03.11.11

HELLO!

PLAN: 1. HOUR
: DIFFERENTIAL EQUATIONS

2. HOUR: KUHN TUCKER.

3. WE WILL SEE, HOPEFULLY MORE DIFFERENTIAL EQUATIONS.

2

d. $\int t e^{t^2} dt = \int \cancel{t} e^{u} \cdot \frac{du}{2\cancel{t}}$

$u = t^2$
$du = 2t\,dt$
$\Rightarrow dt = \frac{du}{2t}$

$= \frac{1}{2} \int e^u du$

$= \frac{1}{2} e^u + C$

$\left(= \frac{1}{2} e^u + C \right)$

$= \frac{1}{2} e^{t^2} + C$

(SUBSTITUTION: REMEMBER **ONLY u.**)

e. $\int \ln t \, dt = \int \overset{u'}{1} \cdot \overset{v}{\ln t} \, dt$ WE DO INTEGRATION BY PARTS...

$\boxed{\text{STANDARD TRICK}}$

$= \underset{u}{t} \cdot \underset{v}{\ln t} - \int \underset{u}{t} \cdot \underset{v'}{\frac{1}{t}} \, dt$

$= t \ln t - \int 1 \, dt$

$= t \ln t - t + C$

3 c.

$$\dot{x} = (2t+1)e^{t^2+t}$$

$$\int \dot{x}\,dt = \int (2t+1)e^{t^2+t}\,dt$$

NOTE:

$u = t^2+t$

GIVES:

$$\frac{du}{dt} = 2t+1$$

$$dt = \frac{du}{2t+1}$$

$$x(t) =$$

THIS IMPLIES THAT

$$\int (2t+1)e^{t^2+t}\,dt$$

$$= \int \cancel{(2t+1)}\, e^u \cdot \frac{du}{\cancel{2t+1}}$$

$$= \int e^u\,du = e^u + C$$

$$= e^{t^2+t} + C$$

IN OTHER WORDS:

$$\underline{x(t) = e^{t^2+t} + C}$$

3d.

$$\dot{x} = \frac{2t+1}{t^2+t+1}$$

$$\int \dot{x}\,dt = \int \frac{2t+1}{t^2+t+1}\,dt$$

$$x(t) = \int \frac{\cancel{2t+1}}{u} \cdot \frac{du}{\cancel{2t+1}}$$

$$\boxed{\begin{aligned} u &= t^2+t+1 \\ du &= (2t+1)\,dt \\ \underline{dt} &= \frac{du}{2t+1} \end{aligned}}$$

$$x(t) = \int \frac{1}{u}\,du$$

$$x(t) = \ln|u| + C$$

$$x(t) = \ln|t^2+t+1| + C$$

[NOTE THAT $t^2+t+1 > 0$

WHY? $t^2+t+1 = 0$

$$t = \frac{-1 \pm \sqrt{1-4\cdot1\cdot1}}{2}$$

NEGATIVE]

$$\underline{x(t) = \ln(t^2+t+1) + C}$$

5. $x = Ct^2$

SHOW THAT THIS SAFISFY

a) $t\dot{x} = 2x$ FOR ALL C.

AND FIND PARTICULAR SOLUTION SAFISFYING $x(1) = 2$.

SOLUTION:

$$x(t) = Ct^2$$

$$\dot{x}(t) = 2Ct$$

a) LEFT HAND SIDE: $t \cdot \dot{x} = t \cdot 2Ct = 2Ct^2$

RIGHT HAND SIDE: $2x = 2 \cdot Ct^2 = 2Ct^2$

$LS = RS \Rightarrow x = Ct^2$ SAFISFIES THE EQUATION FOR ALL x.

PARTICULAR SOLUTION:

$$x(1) = 2$$

SINCE $x(t) = Ct^2$

$$x(1) = C \cdot 1^2 = C = 2$$

HENCE $\underline{C = 2}$

THIS GIVES $\underline{x(t) = 2t^2}$

6. $x^2 \dot{x} = t+1$, $(t, x) = (1, 1)$

SEPARABLE !

THIS MEANS $x(1) = 1$

$$x^2 \dot{x} = t+1$$

$$dt \cdot \Big| \quad x^2 \frac{dx}{dt} = t+1$$

$$\int x^2 \, dx = \int (t+1) \, dt$$

$$3 \cdot \Big| \quad \frac{1}{3} x^3 = \frac{1}{2} t^2 + t + C$$

SMALL POINT C OR 3C DOES NOT MATTER

$$x^3 = \frac{3}{2} t^2 + 3t + C$$

$$x(t) = \sqrt[3]{\frac{3}{2} t^2 + 3t + C}$$

PARTICULAR SOLUTION WITH $(t, x) = (1, 1)$

$$x(1) = \left(\sqrt[3]{\frac{3}{2} \cdot 1^2 + 3 \cdot 1 + C}\right)^3 = 1^3$$

$$\frac{3}{2} + 3 + C = 1$$

$$\frac{9}{2} + C = 1$$

$$C = 1 - \frac{9}{2}$$

$$C = \frac{-7}{2}$$

PARTICULAR SOLUTION: $x(t) = \sqrt[3]{\frac{3}{2} t^2 + 3t - \frac{7}{2}}$

7.b

$$\int \dot{x}\, dt = \int te^{t} - t \, dt$$

$$x(t) = \underbrace{\int te^{t}\, dt}_{\alpha} - \underbrace{\int t \, dt}_{\beta}$$

α
$$\int \overset{v}{t}\,\overset{u'}{e^{t}}\, dt = \overset{v}{t}\,\overset{u}{e^{t}} - \int \overset{v'}{1}\cdot \overset{u}{e^{t}}\, dt$$

$$= te^{t} - \int e^{t}\, dt$$

$$= te^{t} - e^{t} + C$$

β
$$\int t \, dt = \frac{1}{2}t^{2} + C$$

THIS IMPLIES:

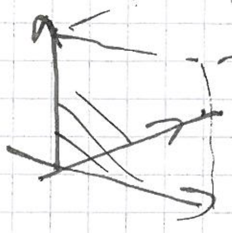
$$\underline{x(t) = te^{t} - e^{t} - \frac{1}{2}t^{2} + C}$$

2. HOUR KUHN – TUCKER

FINAL EXAM 10/12/2010, 4

$f(x,y,z) = xyz$

a. g IS DEFINED ON



$D = \{(x,y,z) : x > 0, y > 0, z > 0\}$

$$g(x,y,z) = \frac{1}{f(x,y,z)} = \frac{1}{xyz} = x^{-1}y^{-1}z^{-1}$$

CONVEX OR CONCAVE ON D?

SOLUTION: WE NEED THE HESSIAN

$$\frac{\partial g}{\partial x} = -1\, x^{-2}y^{-1}z^{-1} \qquad \frac{\partial^2 g}{\partial y \partial x} = (-1)^2 x^{-2}y^{-2}z^{-1}$$

$$\frac{\partial^2 g}{\partial x^2} = 2\, x^{-3}y^{-1}z^{-1} \qquad \frac{\partial^2 g}{\partial z \partial x} = (-1)^2 x^{-2}y^{-1}z^{-2}$$

$$H = \begin{bmatrix} 2x^{-3}y^{-1}z^{-1} & x^{-2}y^{-2}z^{-1} & x^{-2}y^{-1}z^{-2} \\ x^{-2}y^{-2}z^{-1} & 2x^{-1}y^{-3}z^{-1} & x^{-1}y^{-2}z^{-2} \\ x^{-2}y^{-1}z^{-2} & +x^{-1}y^{-2}z^{-2} & 2x^{-1}y^{-1}z^{-3} \end{bmatrix}$$

$$= \frac{1}{x^3y^3z^3} \begin{bmatrix} 2y^2z^2 & \boxed{xyz^2} & xy^2z \\ xyz^2 & 2x^2z^2 & x^2yz \\ xy^2z & x^2yz & 2x^2y^2 \end{bmatrix}$$

NOTE $\frac{1}{x^3y^3z^3} > 0$ ON D

SO WE CAN FORGETT ABOUT THIS FACTOR.

$$\mathcal{D}_1 = 2y^2z^2 \quad > 0$$

$$\underline{\mathcal{D}_2} = 2y^2z^2 \cdot 2x^2z^2 - xyz^2 \cdot xyz^2$$

$$= 4x^2y^2z^4 - x^2y^2z^4 = \underline{3x^2y^2z^4} > 0$$

$$\mathcal{D}_3 = \underline{2y^2z^2}\left(2x^2z^2 \cdot 2x^2y^2 - (x^2yz)^2\right)$$

$$- \boxed{xyz^2}\left(xyz^2 \cdot 2x^2y^2 - xy^2z \cdot x^2yz\right)$$

$$xy^2z\left(xyz^2 \cdot x^2yz - xy^2z \cdot 2x^2y^2\right)$$

$$= \dots = 6(xyz)^4 - (xyz)^4 - (xyz)^4$$

$$= 4(xyz)^4 > 0$$

CONVEX!!

3

b. MAX $f(x,y,z) = xyz$

SUBJ. $x^2+y^2+z^2 \leq 1$

SOLUTION:

REMARK: EXTREME VALUE THEOREM TELLS US THAT THIS PROBLEM HAS A SOLUTION.

(WHY? $f(x,y,z)$ CONT. + $x^2+y^2+z^2 \leq 1$ IS COMPACT.)

(COMPACT = CLOSED + BOUNDED)

(REMARK: CQ IS OK, SINCE

$g(x,y,z) = x^2 + y^2 + z^2$

$\vec{\nabla} g = [2x, 2y, 2z]$

HAS RANK ONE ON

$x^2+y^2+z^2 = 1$)

$$\left(x^2+y^2+z^2=1 \quad x^2=y^2=z^2 \right.$$
$$\left. \Rightarrow \quad 3x^2=1 \right)$$

WE GET $x^2=\frac{1}{3}$

$$x=\pm\frac{1}{\sqrt{3}}$$

THUS $(x,y,z)=\left(\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}}\right)$

(ALL COMBINATIONS)

THESE POINTS GIVE

$$f(x,y,z)=xyz=\pm\left(\frac{1}{\sqrt{3}}\right)^{3}$$
$$=\pm\frac{1}{3\sqrt{3}}$$

THE MAX IS $\frac{1}{3\sqrt{3}}$.

$$\boxed{\begin{gathered}\underbrace{\frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}}\cdot\frac{1}{\sqrt{3}}\\ \frac{1}{3}\cdot\frac{1}{\sqrt{3}}\\ =\frac{1}{3\sqrt{3}}\end{gathered}}$$

$$\mathcal{L} = xyz - \lambda(x^2+y^2+z^2-1)$$

1. $\dfrac{\partial \mathcal{L}}{\partial x} = yz - 2\lambda x = 0$

2. $\dfrac{\partial \mathcal{L}}{\partial y} = xz - 2\lambda y = 0$

3 $\dfrac{\partial \mathcal{L}}{\partial z} = xy - 2\lambda z = 0$

4. $\lambda \geq 0$ $\quad \lambda(x^2+y^2+z^2-1) = 0$ NOT BOTH.

START EASY: $\boxed{\underline{\lambda = 0}}$ $\quad yz = 0$, $xz = 0$, $xy = 0$ $\Rightarrow$ TWO OF x, y, z IS EQUAL TO ZERO.

$\Rightarrow$ WE SOLUTIONS OF THE FORM $(a,0,0)$, $(0,a,0)$ $(0,0,a)$ WITH $a^2 < 1$.

BUT ALL THESE GIVE $f = 0$ $\quad f = xyz$

$\underline{\lambda > 0}$ THEN

1. $yz = 2\lambda x$
2. $xz = 2\lambda y$
3 $xy = 2\lambda z$

ASSUME THAT x, y, z ARE ALL NON ZERO.
(NOTE ONLY INTERESTING CASE, IF WE FIND A ~~\$~~POSITIVE VALUE OF f)

DIVIDE 1./2. THEN $\dfrac{y\cancel{z}}{x\cancel{z}} = \dfrac{\cancel{2\lambda}x}{\cancel{2\lambda}y}$ $\quad x^2 = y^2$

—— 11 — 2/3 THEN $\dfrac{\cancel{x}z}{\cancel{x}y} = \dfrac{\cancel{2\lambda}y}{\cancel{2\lambda}z}$ $\quad z^2 = y^2$

THUS $x^2 = y^2 = z^2$.

<u>CONSTRAINT IS BINDING</u>: $3x^2 = 1$

8.

a. $t\dot{x} = x(1-t)$

$(t_0, x_0) = (1, \frac{1}{e})$

$x(t_0) = x_0$

$x(1) = \frac{1}{e}$

SEPARABLE!

$dt \cdot \Big| \quad t\frac{dx}{dt} = x(1-t)$

$\frac{1}{x} \cdot \Big| \quad t\,dx = x(1-t)dt \quad \Big| \cdot \frac{1}{t}$

$$\frac{1}{x}dx = \frac{1-t}{t}dt$$

$$\int \frac{1}{x}dx = \int\left(\frac{1}{t} - 1\right)dt$$

$$e^{\ln|x|} = e^{(\ln|t| - t + C)}$$

$$|x| = |t|e^{-t} \cdot e^{C}$$

$$x = \pm|t|e^{-t}e^{C}$$

$$= K t e^{-t} \qquad K = \pm e^{C}$$

THE PARTICULAR SOLUTION : $x(1) = \frac{1}{e}$

$$x(1) = K \cdot 1 \cdot e^{-1} = \frac{1}{e}$$

$$\Rightarrow K = 1$$

$$x_p(t) = t e^{-t}$$

HINT: ON THE EXAM: TEST YOUR SOLUTION

8b.

$$(1+t^3)\dot{x} = t^2 x \qquad (t_0, x_0) = (0,2)$$

$$\boxed{x(0) = 2}$$

SEPARABLE!

$$dt \;\Big|\; (1+t^3)\frac{dx}{dt} = t^2 x$$

$$\frac{1}{1+t^3} \;\Big|\; (1+t^3)\,dx = t^2 x\,dt \;\Big|\cdot \frac{1}{x}$$

$$\int \frac{1}{x}\,dx = \int \frac{t^2}{1+t^3}\,dt$$

$$\left[\; \int \frac{t^2}{1+t^3}\,dt = \int \frac{\cancel{t^2}}{u}\,\frac{du}{3\cancel{t^2}} = \frac{1}{3}\int \frac{1}{u}\,du \right.$$

$$u = 1+t^3$$

$$\boxed{du = 3t^2\,dt}$$

$$\left. = \frac{1}{3}\ln|u| + C \;\right]$$

$$\int \frac{1}{x}\,dx = \int \frac{t^2}{1+t^3}\,dt$$

$$e^{\ln|x|} = e^{\left(\frac{1}{3}\ln|1+t^3| + C\right)}$$

$$|x| = e^C\,|1+t^3|^{\frac{1}{3}}$$

$$x(t) = k\,(1+t^3)^{1/3}$$

PARTICULAR SOLUTION: $x(0) = 2$

$$x(0) = k\,(1+0^3)^{1/3} = 2$$

$$k = 2$$

THE SOLUTION IS:

$$\underline{x_p(t) = 2(1+t^3)^{1/3}}$$

[TEST....]

8c.

$$x\dot{x} = t \qquad \cancel{(t_0, x_0) = (0, 2)}$$

$$\cancel{x(0) = 2}$$

$$(t_0, x_0) = (\sqrt{2}, 1)$$

$$x(\sqrt{2}) = 1$$

SEPARABLE

$dt \cdot |$ $$x\frac{dx}{dt} = t$$

$$\int x\,dx = \int t\,dt$$

$2 \cdot |$ $$\frac{1}{2}x^2 = \frac{1}{2}t^2 + C$$

$$x^2 = t^2 + C$$

$$x = \pm\sqrt{t^2 + C}$$

PARTICULAR SOLUTION: $x(\sqrt{2}) = 1$

$$x(\sqrt{2}) = \pm\sqrt{(\sqrt{2})^2 + C} = 1$$

MINUS NOT POSSIBLE

$$\left(\sqrt{2+C}\right)^2 = 1^2$$

$$2 + C = 1$$

$$C = -1$$

THIS SQUARING IMPLIES WE HAVE TO TEST THE SOLUTION:

$$\sqrt{2-1} = 1$$ ☺ **OK!**

$$x_p(t) = +\sqrt{t^2 - 1}$$

8 d.

$$e^{2t}\dot{x} - x^2 - 2x = 1$$

$$\boxed{\begin{array}{c}(t_0, x_0) \\ = (0,0) \\ x(0) = 0\end{array}}$$

$$e^{2t}\dot{x} = x^2 + 2x + 1$$

$$e^{2t}\frac{dx}{dt} = (x+1)^2 \quad \Big| \; dt \cdot$$

$$e^{2t}\,dx = (x+1)^2\,dt \quad \Big| \; e^{-2t} \cdot \quad \Big| \; \frac{1}{(x+1)^2}$$

$$\int (x+1)^{-2}\,dx = \int e^{-2t}\,dt$$

$$\frac{1}{-2+1}(x+1)^{-2+1} = \frac{1}{-2}e^{-2t} + C$$

$$-(x+1)^{-1} = -\frac{1}{2}e^{-2t} + C$$

$$-\frac{1}{x+1} = -\frac{1}{2}e^{-2t} + C$$

⋮ SOLVE FOR x ....