

LECTURE 12 - B

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GRA 6035

MATHEMATICS

PLAIN:

- ① Difference equations
- ② Linear difference equations
 - a) First order
 - b) Second order
- ③ Stability

Reading:

LNE3 23.2

① Difference equations

Differential eqn: Continuous time
 Difference eqn: Discrete time

Defn:

A difference equation is an equation relating

y_t to $y_{t-1}, y_{t-2}, \dots, y_{t-d}$. In general, it has the form

$$y_t = F(t, y_{t-1}, y_{t-2}, \dots, y_{t-d}) \quad d \geq 1$$

The integer d is called the order of the difference eqn.

Ex: $y_t = 2y_{t-1}$ {difference eqn.
of order I.}

Initial condition:
 $y_0 = 1$

$y_1 = 2y_0$
 $y_2 = 2y_1 = 2^2 \cdot y_0 = 4y_0$
 $y_3 = 2 \cdot y_2 = 2^3 \cdot y_0 = 8y_0$
⋮
⋮

$y_1 = 2$
 $y_2 = 2^2 = 4$
 $y_3 = 2^3 = 8$

A difference eqn is also called recurrence relation. A general solution is a closed formula for $y_t = \text{some expression in } t$.

In the example: $y_t = 2^t \cdot y_0$ general solution
 $y_0 = 1 : \underline{\underline{y_t = 2^t}}$ particular solution.

First order difference eqn:

In general: $y_t = F(t, y_{t-1})$

General solution will depend on one undetermined coefficient.

Ex: $y_t = (1+r)y_{t-1} - 100$

$$y_t - y_{t-1} = (1+r)y_{t-1} - 100 - \underline{\underline{y_{t-1}}}$$

$$\underbrace{y_t - y_{t-1}}_{\text{change in } y_t} = \underbrace{r \cdot y_{t-1}}_{\text{expression for the change}} - 100$$

Ex:

$$y_{t+2} = y_{t+1} + y_t \quad , \quad y_0 = y_1 = 1$$

(Fibonacci)

Rewrite: $y_t = y_{t-1} + y_{t-2}$

$$y_0 = 1, y_1 = 1, y_2 = 2, y_3 = 3, y_4 = 5, y_5 = 8, \dots$$

Will get back to this example.

Ex:

Bank account with interest r per time period and starting balance B_0 .

Discrete time:

B_t : balance at time t

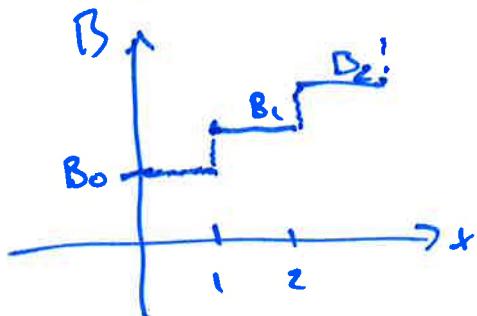
$$B_t - B_{t-1} = r \cdot B_{t-1}$$

$$B_t = B_{t-1} + r \cdot B_{t-1}$$

difference eqn.

Solution:

$$\underline{\underline{B_t = B_0 \cdot (1+r)^t}}$$



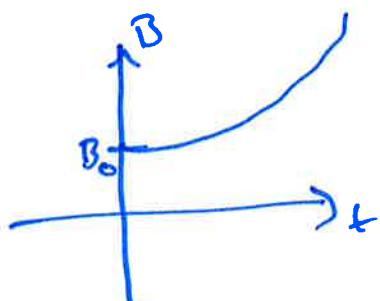
Cont. time:

$B = B(t)$: balance at time t

$$\underline{\underline{B' = r \cdot B}}$$

⋮

$$B(t) = B_0 \cdot e^{rt}$$



a) Linear first order difference eqn

(with const.
coeff.)

$$y_t + a \cdot y_{t-1} = b_t \quad \Downarrow$$

$\left\{ \begin{array}{l} a: \text{number} \\ b_t: \text{expression in } t \end{array} \right.$

$$y_t = b_t - a \cdot y_{t-1} \quad \Downarrow$$

$$y_t - y_{t-1} = b_t - a y_{t-1} - y_{t-1}$$

Solution:

i) Homogeneous case: $b_t = 0$

$$y_t + a \cdot y_{t-1} = 0$$

$$y_t = (-a) \cdot y_{t-1}$$

General solution:

$$y_t = (-a)^t \cdot y_0$$

Can think in terms of
char. eqn:

$$y_t + a y_{t-1} = 0$$

Char.
eqn.: $r + a \cdot 1 = 0$

$$\underline{r = -a}$$

\Downarrow

$$y_t = C \cdot (-a)^t$$

ii) Inhomogeneous case: $b_t \neq 0$

$$y_t + a \cdot y_{t-1} = b_t$$

~~general solution~~

General solution:

$$y_t = y_t^h + y_t^P$$
$$= C \cdot (-a)^t + y_t^P$$

superposition
principle

Examples:

First order linear
difference eqns

$$y_{t+1} + ay_t = b_t$$

Ex I: $y_{t+1} - 1.05y_t = -100$, $y_0 = 5000$

Solve the difference eqn:

$$y_t = y_t^u + y_t^P = \underline{C \cdot 1.05^t + 2000}$$

$$y_t^u = C \cdot (-a)^t = \underline{C \cdot 1.05^t}$$

Interpretation:

$$y_{t+1} - y_t = 0.05y_t - 100$$

$$y_t^P: y_{t+1} - 1.05y_t = -100$$

$$\underline{y_t^P = A}: A - 1.05 \cdot A = -100$$

$$-0.05A = -100$$

$$A = \frac{-100}{-0.05} = \underline{2000}$$

$$y_t^P = \underline{2000}$$

Initial condition: $y_0 = 5000$

$$y_t = C \cdot 1.05^t + 2000$$

$$t=0, y=5000: 5000 = C \cdot 1.05^0 + 2000$$

$$5000 = C + 2000$$

$$C = \underline{+3000}$$

$$\underline{y_t = 3000 \cdot 1.05^t + 2000}$$

$$\underline{\text{Ex 2:}} \quad y_{t+1} - 1.05y_t = 2t - 100$$

$$y_t = y_t^h + y_t^P = \underline{\underline{C \cdot 1.05^t + 1200 - 40t}}$$

Find y_t^P :

$$y_{t+1} - 1.05y_t = \underline{\underline{2t - 100}}$$

$$b_t = 2t - 100$$

$$\begin{aligned} b_{t+1} &= 2(t+1) - 100 \\ &= 2t - 98 \end{aligned}$$

$$\begin{aligned} b_{t+2} &= 2(t+2) - 100 \\ &= 2t - 96 \end{aligned}$$

$$\begin{aligned} (\underline{A}t + \underline{A} + B) - 1.05 \cdot (\underline{A}t + B) \\ = 2t - 100 \end{aligned}$$

$$(-0.05 \cdot A)t + (A - 0.05B)$$

$$= 2t - 100$$

$$-0.05A = 2$$

$$\underline{A = -40}$$

$$A - 0.05B = -100$$

$$-40 - 0.05B = -100$$

$$-0.05B = -60$$

$$\underline{B = 1200}$$

←
insert
in
diff-eqn.

Guess: $y_t = At + B$
 $y_{t+1} = A \cdot (t+1) + B$
 $= At + A + B$

$$y_t^P = \underline{\underline{-40t + 1200}}$$

b) Linear second order difference eqn's (with constant coeff.)

$$y_{t+2} + a \cdot y_{t+1} + b \cdot y_t = f_t$$

a,b numbers
 f_t : expr. in t

i) Homogeneous case:

$$f_t = 0$$

$$y_{t+2} + a y_{t+1} + b y_t = 0$$

$$\text{Char. eqn: } r^2 + ar + b = 0$$

$$r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

Fact: r is a root of the char. eqn.

are exactly the values such that

r^t is a solution to the difference eqn.

Proof: Put in $y_t = r^t$ in the diff. eqn.

$$y_t = r^t \quad y_{t+1} = r^{t+1} \quad y_{t+2} = r^{t+2}$$

$$r^{t+2} + a \cdot r^{t+1} + b \cdot r^t = 0$$

$$r^t \cdot (r^2 + ar + b) = 0$$

char. eqn.

$$\text{Ex: } y_{t+2} - 5y_{t+1} + 6y_t = 0$$

$$\text{Char.eqn: } r^2 - 5r + 6 = 0$$

$$r = \frac{5 \pm \sqrt{25 - 4 \cdot 6}}{2} = \frac{5 \pm 1}{2}$$

$$r_1 = 3, \quad r_2 = 2 \quad \Rightarrow 2^t, 3^t$$

are two solutions

$$\text{General solution: } y_t = C_1 \cdot 2^t + C_2 \cdot 3^t$$

In general, there are three cases:

i) $a^2 - 4b > 0$: Two distinct roots $r_1 \neq r_2$.

$$y_t = C_1 \cdot r_1^t + C_2 \cdot r_2^t$$

ii) $a^2 - 4b = 0$: One double root $r = -\frac{a}{2}$

$$y_t = C_1 \cdot r^t + C_2 \cdot t r^t = (C_1 + C_2 t) r^t$$

iii) $a^2 - 4b < 0$: No (real) roots

$$y_t = (\sqrt{b})^t \cdot (C_1 \cdot \sin(\theta t) + C_2 \cdot \cos(\theta t))$$

where θ is a number such that

$$\cos \theta = -\frac{a}{2\sqrt{b}}$$

ii) Inhomogeneous case : $y_{t+2} + ay_{t+1} + by_t = f_t$

$$y_t = y_t^h + y_t^P,$$

where y_t^h : general solution of

$$y_{t+2} + ay_{t+1} + by_t = 0$$

y_t^P : particular solution of

$$y_{t+2} + ay_{t+1} + by_t = f_t$$

Ex: Fibonacci equation : $y_t = y_{t-1} + y_{t-2}$,
 $y_0 = y_1 = 1$

$$1 \ 1 \ 2 \ 3 \ 5 \ 8 \ 13 \ 21 \ 34 \ \dots$$

$$y_t - y_{t-1} - y_{t-2} = 0$$

$$y_{t+2} - y_{t+1} - y_t = 0$$

put in $t+2$ term

Char. eqn: $r^2 - r - 1 = 0$

$$r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

General solution:

$$y_t = c_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^t + c_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right)^t$$

$$y_0 = 1 : \quad 1 = c_1 + c_2$$

$$y_1 = 1 : \quad 1 = c_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right) + c_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right)$$

$$c_2 = 1 - c_1$$

$$1 = c_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right) + (1-c_1) \cdot \left(\frac{1-\sqrt{5}}{2}\right) \quad | \cdot 2$$

$$2 = c_1 \cdot (1+\sqrt{5}) + (1-c_1)(1-\sqrt{5})$$

$$2 = c_1 \cdot [1+\sqrt{5} - 1+\sqrt{5}] + (1-\sqrt{5})$$

$$2 \cdot \sqrt{5} c_1 = 2 - (1-\sqrt{5}) = 1 + \sqrt{5}$$

$$c_1 = \frac{1+\sqrt{5}}{2\sqrt{5}} \quad c_2 = 1 - \frac{1+\sqrt{5}}{2\sqrt{5}} = \frac{2\sqrt{5}-1-\sqrt{5}}{2\sqrt{5}}$$

$$= \underline{\underline{\frac{\sqrt{5}-1}{2\sqrt{5}}}}$$

$$\underline{\underline{y_t = \frac{\sqrt{5}+1}{2\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^t + \frac{\sqrt{5}-1}{2\sqrt{5}} \cdot \left(\frac{1-\sqrt{5}}{2}\right)^t}}$$

Linear difference equation: Summary

Homogeneous case: Find solution from char. eqn.

i) $y_{t+1} + a y_t = 0 \quad y_t = C \cdot (-a)^t$

ii) $y_{t+2} + a y_{t+1} + b y_t = 0 \quad r^2 + a r + b = 0$
↓
three cases

If $r_1 \neq r_2$: $y_t = C_1 \cdot r_1^t + C_2 \cdot r_2^t$

Inhomogeneous case: $y_t = y_t^h + y_t^P$

How to guess y_t to find y_t^P : $\left\{ \begin{array}{l} y_{t+2} + a y_{t+1} + b y_t = f_t \\ y_t^P = f_t \end{array} \right.$

Look at f_t
 f_{t+1}
 f_{t+2}

Choose y_t such that

- i) y_t has the same form as f_t, f_{t+1}, f_{t+2}
- ii) y_t depends on parameters

If the initial guess does not work, try to multiply it with t .

$$\underline{\text{Ex:}} \quad y_{t+2} - y_{t+1} - 2y_t = 3^t$$

$$y_t = y_t^h + y_t^P = C_1 \cdot 2^t + C_2 \cdot (-1)^t + \underline{\underline{\frac{1}{4} \cdot 3^t}}$$

$$\underline{\underline{y_t^h:}} \quad y_{t+2} - y_{t+1} - 2y_t = 0$$

$$r^2 - r - 2 = 0$$

$$r = \frac{1 \pm \sqrt{1 - 4 \cdot (-2)}}{2} = \frac{1 \pm 3}{2}$$

$$r_1 = \underline{\underline{2}}, \quad r_2 = \underline{\underline{-1}}$$

$$\Rightarrow y_t^h = \underline{\underline{C_1 \cdot 2^t + C_2 \cdot (-1)^t}}$$

$$\underline{\underline{y_t^P:}} \quad y_{t+2} - y_{t+1} - 2y_t = \underline{\underline{3^t}}$$

$$f_t = 3^t$$

$$f_{t+1} = 3^{t+1} = 3 \cdot 3^t$$

$$f_{t+2} = 3^{t+2} = 9 \cdot 3^t$$

$$9A \cdot 3^t - 3A \cdot 3^t - 2 \cdot A \cdot 3^t \\ = 3^t$$

$$3^t / (9A - 3A - 2A) = 3^t$$

$$4A = 1$$

$$\underline{\underline{A = \frac{1}{4}}} \quad \Rightarrow \quad y_t^P = \underline{\underline{\frac{1}{4} \cdot 3^t}}$$

Gwoss:

$$\begin{cases} y_t = A \cdot 3^t \\ y_{t+1} = 3A \cdot 3^t \\ y_{t+2} = 9A \cdot 3^t \end{cases}$$

③ Stability:

Differential equation $\rightarrow y = y(t)$ solution

Difference equation $\rightarrow y = y_t$ solution

It is called globally asymptotically stable

if

$$\bar{y} = \lim_{t \rightarrow \infty} y = \begin{cases} \lim_{t \rightarrow \infty} y(t) \\ \lim_{t \rightarrow \infty} y_t \end{cases}$$

satisfying i) \bar{y} is finite ii) \bar{y} does not depend on the undetermined coefficients

Interpretation:

There is a long-term equilibrium \bar{y} , which is finite and independent of initial conditions.

Ex: $y_{t+2} - y_{t+1} - 2y_t = 3^t$

$$y_t = C_1 \cdot 2^t + C_2 \cdot (-1)^t + \frac{1}{4} \cdot 3^t \rightarrow \infty \quad (\text{as } t \rightarrow \infty)$$

not stable (unstable)

Ex: $y'' - 3y' + 2y = 4$

$$y = y_h + y_p = \underbrace{C_1 \cdot e^t + C_2 \cdot e^{2t}}_{\substack{\text{(unless} \\ \text{C}_1 = C_2 = 0)}} + 2 \rightarrow \pm \infty$$

$y_h: r^2 - 3r + 2 = 0$

$r=1, r=2$

$y_p: y = A = 2$

not globally asymptotically stable

For globally asymptotical stability, it is necessary that "essentially"

$$|r| < 1 \quad \text{for difference eqn's}$$

$$r < 0 \quad \text{" differential equations"}$$

Ex: Systems of difference equations

$$\begin{aligned}x_{t+1} &= 2x_t - y_t \\y_{t+1} &= x_t\end{aligned}$$

similar examples are
Markov chains.

$$x_{t+1} = 2x_t - x_{t-1}$$

$$x_{t+1} - 2x_t + x_{t-1} = 0$$

$$r^2 - 2r + 1 = 0$$

$$\underline{r=1}$$

$$x_t = (c_1 + c_2 t) \cdot 1^t = \underline{\underline{c_1 + c_2 t}}$$

$$y_t = x_{t-1} = c_1 + c_2(t-1)$$

$$= \underline{\underline{c_1 + c_2 t - c_2}}$$