HELLO!

DAG EINAR SOMMERVOLL

PROBLEM SHEET 7

(TOMORROW:

- ENVELOPE TEOREM(S)
- ~~BROD~~ BORDERED HESSIANS
- KUHN TUCKER-CONDITIONS.

1. $f(x,y,z) = x^4 + y^4 + z^4 + x^2 + y^2 + z^2$

(OBS. ALL TERMS NON NEGATIVE
$\Rightarrow$ 0 IS THE LOWEST POSSIBLE VALUE OF f.
OBS. $\boxed{f(0,0,0) = 0}$)

SOLUTION:

I. $\dfrac{\partial f}{\partial x} = 4x^3 + 2x = 0$
$2x(2x^2+1) = 0$

II. $\dfrac{\partial f}{\partial y} = 4y^3 + 2y = 0$
$2y(2y^2+1) = 0$

III. $\dfrac{\partial f}{\partial z} = 4z^3 + 2z = 0$

$2z(2z^2+1) = 0$

I. $\Rightarrow$ $x = 0$ (NOTE $2x^2+1 > 0$)
II $\Rightarrow$ $y = 0$
III. $\Rightarrow$ $z = 0$.

THIS IS A GLOBAL MINIMUM DUE TO $f(x,y,z) \geq 0$,

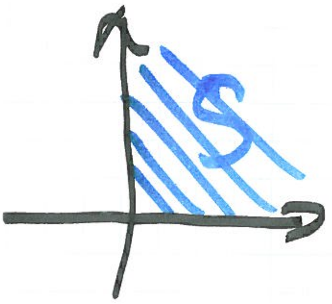
[WE COULD ALSO LOOK AT THE HESSIAN...]

2. ① SHOW

$\cancel{f(x,y)}$

$$f(x,y,\cancel{z}) = x^3 + y^3 - 3x - 2y$$

IS DEFINED ON



$$S = \{(x,y) \;\; x>0, y>0\}$$

STRICTLY CONVEX

② FIND THE GLOBAL MINIMUM.

SOLUTION. (LET US DO ② FIRST.

$$\frac{\partial f}{\partial x} = 3x^2 - 3 = 3(x+1)(x-1) \overset{?}{=} 0$$

$$\frac{\partial f}{\partial y} = 3y^2 - 2 = 3\left(y+\sqrt{\tfrac{2}{3}}\right)\left(y-\sqrt{\tfrac{2}{3}}\right) = 0$$

ONLY ONE POSSIBILITY IN S:

$x = 1 \quad y = \sqrt{\frac{2}{3}}$

STRICTLY CONVEX!

$$H = \begin{bmatrix} 6x & 0 \\ 0 & 6y \end{bmatrix}$$

$D_1 = 6x > 0$ (ON S!)

$D_2 = 36xy > 0$ (ON S!)

$\Rightarrow f$ CONVEX ON S.

(AND THAT $(1, \sqrt{\frac{2}{3}})$ MINIMUM)

3. $C(x,y) = 0.04x^2 - 0.01xy$
$+ 0.01y^2 + 4x + 2y + 500$

x A 13

y B 8

SOLUTION:

$\pi(x,y) = 13x + 8y - C(x,y)$

$= 13x + 8y - (0.04x^2 - 0.01xy$
$+ 0.01y^2 + 4x + 2y + 500)$

$= -0.04x^2 - 0.01y^2 + 0.01xy$
$+ 9x + 6y - 500$

CANDIDATES FOR MAX. PROFITS:

2 I. $\frac{\partial \pi}{\partial x} = -0.08x + \underline{0.01}y + 9 = 0$

II $\frac{\partial \pi}{\partial y} = \boxed{+0.01x - \underline{0.02}y + 6 = 0}$

2I+II: $(-0.16 + 0.01)x + 0 \cdot y + 18 + 6 = 0$

$-0.15x = -24$

$x = \frac{24}{0.15} = \frac{\cancel{24}^{8} \cdot 100}{\cancel{15}_{5}} = \underline{160}$

Y: $x - 2y + 600 = 0$

$160 - 2y + 600 = 0$

$Y = \frac{760}{2} = \underline{380}$

3. CONTINUED:

LET US LOOK AT THE HESSIAN.

$$H = \begin{bmatrix} -0.08 & 0.01 \\ 0.01 & -0.02 \end{bmatrix}$$

$$D_1 = -0.08 < 0$$

$$D_2 = \left(\frac{1}{100}\right)^2 \left[8 \cdot 2 - 1 \cdot 1\right] > 0$$

Π IS CONCAVE, AND

(160, 380) IS A MAX.

4. $f(x,y,z) = x^2 + 2xy + y^2 + z^3$

DEFINED ON $S = \{(x,y,z) \ z > 0\}$

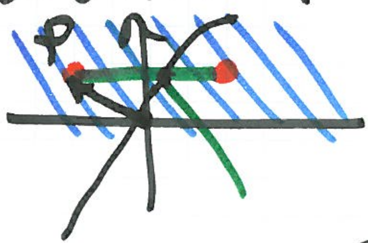
(1) SHOW THAT S IS CONVEX

(2) FIND STAT. POINTS

(3) HESSIAN (4) CONVEX OR CONCAV

(5) EXTREMAL POINTS.

SOLUTION :

(1) 

CONSIDER TWO ARB. POINTS IN S.

~~(a_1, b_1, c_1)~~

R^3

$P(a_1, b_1, c_1)$ AND

$Q(a_2, b_2, c_2)$ $c_1, c_2 > 0$

WE NEED TO PROVE THAT THE LINE BETWEEN P AND Q IS IN S.

BUT THIS LINE IS GIVEN ON VECTOR FORM BY :

$$\vec{OR} = \vec{OP} + t\,\vec{PQ} \qquad t \in [0,1]$$

$$\vec{OR} = [a_1, b_1, c_1] + t\,[a_2 - a_1, b_2 - b_1, c_2 - c_1]$$

(RECALL: $\vec{PQ} = \vec{PO} + \vec{OQ} = -\vec{OP} + \vec{OQ}$

WE NEED TO PROVE THAT R HAS POSITIVE C (THIRD COORDINATE)

BUT THIS FOLLOWS FROM $C = c_1 + t(c_2 - c_1)$

$$C = C_1 + t(C_2 - C_1)$$

$$= \underbrace{(1-t)}_{\geq 0}\,\underbrace{C_1}_{>0} + \underbrace{t}_{\geq 0}\,\underbrace{C_2}_{>0} > 0$$

NOT AT THE SAME TIME

2. HOUR.

4. CONTINUED.

$$f(x,y,z) = \underbrace{\boxed{x^2 + 2xy + y^2}}_{}^{(x+y)^2} + z^3 \implies \text{ALWAYS NON NEG. ON } S.$$

$$\frac{\partial f}{\partial x} = 2x + 2y = 0$$

$$\frac{\partial f}{\partial y} = 2x + 2y = 0$$

$$\frac{\partial f}{\partial z} = 3z^2 = 0$$

IMPOSSIBLE ON S.

$\implies$ NO STATIONARY POINTS ON S.

HESSIAN:

$$H = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 6z \end{bmatrix}$$

$D_1 = 2 > 0$

$D_2 = 0$

$D_3 = 0$

$\implies$ POSITIVE SEMIDEF. ON S

$\implies$ f CONVEX (BUT NOT STRICTLY)

9. $f(x,y,z) = x^3 + 3xy + 3xz + y^3 + 3yz + z^3$

(SYMMETRIC IN x, y, z ....)

1. FIND LOCAL EXT.
2. CLASSIFY.

SOLUTION: WE CONSIDER $g(x,y,z) = \frac{1}{3} f(x,y,z)$

$\cancel{\frac{\partial f}{\partial x} = 3}x$ DO NOT WANT ALL THESE 3.!

$$\text{I} \quad \frac{\partial g}{\partial x} = x^2 + y + z = 0$$

$$\text{II.} \quad \frac{\partial g}{\partial y} = x + y^2 + z = 0$$

$$\text{III.} \quad \frac{\partial g}{\partial z} = x + y + z^2 = 0.$$

$$\text{I}-\text{II}: \quad x^2 - y^2 - (x-y) = 0$$

$$(x-y)(x+y-1) = 0$$

$$\text{I}-\text{III} \quad x^2 - z^2 - (x-z) = 0$$

$$(x-z)(x+z-1) = 0$$

THIS GIVES

1 $\boxed{\begin{array}{l} x-y=0 \\ x-z=0 \end{array}}$ $\boxed{x+y-1=0}$ 2

$\boxed{x+z-1=0}$ 3

1 $x=y=z$

I. $x^2+y+z=0$ BECOMES

$$x^2+x+x=0$$

$$x^2+2x=0 \text{ OR } \begin{array}{l} x=0 \\ x=-2 \end{array}$$

SO WE GET : $(0,0,0)$

$(-2,-2,-2)$

2 NOTE $x+y-1=0$ $x+y=1$

BUT III $x+y+z^2=0$

HENCE $1+z^2=0$ $\Rightarrow$ IMPOSSIBLE

3 SAME ARGUMENT:

II $[x+z+y^2=0] + [x+z=1]$

$\Rightarrow$ IMPOSSIBLE

CLASSIFICATION:

$(0,0,0) \quad (-2,-2,-2)$

HESSIAN OF g:

$$H = \begin{vmatrix} 2x & 1 & 1 \\ 1 & 2y & 1 \\ 1 & 1 & 2z \end{vmatrix}$$

$D_1 = 2 > 0$

$D_2 = 2 \cdot 2 - 1 \cdot 1 = 3 >$

D_3

$$D_3 = 2 \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$$

$$= 2 \cdot (4-1) - (2-1) + (1-2)$$

$$= 6 - 1 - 1 = 4 > 0$$

$$H(0) = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$D_1 = 0$

$D_2 = -1$

$$D_3 = 0 - 1 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= -1(-1) + 1 = 2$$

$\Rightarrow$ SADDLE POINT. $(0,0,0)$

$H(-2,-2,-2)$

$$= \begin{bmatrix} -4 & 1 & 1 \\ 1 & -4 & 1 \\ 1 & 1 & -4 \end{bmatrix}$$

$D_1 = -4 < 0$

$D_2 = (-4)(-4) - 1$
$= 16 - 1 = 15 > 0$

$$D_3 = -4 \begin{vmatrix} -4 & 1 \\ 1 & -4 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & -4 \end{vmatrix} + \begin{vmatrix} 1 & -4 \\ 1 & 1 \end{vmatrix} = -4(15) - (-5) + (+5) < 0$$

$\Rightarrow$ MAX.

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$$f(x,y,z) = x^2 e^x + yz - z^3$$

a. FIND ALL STAT. POINTS

SOLUTION:

$$\text{I} \quad \frac{\partial f}{\partial x} = 2x e^x + x^2 e^x = 0$$

$$x(2+x)e^x = 0$$

$$\text{II} \quad \frac{\partial f}{\partial y} = z = 0 \Rightarrow z=0$$

$$\text{III} \quad \frac{\partial f}{\partial z} = y - 3z^2 = 0$$

$$z=0 \Rightarrow \text{III} \; y = 3z^2 = 0$$

$$\text{I} \Rightarrow x=0 \text{ OR } x=-2.$$

THIS GIVES US TWO POINTS

$\underline{(0,0,0)}$ AND $\underline{(-2,0,0)}$

b. COMPUTE THE HESSIAN:

$$\frac{\partial f}{\partial x} = (2x + x^2)e^x$$

$$\frac{\partial f}{\partial y} = z$$

$$\frac{\partial f}{\partial z} = y - 3z^2$$

$$\frac{\partial^2 f}{\partial x^2} = (2+2x)e^x + (2x+x^2)e^x = (x^2+4x+2)e^x$$

$$H = \begin{bmatrix} (x^2+4x+2)e^x & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -6z \end{bmatrix}$$

CLAS. OF STAT. POINTS:

$(0,0,0)$ $\quad H = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad D_1 = 2 > 0$

$D_2 = 0$

$D_3 = 2 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -2 < 0$

$\Rightarrow$ SADDLE POINT

$(-2,0,0)$ $\quad H = \begin{bmatrix} -2e^{-2} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad \boxed{D_1 = -2e^{-2} < 0}$

$D_2 = 0$

$\boxed{D_3 = -2e^{-2} \cdot -1 > 0}$

$\Rightarrow$ SADDLE POINT

13. $f(x,y,z) = e^{xy+yz-xz}$

a. FIND ALL STAT. POINTS

SOLUTION:

$$\text{I} \quad \frac{\partial f}{\partial x} = (y-z)e^{(\quad)} = 0$$

$$\text{II} \quad \frac{\partial f}{\partial y} = (x+z)e^{(\quad)} = 0$$

$$\text{III} \quad \frac{\partial f}{\partial z} = (y-x)e^{(\quad)} = 0$$

$$\left.\begin{array}{llll} \text{I.} & \Rightarrow & y-z=0 & y=z \\ \text{II} & \Rightarrow & x+z=0 & x=-z \\ \text{III} & \Rightarrow & y-x=0 & y=x \end{array}\right\} \Rightarrow y = \pm z \Rightarrow y=0,\ z=0,\ x=0$$

[DIGRESSION

$$\begin{array}{rrrl} & y & -z & =0 \\ x & & +z & =0 \\ x & -y & & =0 \end{array}$$

THIS IS OF TYPE $Ax=0$ (MATRICES

SINCE A INVERTIBLE ONLY ONE SOLUTION]

CONCLUSION: ONE STAT POINT $(0,0,0)$

b. $g(x,y,z) = \boxed{e^{ax+by+cz}}$

$S = \mathbb{R}^3$

DETERMINE THE VALUES a, b, c SUCH THAT g IS CONVEX

FRUIT FOR THE MIND:

$e^{\text{SOMETHING}}$ e IS CONVEX

SOMETHING IS LINEAR

SOLUTION: $\dfrac{\partial g}{\partial x} = a e^{(\quad)}$

$\dfrac{\partial g}{\partial y} = b e^{(\quad)}$

$\dfrac{\partial g}{\partial z} = c e^{(\quad)}$

SKIPPING COMMON $e^{(\ )}$

$$H = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$$

$D_1 = a^2 \geq 0$

$D_2 = a^2 \cdot b^2 - abab = 0$

$$D_3 = a^2 \begin{vmatrix} b^2 & bc \\ bc & c^2 \end{vmatrix} - ab \begin{vmatrix} ab & bc \\ ac & c^2 \end{vmatrix} + ac \begin{vmatrix} ab & b^2 \\ ac & bc \end{vmatrix}$$

$$D_3 = a^2 \cdot (\underbrace{b^2c^2 - bcbc}_{=0})$$

$$- ab\,(\underbrace{abc^2 - abc^2}_{=0})$$

$$+ ac\,(\underbrace{ab \cdot bc - ab^2c}_{=0}) = 0 .$$

$$D_3 = 0$$

$\Longrightarrow$ CONVEX FOR ALL a, b, c.

ONE THING: CONCAVE?

CONCAVE IF AND ONLY IF

$a^2 = b^2 = c^2 = 0$.

(IN THIS CASE $g(x,y,z) = e^0 = 1$)

14. $f(x,y,z,w) = x^5 + xy^2 - zw$

a. STAT. POINTS:

I $\frac{\partial f}{\partial x} = 5x^4 + y^2 = 0$

II $\frac{\partial f}{\partial y} = 2xy = 0 \Rightarrow \overset{1.}{x=0} \vee \overset{2.}{y=0}$

III $\frac{\partial f}{\partial z} = -w = 0 \Rightarrow w=0$

IV $\frac{\partial f}{\partial w} = -z = 0 \Rightarrow z=0$

1. $x=0 \Rightarrow$ I $5 \cdot 0^4 + y^2 = 0$

$\Rightarrow y = 0$

2. $y=0 \Rightarrow 5 \cdot x^4 + 0^2 = 0$

$\Rightarrow x = 0$

ONE STATIONARY POINT: $(0,0,0,0)$

b. CLASSIFICATION:

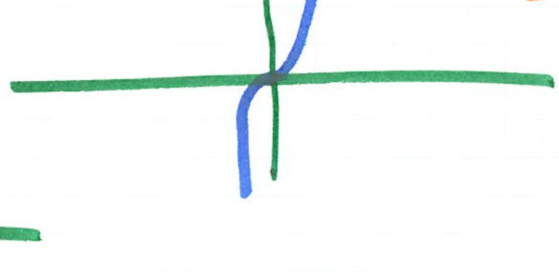
$$H = \begin{bmatrix} 20x^3 & 2y & 0 & 0 \\ 2y & 2x & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

THE POINT $(0,0,0,0)$

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \rightarrow \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} = -1 < 0$$

$\Rightarrow$ SADDLE POINT.

[COMMENT: SADDLE POINTS CAN BE EASY TO "SMOKE OUT"! IN THIS RIGHT FROM START: LOOK AT $f(x,0,0,0) = x^5$



SADDLE OR NOT? SADDLE...]