2B + C + (4A + B)t + At^2\end{aligned}$$

We get $A = 1$, $(4A+B) = 0$ and $2A+2B+C = 0$. We obtain $A = 1$, $B = -4$ and $C = -2A-2B = -2+8 = 6$. Thus the general solution is

$$x(t) = t^2 - 4t + 6 + (C_1 + C_2 t)e^{-t}.$$

We get $\dot{x} = 2t - 4 + C_2 e^{-t} + (C_1 + C_2 t)e^{-t}(-1) = 2t - C_1 e^{-t} + C_2 e^{-t} - tC_2 e^{-t} - 4$. From $x(0) = 0$ we get $6 + C_1 = 0 \implies C_1 = -6$. From $\dot{x}(0) = 1$, we get $-C_1 + C_2 - 4 = 1 \implies C_2 = 5 + C_1 = 5 - 6 = -1$. Thus we have

$$x(t) = t^2 - 4t + 6 - (6+t)e^{-t}.$$

b) We first solve the homogenous equation $\ddot{x} + 4x = 0$. The characteristic equation $r^2 + 4 = 0$ has no solutions, so we put $\alpha = -\frac{1}{2}0 = 0$ and $\beta = \sqrt{4 - \frac{1}{2}0} = 2$. This gives $x_h = e^{\alpha t}(C_1 \cos\beta t + C_2 \sin\beta t) = C_1 \cos 2t + C_2 \sin 2t$. To find a solution of $\ddot{x} + 4x = 4t + 1$ we try $x_p = A + Bt$. This gives $\dot{x}_p = B$ and $\ddot{x}_p = 0$. Putting this into the equation, we find that

$$\ddot{x}_p + 4x_p = 0 + 4(A + Bt) = 4A + 4Bt = 4t + 1.$$

This implies that $B = 1$ and $A = \frac{1}{4}$. Thus

$$x(t) = C_1 \cos 2t + C_2 \sin 2t + \frac{1}{4} + t$$

7 $x = ue^{rt} \implies \dot{x} = \dot{u}e^{rt} + ure^{rt} = e^{rt}(\dot{u} + ur) \implies \ddot{x} = \ddot{u}e^{rt} + \dot{u}re^{rt} + r(\dot{u}e^{rt} + ure^{rt}) = e^{rt}(\ddot{u} + 2r\dot{u} + ur^2)$. From this we get

$$\begin{aligned} \ddot{x} + a\dot{x} + bx &= e^{rt}[(\ddot{u} + 2r\dot{u} + ur^2) + a(\dot{u} + ur) + bu] \\ &= e^{rt}[\ddot{u} + (2r+a)\dot{u} + (r^2 + ar + b)u] \end{aligned}$$

The characteristic equation is assumed to have one solution $r = \frac{-a}{2}$. Putting $r = \frac{-a}{2}$ into the expression we get

$$\ddot{x} + a\dot{x} + bx = e^{rt}\ddot{u}$$

So $x = ue^{rt}$ is a solution if and only if $e^{rt}\ddot{u} = 0 \Leftrightarrow \ddot{u} = 0$. The differential equation $\ddot{u} = 0$ has the general solution $u = A + Bt$. Thus $x = (A + Bt)e^{rt}$ is the general solution of $\ddot{x} + a\dot{x} + bx = 0$.

8

a) Substituting $t = e^s$ transforms the equation into $x''(s) + 4x'(s) + 3x'(s) = 0$. The characteristic equation is $r^2 + 4r + 3 = 0$. The solutions are $r = -3, -1$. Thus $x(s) = C_1 e^{-3t} + C_2 e^{-t}$. Substituting $s = \ln t$ gives $x(t) = C_1 t^{-3} + C_2 t^{-1}$.

b) Substituting $t = e^s$ transforms the equation into $x''(s) - 4x'(s) + 3x'(s) = (e^s)^2 = e^{2s}$. First we solve the homogenous equation $x''(s) - 5x'(s) + 3x'(s) = 0$. The characteristic equation is $r^2 - 4r + 3 = 0$, and has the solutions $r = 1$ and $r = 3$. Thus $x_h = C_1 e^s + C_2 e^{3s}$. To find a particular solution of $x''(s) - 4x'(s) + 3x(s) = (e^s)^2 = e^{2s}$ we try $x_p = Ae^{2s}$. We have $x_p' = 2Ae^{2s}$ and $x_p'' = 4Ae^{2s}$. Substituting this into the equation, gives

$$\begin{aligned} x''(s) - 4x'(s) + 3x(s) &= 4Ae^{2s} - 4 \cdot 2Ae^{2s} + 3 \cdot Ae^{2s} \\ &= -Ae^{2s} \end{aligned}$$

Thus we get $A = -1$, and

$$x(s) = C_1 e^s + C_2 e^{3s} - e^{2s}$$

Substituting $s = \ln t$ gives

$$x(t) = C_1 t + C_2 t^3 - t^2.$$

9 If $a \neq 0$ we get the general solution

$$x = 100 \frac{e^{bt}}{2ab - 3a^2 + b^2} + C_1 e^{at} + C_2 e^{-3at}$$

provided that $2ab - 3a^2 + b^2 \neq 0$. When $a = 0$ and $b \neq 0$ we get the general solution

$$x = C_1 + \frac{100}{b^2} e^{bt} + C_2 t$$

There are also some other cases to consider, see answers in FMEA ex.6.3.9.

10 The homogeneous equation $y'' + 2y' - 35y = 0$ has characteristic equation $r^2 + 2r - 35 = 0$ and roots $r = 5$ and $r = -7$, so $y_h = C_1 e^{5t} + C_2 e^{-7t}$. We try to find a particular solution of the form $y = Ae^t + B$, which gives

$$y' = y'' = Ae^t$$

Substitution in the differential equation gives

$$Ae^t + 2Ae^t - 35(Ae^t + B) = 11e^t - 5 \Leftrightarrow -32A = 11 \text{ and } -35B = -5$$

This gives $A = -11/32$ and $B = 1/7$. Hence the general solution of the differential equation is $y = y_h + y_p = C_1 e^{5t} + C_2 e^{-7t} - \frac{11}{32} e^t + \frac{1}{7}$

11 The homogeneous equation $y'' + y' - 6y = 0$ has characteristic equation $r^2 + r - 6 = 0$ and roots $r = 2$ and $r = -3$, so $y_h = C_1 e^{2t} + C_2 e^{-3t}$. We try to find a particular solution of the form $y = (At + B)e^t$, which gives

$$y' = (At + A + B)e^t, \quad y'' = (At + 2A + B)e^t$$

Substitution in the differential equation gives

$$(At + 2A + B)e^t + (At + A + B)e^t - 6(At + B)e^t = te^t \Leftrightarrow -4A = 1 \text{ and } 3A - 4B = 0$$

This gives $A = -1/4$ and $B = -3/16$. Hence the general solution of the differential equation is $y = y_h + y_p = C_1 e^{2t} + C_2 e^{-3t} - (\frac{1}{4}t + \frac{3}{16})e^t$