SEP 14 2011
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Problem 1.8:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ -1 & a & -21 & 2 \\ 3 & 7 & a & b \end{array}\right)$$

Compute $|A| = \begin{vmatrix} 1 & 2 & 3 \\ -1 & a & -21 \\ 3 & 7 & a \end{vmatrix} = 1\cdot(a^2 + 147) \overset{-(-1)}{+1}\cdot(2a-21) + 3(-42-3a)$

$$= a^2 - 7a + \cancel{147} \cancel{-21} \cancel{-126} = 0$$

$a = 0$ or $a = 7$

If $a \neq 0$, $a \neq 7$, then there is one solution. (since $|A| \neq 0$)

$\underline{a = 0}$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ -1 & 0 & -21 & 2 \\ 3 & 7 & 0 & b \end{array}\right) \begin{array}{l} \\ 1 \\ -3 \end{array} \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 2 & -18 & 3 \\ 0 & 1 & -9 & b-3 \end{array}\right) \begin{array}{l} \\ -1/2 \\ \end{array}$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 2 & -18 & 3 \\ 0 & 0 & 0 & b-\frac{9}{2} \end{array}\right)$$

$\underline{b = \frac{9}{2}}$: infinitely many solutions

$\underline{b \neq \frac{9}{2}}$: no solutions

$a \neq 0, a \neq 7$: one solution
$a = 0, b = \frac{9}{2}$: one degree of freedom
$a = 0, b \neq \frac{9}{2}$: no solution
$a = 7, b = \frac{10}{3}$: one degree of freedom
$a = 7, b \neq \frac{10}{3}$: no solutions

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ -1 & 7 & -21 & 2 \\ 3 & 7 & 7 & b \end{array}\right) \begin{array}{l} \\ 1 \\ -3 \end{array}$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 9 & -18 & 3 \\ 0 & 1 & -2 & b-3 \end{array}\right) \begin{array}{l} \\ -\frac{1}{9} \\ \end{array}$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 9 & -18 & 3 \\ 0 & 0 & 0 & b-\frac{10}{3} \end{array}\right)$$

Problem 1.14:

4×6 linear system homogeneous $\Longrightarrow$ non-trivial solutions

$$4\left\{\left(\underbrace{\begin{array}{cccccc} \odot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \odot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \odot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \odot & \cdot & \cdot \end{array}}_{6}\left|\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right.\right)\right.$$

augmented matrix

maximally 4 pivot positions

$\Downarrow$

at least $6-4=2$ free variables

$\Downarrow$

non-trivial solutions

Problem 2.16 d):

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{Find } A^{-1}.$$

zero

$$\left(\begin{array}{ccc|ccc} 3 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \Big]{-1} \rightarrow \left(\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} :3 \\ \\ :2 \end{array}$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & -1/3 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right) = (I \mid A^{-1})$$

$$A^{-1} = \begin{pmatrix} 1/3 & -1/3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$$

If you cannot get I in the first part, it means that A^{-1} does not exist.

Problem 2.19.

Give an example s.t. $rk(AB) \neq rk(BA)$.

Hint: Use A, B : 2×2-matrices.

$$A = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \qquad B = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$$

If $rk(AB) = 2$: $|AB| \neq 0$ but $|AB| = |A|\cdot|B|$
$= |B|\cdot|A| = |BA|$

$$\Rightarrow rk(BA) = 2$$

Conclusion: We must find A, B s.t. $\begin{cases} rk(AB) = 1 \\ rk(BA) = 0 \end{cases}$

i.e. $BA = 0$, $AB \neq 0$

Try: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\cancel{B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}$ $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

~~$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $rk(AB) = 1$~~

~~$BA = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ not good~~

$AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $rk\ AB = 1$

$BA = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $rk\ BA = 0$

Problem 3.4.

$\underline{a}, \underline{b}, \underline{c}$ linearly independent m-vectors

a) Show that $\underline{a}+\underline{b}$, $\underline{b}+\underline{c}$, $\underline{a}+\underline{c}$ are lin. independent.

$$c_1 \cdot \underline{v_1} + c_2 \cdot \underline{v_2} + c_3 \cdot \underline{v_3} = \underline{0}$$

$$c_1 \cdot (\underline{a} + \underline{b}) + c_2 \cdot (\underline{b} + \underline{c}) + c_3 \cdot (\underline{a} + \underline{c}) = \underline{0}$$

$$(c_1 + c_3)\underline{a} + (c_1 + c_2)\underline{b} + (c_2 + c_3) \cdot \underline{c} = \underline{0}$$

$$x \cdot \underline{a} + y \cdot \underline{b} + z \cdot \underline{c} = \underline{0} \Rightarrow x = y = z = 0$$

$$\begin{aligned} c_1 + c_3 &= 0 \\ c_1 + c_2 &= 0 \\ c_2 + c_3 &= 0 \end{aligned} \qquad \begin{aligned} c_1 \quad\quad + c_3 &= 0 \\ c_1 + c_2 \quad\quad &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1 \cdot 1 - 1 \cdot (-1) = 2 \neq 0$$

$\Rightarrow$ one solution $\underline{c_1 = c_2 = c_3 = 0}$

Conclusion:

Only trivial solution $c_1 = c_2 = c_3 = 0$ $\Big\}$ $\Rightarrow$ $\underline{a}+\underline{b}$, $\underline{b}+\underline{c}$, $\underline{a}+\underline{c}$ are linearly independent.

Problem 3.5: $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$

(1) One of the vectors (say $\underline{v}_1$) can be written as a linear comb. of the others

(2) $x_1\underline{v}_1 + \dots + x_n\underline{v}_n = \underline{0}$ has non-trivial solutions

Show that (1) $\Leftrightarrow$ (2).

a) (1) $\Rightarrow$ (2):
$$\underline{v}_1 = a_2\underline{v}_2 + a_3\underline{v}_3 + \dots + a_n\underline{v}_n$$
$$\Downarrow$$
$$\underline{0} = -\underline{v}_1 + a_2\underline{v}_2 + \dots + a_n\underline{v}_n$$
$$\Downarrow$$
$(-1, a_2, a_3, a_4, \dots, a_n)$ is a non-trivial solution.

b) (2) $\Rightarrow$ (1): Let $x_1\underline{v}_1 + x_2\underline{v}_2 + \dots + x_n\underline{v}_n = \underline{0}$

with one $x_i \neq 0$, say $x_1 \neq 0$.
$$\Downarrow$$
$$x_1\underline{v}_1 = -x_2\underline{v}_2 - x_3\underline{v}_3 - \dots - x_n\underline{v}_n$$
$$\Downarrow$$
$$\underline{v}_1 = \left(-\frac{x_2}{x_1}\right)\underline{v}_2 - \left(\frac{x_3}{x_1}\right)\underline{v}_3 - \dots - \frac{x_n}{x_1}\underline{v}_n$$

$\underline{v}_1$ is a linear comb. of the other vectors

Problem 3.7:

n vectors in $\mathbb{R}^m$ are linearly dependent if $\underline{n > m}$.

$$\underline{v}_1, \underline{v}_2, \underline{v}_3, \ldots\ldots, \underline{v}_n \rightsquigarrow A = \underbrace{\left(\underline{v}_1 \,\middle|\, \underline{v}_2 \,\middle|\, \underline{v}_3 \,\middle|\, \cdots \,\middle|\, \underline{v}_n \right)}_{n} \Big\} m$$

$n = 5$, $m = 3$: $\underbrace{\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}}_{5} \Big\} 3$

$$\boxed{\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\} \text{ lin. independent} \iff \operatorname{rk} A = n}$$

Fact: If $n > m$, then $\operatorname{rk} A \leq m < n$.

$\Rightarrow$ the vectors are <u>linearly dependent</u>.

Problem 3.11:

It is given $\text{rk}(A^TA) = \text{rk}\,A$.

a) $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \\ 4 & 8 \end{pmatrix}$ $\quad A^T \cdot A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \\ 4 & 8 \end{pmatrix}$

$= \begin{pmatrix} 30 & 60 \\ 60 & 120 \end{pmatrix}$

$|A^TA| = 30 \cdot 120 - 60^2 = 0$

$\Rightarrow \text{rk}\, A^TA < 2$

$\Rightarrow \text{rk}\, A^TA = 1 \Rightarrow \underline{\underline{\text{rk}\, A = 1}}$

Problem 3.12:

A $m \times n$ - matrix. Nullspace of A = solution of $A\underline{x} = \underline{0}$.

a) Nullspace of A = Nullspace of A^TA.

Show that: $\boxed{\text{Solutions of } A\underline{x} = \underline{0}} = \boxed{\text{Solutions of } A^TA\underline{x} = \underline{0}}$

1) $\underline{x}$ solution of $A\underline{x} = \underline{0}$

$A^TA\underline{x} = A^T \cdot \underline{0} = \underline{0}$ $\Rightarrow$ Solution of $A^TA\underline{x} = \underline{0}$.

2) $\underline{x}$ solution of $A^TA\underline{x} = \underline{0}$

$\underline{x}^T A^TA\underline{x} = \underline{x}^T \cdot \underline{0} = \underline{0}$

$(A\underline{x})^T (A\underline{x}) = \underline{0}$

$u_1^2 + u_2^2 + \dots + u_m^2 = 0$

$u_1 = u_2 = \dots = u_m = 0$

$\underline{x}$ solution of $A\underline{x} = \underline{0}$. $\Leftarrow$ $A\underline{x} = \underline{0}$

$\left\{ \begin{array}{l} A\underline{x} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} \\ (A\underline{x})^T = (u_1\ u_2\ \dots\ u_m) \\ (u_1\ u_2 \dots u_m) \cdot \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} = \end{array} \right.$

$= u_1^2 + u_2^2 + \dots + u_m^2$

b) Show that $\operatorname{rk} A = \operatorname{rk} A^T A$. A $m \times n$ matrix.

$$m\left\{\underbrace{\begin{pmatrix} \vdots & \cdots & \cdots \\ \vdots & \cdots & \cdots \end{pmatrix}}_{n}\right.$$

$A \cdot \underline{x} = \underline{0}$	$\underset{n\times m}{A^T}\,\underset{m\times n}{A}\,\underline{x} = \underline{0}$
no. free variables	no. free variables
$= n - \operatorname{rk} A$	$= n - \operatorname{rk}(A^T A)$

The two systems have the same solutions

$$n - \operatorname{rk} A = n - \operatorname{rk}(A^T A)$$

$$\underline{\underline{\operatorname{rk} A = \operatorname{rk}(A^T A)}}$$

Problem 1.16:

$$A = \begin{pmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 7 \\ 6 & 9 & -5 & -2 & 4 \end{pmatrix} \begin{matrix} \\ \leftarrow \cdot(-2) \\ \leftarrow \cdot(-3) \end{matrix} \quad \text{rk } A =$$

$$\downarrow$$

$$\begin{pmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & -6 & 4 & 10 & -20 \end{pmatrix} \begin{matrix} \\ \\ \leftarrow \cdot(-2) \end{matrix}$$

$$\downarrow$$

$$\begin{pmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 0 & -6 \end{pmatrix} \Rightarrow \text{rk } A = \underline{\underline{3}}$$

echelon form

Problem 2.11:

$$\begin{vmatrix} 3 & 1 & 5 \\ 9 & 3 & 15 \\ -3 & -1 & -5 \end{vmatrix} \begin{matrix} \\ \leftarrow \cdot(-3) \\ \\ \end{matrix} = \begin{vmatrix} 3 & 1 & 5 \\ 0 & 0 & 0 \\ -3 & -1 & -5 \end{vmatrix} = \underline{\underline{0}}$$

Problem 2.20

b) $$\begin{aligned} x_1 + x_2 - x_3 + x_4 &= 2 \\ 2x_1 - x_2 + x_3 - 3x_4 &= 1 \end{aligned} \qquad \left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 2 \\ 2 & -1 & 1 & -3 & 1 \end{array}\right)$$

$$\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -1 - 2 = -3 \neq 0 \qquad \begin{cases} \text{rk } A = 2 \\ \text{rk } \hat{A} = 2 \end{cases}$$

$\text{rk } A = \text{rk } \hat{A}$: there are solutions

$n - \text{rk } A = 4 - 2 = 2$ degrees of freedom

x_1, x_2 : basic

x_3, x_4 : free

$$x_1 + x_2 = x_3 - x_4 + 2 \Rightarrow x_2 = x_3 - x_4 + 2 - \left(\frac{2}{3}x_4 + 1\right)$$
$$2x_1 - x_2 = -x_3 + 3x_4 + 1$$

$$3x_1 = 2x_4 + 3 \Rightarrow x_1 = \underline{\frac{2}{3}x_4 + 1}$$

$$x_2 = \underline{x_3 - \frac{5}{3}x_4 + 1}$$

$$\left\{\begin{array}{l} x_1 = \frac{2}{3}x_4 + 1 \\ x_2 = x_3 - \frac{5}{3}x_4 + 1 \\ x_3 \quad \text{free} \\ x_4 \quad \text{free} \end{array}\right.$$

c) $\left(\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 1 \\ 2 & 1 & -1 & 3 & 3 \\ 1 & 5 & -8 & 1 & 1 \\ 4 & 5 & -7 & 7 & 7 \end{array}\right)$

$$\begin{vmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 3 \\ 1 & 5 & -8 & 1 \\ 4 & 5 & -7 & 7 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & 1 \\ 0 & 6 & -10 & 0 \\ 0 & 0 & 3 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & 1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 3 & 3 \end{vmatrix} = 1 \cdot 3 \cdot \begin{vmatrix} 0 & -2 \\ 3 & 3 \end{vmatrix}$$

$= 3 \cdot 6 = 18 \neq 0. \Rightarrow \mathrm{rk}\, A = 4, \mathrm{rk}\, \hat{A} = 4$

$n - \mathrm{rk}\, A = 4 - 4 = 0$

$\Downarrow$

one solution.

Find solution via Gauss.

Problem 3.14:

$$\begin{pmatrix} -4 \\ 3 \\ h \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} + y \cdot \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix} + z \cdot \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} -4 \\ 3 \\ h \end{pmatrix}}_{\underline{w}} \quad \underbrace{\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}}_{\underline{v}_1} \quad \underbrace{\begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix}}_{\underline{v}_2} \quad \underbrace{\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}}_{\underline{v}_3}$$

$\underline{w}$ lin comb of $\underline{v}_1, \underline{v}_2, \underline{v}_3$ $\Longleftrightarrow$ the vecter equatoin has solutions

$$\begin{pmatrix} 1 & 5 & -3 \\ -1 & -4 & 1 \\ -2 & -7 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ h \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{array}\right) \xrightarrow{} \left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{array}\right) \xrightarrow{} \left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{array}\right)$$

$\underline{h=5}$: One free var $\Longrightarrow$ w is lin comb.

$\underline{h \neq 5}$: no solutions $\Longrightarrow$ w is not. lin comb.

Alt. c