Problem Sheet 12 with Solutions
GRA 6035 Mathematics

BI Norwegian Business School

Problems

1. Find the solution of the difference equation $x_{t+1} = 2x_t + 4$ with $x_0 = 1$.

2. Find the solution of the difference equation $w_{t+1} = (1+r)w_t + y_{t+1} - c_{t+1}$ when $r = 0.2$, $w_0 = 1000$, $y_t = 100$ and $c_t = 50$.

3. Prove by direct substitution that the following sequences in t are solutions of the associated difference equations when A, B are constants:

a) $x_t = A + B \cdot 2^t$ is a solution of $x_{t+2} - 3x_{t+1} + 2x_t = 0$
b) $x_t = A \cdot 3^t + B \cdot 4^t$ is a solution of $x_{t+2} - 7x_{t+1} + 12x_t = 0$

4. Find the general solution of the difference equation $x_{t+2} - 2x_{t+1} + x_t = 0$.

5. Find the general solution of the difference equation $3x_{t+2} - 12x_t = 4$.

6. Find the general solution of the following difference equations:

a) $x_{t+2} - 6x_{t+1} + 8x_t = 0$
b) $x_{t+2} - 8x_{t+1} + 16x_t = 0$
c) $x_{t+2} + 2x_{t+1} + 3x_t = 0$

7. Find the general solution of the difference equation $x_{t+2} + 2x_{t+1} + x_t = 9 \cdot 2^t$.

8. A model for location uses the difference equation

$$D_{t+2} - 4(ab+1)D_{t+1} + 4a^2b^2D_t = 0$$

where a, b are constants and D_t is the unknown sequence. Find the solution of this equation assuming that $1 + 2ab > 0$.

9. Is the difference equation $x_{t+2} - x_{t+1} - x_t = 0$ globally asymptotically stable?

10. Final Exam in GRA6035 10/12/2007, Problem 3

a) Find the solution of $\dot{x} = (t-2)x^2$ that satisfies $x(0) = 1$.
b) Find the general solution of the differential equation $\ddot{x} - 5\dot{x} + 6x = e^{7t}$.
c) Find the general solution of the differential equation $\dot{x} + 2tx = te^{-t^2+t}$.
d) Find the solution of $3x^2e^{x^3+3t}\dot{x} + 3e^{x^3+3t} - 2e^{2t} = 0$ with $x(1) = -1$.

11. Final Exam in GRA6035 10/12/2010, Problem 3a
You borrow an amount K. The interest rate per period is r. The repayment is 500 in the first period, and increases with 10 for each subsequent period. Show that the outstanding balance b_t after period t satisfies the difference equation

$$b_{t+1} = (1+r)b_t - (500 + 10t), \quad b_0 = K$$

and solve this difference equation.

12. Mock Final Exam in GRA6035 12/2010, Problem 3

a) Find the solution of $y' = y(1-y)$ that satisfies $y(0) = 1/2$.
b) Find the general solution of the differential equation

$$\big(\ln(t^2+1)-2\big)\,y' = 2t - \frac{2ty}{t^2+1}$$

c) Solve the difference equation

$$p_{t+2} = \frac{2}{3}\,p_{t+1} + \frac{1}{3}\,p_t, \quad p_0 = 100, \quad p_1 = 102$$

13. Final Exam in GRA6035 30/05/2011, Problem 3a
Solve the difference equation $x_{t+1} = 3x_t + 4$, $x_0 = 2$ and compute x_5.

Solutions

1 We write the difference equation $x_{t+1} - 2x_t = 4$, and see that it is a first order linear inhomogeneous equation. The homogeneous solution is $x_t^h = C \cdot 2^t$ since the characteristic equation is $r - 2 = 0$, so that $r = 2$. We look for a particular solution of the form $x_t^p = A$ (constant), and see that $A - 2A = 4$, so that $A = -4$ and $x_t^p = -4$. Hence the general solution is

$$x_t = x_t^h + x_t^p = C \cdot 2^t - 4$$

The initial condition $x_0 = 1$ gives $C \cdot 1 - 4 = 1$, or $C = 5$. The solution is therefore $x_t = \mathbf{5 \cdot 2^t - 4}$.

2 We write the difference equation $w_{t+1} - 1.2w_t = 50$, and see that it is a first order linear inhomogeneous equation. The homogeneous solution is $w_t^h = C \cdot 1.2^t$ since the characteristic root is 1.2. We look for a particular solution of the form $w_t^p = A$ (constant), and see that $A - 1.2A = 50$, so that $A = -250$ and $x_t^p = -250$. Hence the general solution is

$$w_t = w_t^h + w_t^p = C \cdot 1.2^t - 250$$

The initial condition $w_0 = 1000$ gives $C \cdot 1 - 250 = 1000$, or $C = 1250$. The solution is therefore $w_t = \mathbf{1250 \cdot 1.2^t - 250}$.

3 We compute the left hand side of the difference equations to check that the given sequences are solutions:

a) $(A + B \cdot 2^{t+2}) - 3(A + B \cdot 2^{t+1}) + 2(A + B \cdot 2^t) = (A - 3A + 2A) + (4B - 6B + 2B) \cdot 2^t = 0$
b) $(A \cdot 3^{t+2} + B \cdot 4^{t+2}) - 7(A \cdot 3^{t+1} + B \cdot 4^{t+1}) + 12(A \cdot 3^t + B \cdot 4^t) = (9A - 21A + 12A) \cdot 3^t + (16B - 28B + 12B) \cdot 2^t = 0$

We see that the given sequence is a solution in each case.

4 The difference equation $x_{t+2} - 2x_{t+1} + x_t = 0$ is a second order linear homogeneous equation. The characteristic equation is $r^2 - 2r + 1 = 0$ and has a double root $r = 1$, and therefore the general solution is

$$x_t = C_1 \cdot 1^t + C_2 t \cdot 1^t = \mathbf{C_1 + C_2 t}$$

5 We write the difference equation $3x_{t+2} - 12x_t = 4$ as $x_{t+2} - 4x_t = 1$. It is a second order linear inhomogeneous equation. We first find the homogeneous solution: The characteristic equation is $r^2 - 4 = 0$ and has roots $r = \pm 2$, and therefore the homogeneous solution is $x_t = C_1 \cdot 2^t + C_2 \cdot (-2)^t$. For the particular solution, we see that $f_t = 4$ in the original difference equation $3x_{t+2} - 12x_t = 4$, so we guess $x_t^p = A$, a constant. This gives $x_t = A$ and $x_{t+2} = A$, so $3A - 12A = 4$, or $A = -4/9$. Hence the particular solution is $x_t^p = -4/9$, and the general solution is

$$x_t = x_t^h + x_t^p = \mathbf{C_1 \cdot 2^t + C_2 \cdot (-2)^t - 4/9}$$

6 In each case, we solve the characteristic equation to find the general solution:

a) The characateristic equation of $x_{t+2} - 6x_{t+1} + 8x_t = 0$ is $r^2 - 6r + 8 = 0$, and has roots $r = 2,4$. Therefore, the general solution is $x_t = \mathbf{C_1 \cdot 2^t + C_2 \cdot 4^t}$.
b) The characateristic equation of $x_{t+2} - 8x_{t+1} + 16x_t = 0$ is $r^2 - 8r + 16 = 0$, and has a double root $r = 4$. Therefore, the general solution is $x_t = \mathbf{C_1 \cdot 4^t + C_2\, t \cdot 4^t}$.
c) The characateristic equation of $x_{t+2} + 2x_{t+1} + 3x_t = 0$ is $r^2 + 2r + 3 = 0$, and has roots given by

$$r = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 3}}{2} = -1 \pm \sqrt{-8}/2$$

Hence there are no real roots. We have $a = 2$ and $b = 3$, so the general solution is $x_t = (\sqrt{\mathbf{3}})^{\mathbf{t}}(\mathbf{C_1} \cos(\mathbf{2.186t}) + \mathbf{C_2} \sin(\mathbf{2.186t}))$ since we have that $\cos(2.186) \simeq -1/\sqrt{3}$.

7 The difference equation $x_{t+2} + 2x_{t+1} + x_t = 9 \cdot 2^t$ is a second order linear inhomogeneous equation. We first find the homogeneous solution, and therefore consider the homogeneous equation $x_{t+2} + 2x_{t+1} + x_t = 0$. The characteristic equation is $r^2 + 2r + 1 = 0$ and it has a double root $r = -1$. Therefore the homogeneous solution is $x_t^h = C_1 \cdot (-1)^t + C_2\, t \cdot (-1)^t = (C_1 + C_2\, t)(-1)^t$. We then find a particular solution of the inhomogeneous equation $x_{t+2} + 2x_{t+1} + x_t = 9 \cdot 2^t$, and look for a solution of the form $x_t = A \cdot 2^t$. This gives

$$A \cdot 2^{t+2} + 2(A \cdot 2^{t+1}) + (A \cdot 2^t) = 9 \cdot 2^t \quad \Rightarrow \quad (4A + 4A + A) \cdot 2^t = 9 \cdot 2^t$$

This gives $9A = 9$ or $A = 1$, and the particular solution is $x_t^p = 1 \cdot 2^t = 2^t$. Hence the general solution is

$$x_t = x_t^h + x_t^p = (\mathbf{C_1} + \mathbf{C_2\, t}) \cdot (-\mathbf{1})^{\mathbf{t}} + \mathbf{2^t}$$

8 The difference equation $D_{t+2} - 4(ab+1)D_{t+1} + 4a^2b^2D_t = 0$ is a linear second order homogeneous equation. Its characteristic equation is $r^2 - 4(ab+1)r + 4a^2b^2 = 0$, and it has roots given by

$$r = \frac{4(ab+1) \pm \sqrt{16(ab+1)^2 - 4 \cdot 4a^2b^2}}{2} = 2(ab+1) \pm 2\sqrt{2ab+1}$$

Since we assume that $1 + 2ab > 0$, there are distinct characteristic roots $r_1 \neq r_2$ given by

$$r_1 = 2(ab + \sqrt{2ab+1}), \quad r_2 = 2(ab - \sqrt{2ab+1})$$

and the general solution is

$$D_t = C_1 \cdot r_1^t + C_2 \cdot r_2^t = \mathbf{2^t}(\mathbf{C_1} \cdot (\mathbf{ab} + \sqrt{\mathbf{2ab+1}})^{\mathbf{t}} + \mathbf{C_2} \cdot (\mathbf{ab} - \sqrt{\mathbf{2ab+1}})^{\mathbf{t}})$$

9 The difference equation $x_{t+2} - x_{t+1} - x_t = 0$ is a linear second order homogeneous equation, with characteristic equation $r^2 - r - 1 = 0$ and characteristic roots given by

$$r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Hence it has two distinct characteristic roots $r_1 \neq r_2$ given by

$$r_1 = \frac{1+\sqrt{5}}{2} \simeq 1.618, \quad r_2 = \frac{1-\sqrt{5}}{2} \simeq -0.618$$

and the general solution is $x_t = C_1 \cdot r_1^t + C_2 \cdot r_2^t$. It is globally asymptotically stable if $x_t \to 0$ as $t \to \infty$ for all values of C_1, C_2, and this is not the case since $r_1 > 1$. In fact, $x_t \to \pm\infty$ as $t \to \infty$ if $C_1 \neq 0$. Therefore, the difference equation is **not globally asymptotically stable**.

10 Final Exam in GRA6035 10/12/2007, Problem 3

a) We have $\dot{x} = (t-2)x^2 \implies \frac{1}{x^2}\dot{x} = t-2 \implies \int \frac{1}{x^2}dx = \int (t-2)dt \implies -\frac{1}{x} = \frac{1}{2}t^2 - 2t + C \implies x = \frac{-2}{t^2-4t+2C}$. The initial condition $x(0) = \frac{-2}{2C} = \frac{-1}{C} = 1 \implies C = -1 \implies x(t) = \frac{-2}{t^2-4t-2}$.

b) We have $\ddot{x} - 5\dot{x} + 6x = 0, r^2 - 5r + 6 = 0 \implies r = 3, r = 2 \implies x_h(t) = Ae^{2t} + Be^{3t}$, and $x_p = Ce^{7t} \implies \dot{x}_p = 7Ce^{7t} \implies \ddot{x}_p = 49Ce^{7t}$ gives $\ddot{x}_p - 5\dot{x}_p + 6x_p = Ce^{7t}(49 - 5\cdot 7 + 6) = 20Ce^{7t} = 1 \implies C = \frac{1}{20}$. Hence $x(t) = Ae^{2t} + Be^{3t} + \frac{1}{20}e^{7t}$.

c) Integrating factor $e^{t^2} \implies xe^{t^2} = \int te^{-t^2+t}e^{t^2}dt = \int te^t dt = te^t - e^t + C \implies x(t) = (te^t - e^t + C)e^{-t^2}$.

d) We have $\frac{\partial}{\partial t}(3x^2e^{x^3+3t}) = 9x^2e^{3t+x^3}$ and $\frac{\partial}{\partial x}(3e^{x^3+3t} - 2e^{2t}) = 9x^2e^{3t+x^3}$, so the differential equation is exact. We look for h with $h'_x = 3x^2e^{x^3+3t} \implies h = e^{x^3+3t} + \alpha(t) \implies h'_t = 3e^{x^3+3t} + \alpha'(t)$. But $h'_t = 3e^{x^3+3t} + \alpha'(t) = 3e^{x^3+3t} - 2e^{2t} \implies \alpha'(t) = -2e^{2t} \implies \alpha(t) = -e^{2t} + C \implies h = e^{x^3+3t} - e^{2t} + C$. This gives solution in implicit form

$$h = e^{x^3+3t} - e^{2t} = K$$

The initial condition $x(1) = -1 \implies e^{(-1)^3+3} - e^2 = K \implies K = 0 \implies e^{x^3+3t} - e^{2t} = 0 \implies e^{x^3+3t} = e^{2t} \implies x^3 + 3t = 2t \implies x^3 = -t \implies x(t) = \sqrt[3]{-t}$.

11 Final Exam in GRA6035 10/12/2010, Problem 3a

We have $b_{t+1} - b_t = rb_t - s_{t+1}$, where $s_{t+1} = 500 + 10t$ is the repayment in period $t+1$. Hence we get the difference equation

$$b_{t+1} = (1+r)b_t - (500+10t), \quad b_0 = K$$

The homogenous solution is $b_t^h = C(1+r)^t$. We try to find a particular solution of the form $b_t = At + B$, which gives $b_{t+1} = At + A + B$. Substitution in the difference equation gives

$$At + A + B = (1+r)(At+B) - (500+10t) = ((1+r)A - 10)t + (1+r)B - 500$$

and this gives $A = 10/r$ and $B = 10/r^2 + 500/r$. Hence the solution of the difference equation is

$$b_t = b_t^h + b_t^p = C(1+r)^t + \frac{10}{r}t + \frac{10}{r^2} + \frac{500}{r}$$

The initial value condition is $K = C + 10/r^2 + 500/r$, hence we obtain the solution

$$b_t = \left(\mathbf{K} - \frac{\mathbf{10}}{\mathbf{r^2}} - \frac{\mathbf{500}}{\mathbf{r}}\right)(\mathbf{1}+\mathbf{r})^{\mathbf{t}} + \frac{\mathbf{10}}{\mathbf{r}}\mathbf{t} + \frac{\mathbf{10}}{\mathbf{r^2}} + \frac{\mathbf{500}}{\mathbf{r}}$$

12 Mock Final Exam in GRA6035 12/2010, Problem 3

See handwritten solution on the coarse page for GRA 6035 Mathematics 2010/11.

13 Final Exam in GRA6035 30/05/2011, Problem 3a

We have $x_{t+1} - 3x_t = 4$, and the homogenous solution is $x_t^h = C \cdot 3^t$. We try to find a particular solution of the form $x_t = A$, and substitution in the difference equation gives $A = 3A + 4$, so $A = -2$ is a particular solution. Hence the solution of the difference equation is

$$x_t = x_t^h + x_t^p = C \cdot 3^t - 2$$

The initial value condition is $2 = C - 2$, hence we obtain the solution

$$x_t = \mathbf{4} \cdot \mathbf{3^t} - \mathbf{2}$$

This gives $x_5 = \mathbf{970}$.