Problem Sheet 6 with Solutions
GRA 6035 Mathematics

BI Norwegian Business School

Problems

1. Consider the four sets in [FMEA] Section 2.2 Problem 1, and determine which sets are convex.

2. Sketch each set and determine if it is open, closed, bounded or convex:

a) $\{(x,y) : x^2+y^2 < 2\}$
b) $\{(x,y) : x^2+y^2 > 8\}$
c) $\{(x,y) : xy \leq 1\}$
d) $\{(x,y) : x \geq 0, y \geq 0\}$
e) $\{(x,y) : x \geq 0, y \geq 0, xy \geq 1\}$
f) $\{(x,y) : \sqrt{x}+\sqrt{y} \leq 2\}$

3. Consider the functions in [FMEA] Section 2.3 Problem 1, and determine which functions are convex and concave.

4. Compute the Hessian of the function $f(x,y) = x - y - x^2$, and show that f is a concave function defined on $D_f = \mathbb{R}^2$. Determine if

$$g(x,y) = e^{x-y-x^2}$$

is a convex or a concave function on $\mathbb{R}^2$.

5. Let $f(x,y) = ax^2+bxy+cy^2+dx+ey+f$ be the general polynomial in two variables of degree two. For which values of the parameters is this function (strictly) convex and (strictly) concave?

6. Determine the values of the parameter a for which the function

$$f(x,y) = -6x^2+(2a+4)xy-y^2+4ay$$

is convex and concave on $\mathbb{R}^2$.

7. The function $f(x,y,z) = \ln(xyz)$ is defined on $D_f = \{(x,y,z) : x > 0, y > 0, z > 0\}$. Determine if this function is convex or concave.

8. Consider the function $f(x,y) = x^4+16y^4+32xy^3+8x^3y+24x^2y^2$ defined on $\mathbb{R}^2$. Determine if this function is convex or concave.

9. Consider the function $f(x,y) = e^{x+y}+e^{x-y}$ defined on $\mathbb{R}^2$. Determine if this function is convex or concave.

10. Midterm in GRA6035 24/09/2010, Problem 7
Consider the function

$$f(x_1,x_2,x_3) = 3x_1^2+2x_1x_2+3x_2^2+x_3^2+x_1-x_2$$

defined on $\mathbb{R}^3$. **Which statement is true?**

a) f is a convex function but not a concave function
b) f is a convex function and a concave function
c) f is not a convex function but a concave function
d) f is neither a convex nor a concave function
e) I prefer not to answer.

11. Mock Midterm in GRA6035 09/2010, Problem 7
Consider the function

$$f(x_1,x_2) = 3 - a \cdot Q(x_1,x_2)$$

defined on $\mathbb{R}^2$, where $a \in \mathbb{R}$ is a number and Q is a positive definite quadratic form. **Which statement is true?**

a) f is convex for all values of a
b) f is concave for all values of a
c) f is convex if $a \geq 0$ and concave if $a \leq 0$
d) f is convex if $a \leq 0$ and concave if $a \geq 0$
e) I prefer not to answer.

12. Midterm in GRA6035 24/05/2011, Problem 7
Consider the function

$$f(x_1,x_2,x_3) = -x_1^2 + 2x_1x_2 - 3x_2^2 - x_3^2 - x_1 - x_3$$

defined on $\mathbb{R}^3$. **Which statement is true?**

a) f is a convex function but not a concave function
b) f is a convex function and a concave function
c) f is not a convex function but a concave function
d) f is neither a convex nor a concave function
e) I prefer not to answer.

Solutions

1 The sets in a) and d) are convex, while the sets in b) and c) are not.

2 We see that the sets have the following type:

a) The set $\{(x,y) : x^2 + y^2 < 2\}$ is convex, open and bounded, but not closed.
b) The set $\{(x,y) : x^2 + y^2 > 8\}$ is open, but not closed, bounded or convex.
c) The set $\{(x,y) : xy \leq 1\}$ is closed, but not open, bounded or convex.
d) The set $\{(x,y) : x \geq 0, y \geq 0\}$ is closed and convex, but not open or bounded.
e) The set $\{(x,y) : x \geq 0, y \geq 0, xy \geq 1\}$ is closed and convex, but not open or bounded.
f) The set $\{(x,y) : \sqrt{x} + \sqrt{y} \leq 2\}$ is closed and bounded, but not open or convex.

3 The function in a) is convex, the functions in b) and c) are concave.

4 The Hessian of the function $f(x,y) = x - y - x^2$ is given by

$$H(f) = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$$

since $f'_x = 1 - 2x$ and $f'_y = -1$. Since $\Delta_1 = -2, 0 \leq 0$ and $\Delta_2 = 0 \geq 0$, the function f is concave on $D_f = \mathbb{R}^2$. The Hessian of the function $g(x,y) = e^{x-y-x^2}$ is given by

$$H(g) = \begin{pmatrix} e^{x-y-x^2}(1-2x)^2 + e^{x-y-x^2}(-2) & e^{x-y-x^2}(-1)(1-2x) \\ e^{x-y-x^2}(-1)(1-2x) & e^{x-y-x^2}(-1)(-1) \end{pmatrix}$$

by the product rule, since $g'_x = e^{x-y-x^2}(1-2x)$ and $g'_y = e^{x-y-x^2}(-1)$. This gives

$$H(g) = e^{x-y-x^2} \begin{pmatrix} 4x^2 - 4x - 1 & 2x - 1 \\ 2x - 1 & 1 \end{pmatrix}$$

This gives $D_2 = (e^{x-y-x^2})^2(4x^2 - 4x - 1 - (2x-1)^2) = -2(e^{x-y-x^2})^2 < 0$, and this means that g is neither convex nor concave.

5 The Hessian matrix of the function $f(x,y) = ax^2 + bxy + cy^2 + dx + ey + f$ is given by

$$H(f) = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$$

This means that $D_1 = 2a$ and $D_2 = 4ac - b^2$. Therefore, f is strictly convex if and only if $a > 0$ and $4ac - b^2 > 0$, and f is strictly concave if and only if $a < 0$ and $4ac - b^2 > 0$. The remaining principal minors are $\Delta_1 = 2c$, and this means that f is convex if and only if $a \geq 0, c \geq 0, 4ac - b^2 \geq 0$, and that f is concave if and only if $a \leq 0, c \leq 0, 4ac - b^2 \geq 0$.

6 The Hessian matrix of the function $f(x,y) = -6x^2 + (2a+4)xy - y^2 + 4ay$ is given by

$$H(f) = \begin{pmatrix} -12 & 2a+4 \\ 2a+4 & -2 \end{pmatrix}$$

Hence the leading principal minors are $D_1 = -12$ and $D_2 = 24 - (2a+4)^2 = -4a^2 - 16a + 8$, and the remaining principal minor is $\Delta_1 = -2$. We have that

$$-4a^2 - 16a + 8 = -4(a^2 + 4a - 2) \geq 0 \quad \Leftrightarrow \quad -2 - \sqrt{6} \leq a \leq -2 + \sqrt{6}$$

This implies that f is concave if and only if $-2 - \sqrt{6} \leq a \leq -2 + \sqrt{6}$, and that f is never convex.

7 Since $f'_x = yz/xyz = 1/x$, we have $f'_y = 1/y$ and $f'_z = 1/z$ in the same way (or by the remark $\ln(xyz) = \ln(x) + \ln(y) + \ln(z)$). The Hessian becomes

$$H(f) = \begin{pmatrix} -1/x^2 & 0 & 0 \\ 0 & -1/y^2 & 0 \\ 0 & 0 & -1/z^2 \end{pmatrix}$$

This matrix has $D_1 = -1/x^2 < 0$, $D_2 = 1/(x^2y^2) > 0$, $D_3 = -1/(x^2y^2z^2) < 0$ on $D_f = \{(x,y,z) : x > 0, y > 0, z > 0\}$, hence f is concave.

8 Since $f'_x = 4x^3 + 32y^3 + 24x^2y + 48xy^2$ and $f'_y = 64y^3 + 96xy^2 + 8x^3 + 48x^2y$, we have

$$H(f) = \begin{pmatrix} 12x^2 + 48xy + 48y^2 & 96y^2 + 24x^2 + 96xy \\ 96y^2 + 24x^2 + 96xy & 192y^2 + 192xy + 48x^2 \end{pmatrix}$$

Completing the squares, we see that

$$H(f) = \begin{pmatrix} 12(x+2y)^2 & 24(x+2y)^2 \\ 24(x+2y)^2 & 48(x+2y)^2 \end{pmatrix} = 12(x+2y)^2 \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

This gives $\Delta_1 = 12(x+2y)^2, 48(x+2y)^2 \geq 0$ and $\Delta_2 = 144(x+2y)^4(4-4) = 0$. This implies that f is convex.

9 We have $f'_x = e^{x+y} + e^{x-y}$ and $f'_y = e^{x+y} - e^{x-y}$, and the Hessian is given by

$$H(f) = \begin{pmatrix} e^{x+y} + e^{x-y} & e^{x+y} - e^{x-y} \\ e^{x+y} - e^{x-y} & e^{x+y} + e^{x-y} \end{pmatrix}$$

This implies that $D_1 = e^{x+y} + e^{x-y} > 0$ and that we have

$$D_2 = (e^{x+y} + e^{x-y})^2 - (e^{x+y} + e^{x-y})^2 = 4e^{x+y}e^{x-y} = 4e^{2x} > 0$$

Hence the function f is convex.

10 Midterm in GRA6035 24/09/2010, Problem 7

The function f is a sum of a linear function and a quadratic form with symmetric matrix

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since A has eigenvalues $\lambda = 1, 2, 4$, the quadratic form is positive definite and therefore convex (but not concave). Hence the correct answer is alternative **A**.

11 Mock Midterm in GRA6035 09/2010, Problem 7

The function f is a sum of a constant function and the quadratic form $-aQ(x_1,x_2)$. Since Q is positive definite, it is convex, and $-Q$ is concave. If $a \geq 0$, then $-aQ(x_1,x_2) = a(-Q(x_1,x_2))$ is concave. If $a \leq 0$, then $-a \geq 0$ and $-aQ(x_1,x_2)$ is convex. The correct answer is alternative **D**.

12 Midterm in GRA6035 24/05/2011, Problem 7

The function f is a sum of a linear function and a quadratic form with symmetric matrix

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Since A has eigenvalues $\lambda = -2 \pm \sqrt{2}$ and $\lambda = -1$, the quadratic form is negative definite and therefore concave (but not convex). Hence the correct answer is alternative **C**.