HELLO **HELLO**

TWO LEFT OVERS: 10 – 11

10. $\mathcal{L} = 100 - x^2 - y^2 - z^2 - \lambda(x + 2y + z - a)$

I $\quad \mathcal{L}'_x = -2x - \lambda = 0$

II $\quad \mathcal{L}'_y = -2y - 2\lambda = 0$

III $\quad \mathcal{L}'_z = -2z - \lambda = 0$

I – III: $\quad -2x - (-2z) = 0 \Rightarrow -2x = -2z$

$\Rightarrow \underline{x = z}$

FROM I: $\quad -2x - \lambda = 0$

$\Rightarrow \lambda = -2x$

THEN II GIVES: $-2y - 2\lambda = -\cancel{2}y - \cancel{2}(-2x) = 0$

THAT IS $\quad -y = -2x \Rightarrow y = 2x$

NOW WE CAN USE THE CONSTRAINT:

$x + 2y + z = a$

$x + 2 \cdot 2x + x = a$

$6x = a \Rightarrow \boxed{x = \frac{a}{6} \quad y = \frac{2a}{6} \quad z = \frac{a}{6}}$

WHAT ABOUT λ?

$\lambda = -2x = -2 \cdot \frac{a}{6} = \underline{-\frac{a}{3}}$

THE POINT OF THIS EXERCISE IS TO SEE THAT $\frac{\partial f}{\partial a}$ (THE DERIVATIVE OF THE VALUE FUNCTION IS λ)

SO:

WE CALCULATE THE VALUE FUNCTION:

$$\underline{f(a)} = f\left(\frac{a}{6}, \frac{2a}{6}, \frac{a}{6}\right)$$

$$= 100 - \left(\frac{a}{6}\right)^2 - \left(\frac{2a}{6}\right)^2 - \left(\frac{a}{6}\right)^2$$

$$= 100 - \frac{a^2 + 4a^2 + a^2}{6^2}$$

$$= 100 - \frac{6a^2}{36} = \underline{100 - \frac{a^2}{6}}$$

THE DERIVATIVE: $\frac{\partial f}{\partial a} = -\frac{2a}{6} = -\frac{a}{3}$

(WHICH IS EQUAL TO λ ☺)

11. $x+4y+z$

$$\mathcal{L} = x+4y+z - \lambda(x^2+y^2+z^2 - 2(6)) - \mu(x+2y+3z)$$

FIRST ORDER COND:

$$\mathcal{L}_x = 1 - 2\lambda x - \mu = 0$$

$$\mathcal{L}_y = 4 - 2\lambda y - 2\mu = 0$$

$$\mathcal{L}_z = 1 - 2\lambda z - 3\mu = 0$$

NOTE: $\lambda = 0$ NOT POSSIBLE.

WHY? THEN $1-\mu=0$ AND $1-3\mu=0$.

SO $\boxed{\lambda \neq 0.}$

CONSIDER $2\lambda \,|\, x + 2y + 3z = 0$

$$\Rightarrow \cancel{2\lambda_* x + 2\lambda_* 2y + 3\lambda_* z = 0}$$

$$2\lambda x + 2\lambda \cdot 2y + 2\lambda \cdot 3z = 0$$

$$\boxed{2\lambda x + 4\lambda y + 6\lambda z = 0}$$

NOTE: $-2\lambda x = \mu - 1$ OR $2\lambda x = 1-\mu$

$-2\lambda y = 2\mu - 4$ OR $2\lambda y = 4-2\mu$

$-2\lambda z = 3\mu - 1$ OR $2\lambda z = 1-3\mu$

THIS IMPLIES: $2\lambda x + 4\lambda y + 6\lambda z = 0$

BECOMES $1-\mu + 2(4-2\mu) + 3(1-3\mu) = 0$

$$1+8+3-(1+4+9)\mu = 0$$

$$12 - 14\mu = 0$$

$$\underline{\mu} = \frac{12}{14} = \underline{\frac{6}{7}}$$

WE WILL USE THIS TO FIND x, y, z AS A FUNCTION OF λ, AND USE THE SECOND CONSTRAINT TO FIND λ, AND THEN WE ARE THRU. -- I. $\mathcal{L}_x = 1 - 2\lambda x - \mu = 0$

$$\mathcal{L}_y = 4 - 2\lambda y - 2\mu = 0$$

$$\mathcal{L}_z = 1 - 2\lambda z - 3\mu = 0$$

$\mu = \frac{6}{7} \Rightarrow$ I BECOMES $1 - 2\lambda x - \frac{6}{7} = 0$

$$\Rightarrow \quad -2\lambda x = -\frac{1}{7}$$

$$\boxed{x = \frac{1}{14\lambda}}$$

II BECOMES $4 - 2\lambda \cdot y - 2 \cdot \frac{6}{7} = 0$

$$-2\lambda y = \frac{12 - 28}{7}$$

$$y = \frac{16}{2 \cdot 7\lambda}$$

$$\boxed{y = \frac{8}{7\lambda}}$$

III BECOMES $1 - 2\lambda z - 3 \cdot \frac{6}{7} = 0$

$$-2\lambda z = \frac{18 - 7}{7}$$

$$\boxed{z = \frac{-11}{14\lambda}}$$

WE ~~KNOW~~ KNOW:

$$x = \frac{1}{14\lambda}, \quad y = \frac{8}{7\lambda} = \frac{16}{14\lambda}, \quad z = \frac{-11}{14\lambda}$$

CONSTRAINT: $x^2 + y^2 + z^2 = 216$

GIVES

$$\left(\frac{1}{14\lambda}\right)^2 + \left(\frac{16}{14\lambda}\right)^2 + \left(\frac{-11}{14\lambda}\right)^2 = 216$$

$$\left(\frac{1}{14\lambda}\right)^2 \left[1 + 16^2 + 11^2\right] = 216$$

$$\left(\frac{1}{14\lambda}\right)^2 \left[1 + 256 + 121\right] = 216$$

$$\left(\frac{1}{14\lambda}\right)^2 \left[378\right] = 216$$

$$\left[14\lambda\right]^2 = \frac{378}{216}$$

$$\lambda = \pm\sqrt{\frac{378}{14 \cdot 216}}$$

$$= \pm\frac{\sqrt{7}}{28}$$

WHICH GIVES

$$x = \frac{1}{14} \cdot \pm\left(\frac{28}{\sqrt{7}}\right) = \pm\frac{2}{\sqrt{7}}$$

$$y = \frac{16}{14} \cdot \pm\left(\frac{28}{\sqrt{7}}\right) = \pm\frac{32}{\sqrt{7}}$$

$$z = \frac{-11}{14} \pm\left(\frac{28}{\sqrt{7}}\right) = \pm\frac{-22}{\sqrt{7}}$$

WE ARE INTERESTED IN THE MAXIMUM

$$\max f(x,y,z) = x + 4y + z \quad \text{GIVEN} \quad x^2 + y^2 + z = 216$$

$$x + 2y + 3z = 0$$

$$\mathcal{L} = \underline{x + 4y + z} - \lambda (x^2 + y^2 + z^2 - 216) - \underline{\mu (x + 2y + 3z)}$$

$\mathcal{L}$ IS CONCAVE WHEN $\lambda > 0$.

$\mathcal{L}$ IS CONVEX WHEN $\lambda < 0$.

$\Rightarrow$ $\lambda > 0$ CORRESPONDS TO THE MAX.

SOLUTION: $\boxed{\lambda = \frac{\sqrt{7}}{28} \qquad \mu = \frac{6}{7}}$

NEED THIS IN b.

$$\boxed{\begin{aligned} x &= \frac{2}{\sqrt{7}} \\ y &= \frac{32}{\sqrt{7}} \\ z &= \frac{-22}{\sqrt{7}} \end{aligned}}$$

THE MAX VALUE IS:

$$f\left(\frac{2}{\sqrt{7}}, \frac{32}{\sqrt{7}}, \frac{-22}{\sqrt{7}}\right) \overset{\text{COMPUTATION}}{=} \frac{108}{7}\sqrt{7}$$

(≈ 40.820)

b) NEW MAX VALUE IF CONSTRAINTS ARE CHANGED TO:

$x^2 + y^2 + z^2 = 215$ (WAS 216)

$x + 2y + z = 0.1$ (WAS 0)

NOTE THAT THE LAGRANGE MULTIPLIERS, λ OG μ ARE JUST THE PARTIAL DERIVATIVES OF THE VALUE FUNCTION:

$$\text{CHANGE IN THE MAX} = \lambda(215-216) + \mu(0.1-0)$$

$$= \frac{\sqrt{7}}{28}(-1) + \frac{6}{7} \cdot 0.1$$

$$\approx 0.009$$

WHICH GIVES $f_{NEW} \approx 40.811$

(GUSTAVSEN/ERIKSEN)

7.1. (3.1.4 IN THE BOOK)

FIND $x^*(r)$, $y^*(r)$ SUCH THAT $x = x^*(r)$ $y = y^*(r)$ SOLVE THE PROBLEM:

$$\max_{x,y} f(x,y,r) = \max(-x^2 - xy - 2y^2 + 2rx + 2ry)$$

(r IS A PARAMETER)

$$\left(\text{VERIFY: } \frac{\partial f^*(r)}{\partial r} = \left[\frac{\partial f(x,r)}{\partial r}\right]_{x = x^*(r)}\right)$$

SOLUTION:

$$\text{I. } \frac{\partial f}{\partial x} = -2x - y + 2r = 0$$

$$\text{II } \frac{\partial f}{\partial y} = -x - 4y + 2r = 0$$

$$\text{I} - 2\text{II}: \quad 0x - y + 8y - 2r = 0$$

$$7y = 2r$$

$$\underline{y = \frac{2}{7}r}$$

$$\text{II GIVES} \quad -x - 4\left(\frac{2}{7}r\right) + 2r = 0$$

$$x = 2r - \frac{8}{7}r = \frac{14-8}{7}r$$

$$= \underline{\frac{6}{7}r}$$

(f CONCAVE ⇒ MAX)

$$\boxed{(x^*, y^*) = \left(\frac{6}{7}r, \frac{2}{7}r\right)}$$

NOW COMES THE POINT:

VERIFY $\underbrace{\frac{\partial f^*(r)}{\partial r}}_{\beta} = \underbrace{\left[\frac{\partial f(x,r)}{\partial r}\right]_{x=x^*(r)}}_{\alpha}$

α $\frac{\partial f(x,r)}{\partial r} = \frac{\partial}{\partial r}\left[\cancel{-x^2-xy-2y^2}+2rx+2ry\right]$

$= 2x+2y$

EVALUATED AT $x = x^*(r)$

$\left[\frac{\partial f(x,r)}{\partial r}\right]_{x=x^*(r)} = 2x^*+2y^*$

$= 2\cdot\frac{6}{7}r+2\cdot\frac{2}{7}r$

$= \frac{12r+\cancel{4}\,4r}{7} = \cancel{\frac{14r}{7}}$

$= \frac{16}{7}r$

β $f^*(r) = f\left(\frac{6}{7}r,\frac{2}{7}r,r\right)$

$= -\left(\frac{6}{7}r\right)^2-\frac{6}{7}\cdot\frac{2}{7}r^2-2\left(\frac{2}{7}\right)^2r^2$

$+2r\frac{6}{7}r+2r\cdot\frac{2}{7}r$

$= \dots = \frac{8}{7}r^2$

$\frac{\partial f^*(r)}{\partial r} = \frac{8}{7}\cdot 2r = \frac{16}{7}r$

YES!!

30.05.11

QUESTION 4.

$$f(x,y) = xy\,e^{x+y}$$

$$D_f = \{(x,y)\ (x+1)^2 + (y+1)^2 \le 1\}$$

a. COMPUTE THE HESSIAN

$$\frac{\partial f}{\partial x} = y\,e^{x+y} + xy\,e^{x+y} \cdot 1 = y(x+1)e^{x+y}$$

$$\frac{\partial^2 f}{\partial x^2} = y\,e^{x+y} + y(x+1)\,e^{x+y} \cdot 1 = y(x+2)e^{x+y}$$

$$\frac{\partial f}{\partial y} = \qquad = x(y+1)e^{x+y}$$

$$\frac{\partial^2 f}{\partial y^2} = \qquad = x(y+2)e^{x+y}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \left[y(x+1)e^{x+y}\right]'_y = (x+1)\,e^{x+y} + y(x+1)e^{x+y}$$

$$= (x+1)e^{x+y}[1+y]$$

$$= (x+1)(1+y)\,e^{x+y}$$

$$= (x+1)(y+1)\,e^{x+y}$$

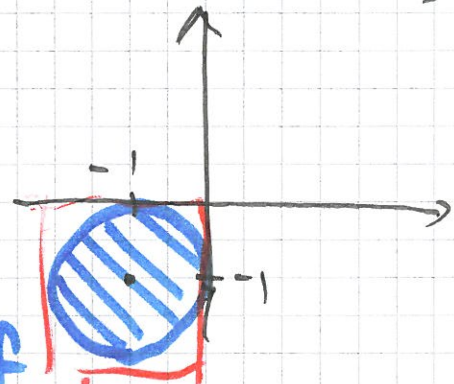
THIS GIVES THE FOLLOWING HESSIAN:

$$H = \begin{bmatrix} (x+2)y\,e^{x+y} & (x+1)(y+1)e^{x+y} \\ (x+1)(y+1)e^{x+y} & x(y+2)e^{x+y} \end{bmatrix}$$

NEXT QUESTION: f CONVEX? CONCAVE?

ON $D_f = \{(x,y) \quad (x+1)^2+(y+1)^2 \le 1\}$

ACCORDING TO TH. 2.3.2 IT IS ENOUGH TO CONSIDER THE PRINCIPAL MINORS:



D_f

$$D_1 = (x+2)y\,e^{x+y}$$

NOTE $x \in D_f \Rightarrow -1 \le x+1 \le 1$

[WHY? $(x+1)^2 + \cancel{(y+1)^2} \le 1$

$\Rightarrow \quad (x+1)^2 \le 1$

$\Rightarrow \quad -1 \le x+1 \le 1$]

$$-1 \le x+1 \le 1$$

$$-1+1 \le x+2 \le 1+1$$

$$0 \le x+2 \le 2$$

EVERYTHING ON THE NEGATIVE SIDE

$$\Rightarrow D_1 = (x+2)y\,e^{x+y} \le 0$$

(NOTE THAT y IS NEGATIVE)

WHAT ABOUT D_2?

$$H = \begin{bmatrix} (x+2)y\,e^{x+y} & (x+1)(y+1)e^{x+y} \\ (x+1)(y+1)e^{x+y} & x(y+2)e^{x+y} \end{bmatrix}$$

$$D_2 = (x+2)y \cdot x(y+2)\, e^{(\,)} \cdot e^{(\,)} - (x+1)^2 (y+1)^2 e^{(\,)} e^{(\,)}$$

$$= \left(\cancel{xy}\, x(x+2) \cdot y(y+2) - (x+1)^2 \cdot (y+1)^2 \right) e^{(\,)} \cdot e^{(\,)}$$

TRICK BOX:

$9 \cdot 7 = (8+1)(8-1) = 8^2 - 1^2 = 63$

$x(x+2)$
$= (x+1)(x+1) - 1 \cdot 1$
$= (x+1)^2 - 1$

$(x+1)^2 = x^2 + 2x + 1 - 1$
$= x + 2x = x(x+2)$

LET US USE THIS:

$$D_2 = \left[x(x+2) \cdot y(y+2) - (x+1)^2 \cdot (y+1)^2 \right] e^{(\,)} e^{(\,)}$$

$$= \left[((x+1)^2 - 1)[(y+1)^2 - 1] - (x+1)^2 \cdot (y+1)^2 \right] e^{(\,)} e^{(\,)}$$

$=$

$$D_2 = \left[\cancel{(x+1)^2 (y+1)^2} - \cancel{2}(x+1)^2 - \cancel{2}(y+1)^2 + 1 - \cancel{(x+1)^2 \cdot (y+1)^2} \right] e^{(\,)} e^{(\,)}$$

$$= \left[-\cancel{2}(x+1)^2 - \cancel{2}(y+1)^2 + 1 \right] e^{(\,)} e^{(\,)}$$

NOW REMEMBER : $(x+1)^2 + (y+1)^2 \leq 1$

$$\Rightarrow \geq 0$$

$$\left[-(x+1)^2 - (y+1)^2 \right]$$

$$\left[D_f = \{(x,y) \ (x+1)^2 + (y+1)^2 \leq 1 \right]$$

$D_1 \leq 0$, $D_2 \geq 0 \Rightarrow$ <u>CONCAVE</u> .