# LECTURE 13 - I

**GRA 6035**
MATHEMATICS

Eivind Eriksen    Nov 21 2013

## REVIEW:

(1) Matrix methods

(2) Unconstrained optimization

**Exam Problems**

Final 12/2012 Problem 1

— " — Problem 2

## (1) Matrix methods

Basic techniques

$$\boxed{\begin{array}{l}\text{a) Gaussian elimination}\\ \text{b) Determinants}\end{array}}$$

### i) Linear independence of vectors

m-vectors $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n \rightsquigarrow A = \left( \underline{v}_1 \,\vdots\, \underline{v}_2 \,\vdots\, \cdots \,\vdots\, \underline{v}_n \right)$

Are $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$ linearly independent?    $m \times n$-matrix

**Fact 1:**
If $m = n$ ($A$ square matrix), then $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$ are linearly independent $\iff |A| \neq 0$

**Fact 2:**
The vectors $\{\underline{v}_1, \dots, \underline{v}_n\}$ are linearly independent if and only if $A \cdot \underline{x} = \underline{0}$ only have the trivial solution $\underline{x} = \underline{0}$.

• Ex: $A = \begin{pmatrix} 1 & t & -2 \\ 2 & 4 & -t \\ -t & -4 & -4 \end{pmatrix}$   $t$ is a parameter

$\underline{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -t \end{pmatrix}$  $\underline{v}_2 = \begin{pmatrix} t \\ 4 \\ -4 \end{pmatrix}$  $\underline{v}_3 = \begin{pmatrix} -2 \\ -t \\ -4 \end{pmatrix}$

Using determinants:

$$\begin{vmatrix} 1 & t & -2 \\ 2 & 4 & -t \\ -t & -4 & -4 \end{vmatrix} = 1\cdot(-16-4t) - 2(-4t-8) - t(-t^2+8)$$

$$= \cancel{-16} - 4t + \cancel{8t} + \cancel{16} + t^3 - \cancel{8t} = \underline{t^3 - 4t}$$

$\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ linearly independent $\Leftrightarrow$ $|A| = t^3 - 4t \neq 0$

— " — dependent  $|A| = t^3 - 4t = 0$

$t(t^2-4) = 0$

$\underline{t=0}$  $t = \underline{\pm 2}$

$\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ lin. independent $\Leftrightarrow$ $\underline{t \neq 0, 2, -2}$

What if $t=-2$:  $|A| = 0$, $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ lin. dependent

$$A = \begin{pmatrix} 1 & -2 & -2 \\ 2 & 4 & 2 \\ 2 & -4 & -4 \end{pmatrix} \xrightarrow{-2,\ -2} \begin{pmatrix} 1 & -2 & -2 \\ 0 & 8 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

$A \cdot \underline{x} = \underline{0}$ has one free variable ($z$)

→ $\{\underline{v}_1, \underline{v}_2\}$ are linearly independent  (pivot positions in col. 1 and 2)

→ $\underline{v}_3$ is a linear combination of $\{\underline{v}_1, \underline{v}_2\}$  (no pivot position in col 3)

ii) Rank

A $m \times n$-matrix: $rk(A)$ = max number of linearly independent column vectors in $A$

Fact I:

If $m=n$, then $\begin{cases} rk\,A = n \iff |A| \neq 0 \\ rk\,A < n \iff |A| = 0 \end{cases}$

Fact 2:

$rk(A) = \#$ pivot positions in $A$

Ex: $A = \begin{pmatrix} 1 & t & -2 \\ 2 & 4 & -t \\ -t & -4 & -4 \end{pmatrix}$

$|A| = t^3 - 4t \Rightarrow \begin{cases} |A| = 0 \iff t = 0, 2, -2 \to rk\,A < 3 \\ |A| \neq 0 \iff t \neq 0, 2, -2 \to rk\,A = 3 \end{cases}$

If $t=-2$: $A = \begin{pmatrix} 1 & -2 & -2 \\ 2 & 4 & 2 \\ 2 & -4 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -2 \\ 0 & 8 & 6 \\ 0 & 0 & 0 \end{pmatrix}$ $rk\,A = 2$ when $t=-2$

$\underbrace{\begin{vmatrix} 1 & -2 \\ 2 & 4 \end{vmatrix}}_{\text{2-minor}} = 4 + 4 = 8 \neq 0$ $rk\,A = 2$ when $t=-2$

$\boxed{rk\,A = \text{maximal order of a non-zero minor of } A}$

## iii) Eigenvalues and eigenvectors

$A$ $n\times n$-matrix: If $A\underline{x} = \lambda \underline{x}$ with $\underline{x} \neq \underline{0}$ then

$$\begin{cases} \text{the number } \lambda \text{ is a } \underline{\text{eigenvalue}} \\ \text{the vector } \underline{x} \text{ is a } \underline{\text{eigenvector}} \end{cases}$$

Fact I: The eigenvalues are the solutions of the characteristic equation

$$\boxed{\det(A - \lambda I) = 0}$$

Fact 2: The eigenvectors for $A$ with eigenvalue $\lambda^*$ are the solutions of

$$\boxed{(A - \lambda^* I)\underline{x} = \underline{0}}$$

Fact 3: There is a diagonal matrix $D$ and a invertible matrix $P$ such that

$$\boxed{P^{-1}AP = D} \qquad \text{(diagonalization)}$$

if and only if
i) there are $n$ eigenvalues (when you count with multiplicities)
ii) there are $n$ linearly independent eigenvectors

If this is the case,

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix} \qquad P = \left( \underline{v_1} \,\middle|\, \underline{v_2} \,\middle|\, \cdots \,\middle|\, \underline{v_n} \right)$$

Ex: $A = \begin{pmatrix} 1 & -2 & -2 \\ 2 & 4 & 2 \\ 2 & -4 & -4 \end{pmatrix}$

i) Eigenvalues:

$$\begin{vmatrix} 1-\lambda & -2 & -2 \\ 2 & 4-\lambda & 2 \\ 2 & -4 & -4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\cdot(\lambda^2 - 0\cdot\lambda - 8) - 2\left(-2(\cancel{-4}-\lambda) \cancel{-8}\right) + 2\left(-4 + 2(4-\lambda)\right) = 0$$

$$(1-\lambda)(\lambda^2-8) - 4\lambda - 8 + 16 - 4\lambda = 0$$

$$-\lambda^3 + \lambda^2 + \cancel{8\lambda} \cancel{-8} \cancel{-4\lambda} \cancel{+8} \cancel{-4\lambda} = 0$$

$$-\lambda^3 + \lambda^2 = 0 \qquad \lambda^2(-\lambda+1) = 0$$

$$\underline{\lambda_1 = 0}, \underline{\lambda_2 = 0}, \underline{\lambda_3 = 1}$$

ii) Eigenvectors:

$\lambda = 0$: $\begin{pmatrix} 1 & -2 & -2 \\ 2 & 4 & 2 \\ 2 & -4 & -4 \end{pmatrix} \underline{x} = \underline{0}$

$\uparrow$

$A - \lambda \cdot I$, $\lambda = 0$

$$\left(\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 2 & 4 & 2 & 0 \\ 2 & -4 & -4 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 0 & 8 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 - 2x_2 - 2x_3 = 0$

$8x_2 + 6x_3 = 0$

$x_3$ free

$x_2 = -\frac{6}{8} x_3$

$x_2 = -3/4\, x_3$

$x_1 = 2(-3/4\, x_3) + 2x_3$

$x_1 = \frac{1}{2} x_3$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x_3 \\ -3/4\, x_3 \\ x_3 \end{pmatrix} = x_3 \cdot \begin{pmatrix} 1/2 \\ -3/4 \\ 1 \end{pmatrix}$$

## Eigenvectors with $\lambda = 0$:

$$\underline{x} = x_3 \cdot \begin{pmatrix} 1/2 \\ -3/4 \\ 1 \end{pmatrix} \longrightarrow$$

Just one linearly independent eigenvector

$$\underline{v_1} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$$

## Is A diagonalizable?

Eigenvalues: $\lambda = 0, 0, 1 \rightsquigarrow D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Eigenvectors: <u>not</u> enough eigenvectors

$$P = \begin{pmatrix} 2 & X & \\ -3 & X & \\ 4 & X & \end{pmatrix}$$

$\lambda = 0$ have mult. 2

but only <u>one</u> lin. independent eigenvector for $\lambda = 0$.

A <u>not diagonalizable</u>

## v) Definiteness

A: symmetric $n \times n$-matrix

Leading principal minors: $D_1, D_2, \ldots, D_n$

Principal minors: $\Delta_1, \Delta_2, \ldots, \Delta_n$

<u>Fact:</u>

1) $D_1 > 0, D_2 > 0, \ldots, D_n > 0 \iff$ A positive definite
2) $D_1 < 0, D_2 > 0, D_3 < 0, \ldots \iff$ A negative definite
3) $D_1 \geq 0, D_2 \geq 0, \ldots, D_n \geq 0$ and at least one is zero $\Rightarrow$ A may be positive semidefinite
4) $D_1 \leq 0, D_2 \geq 0, D_3 \leq 0, \ldots$ and — " — $\Rightarrow$ A may be negative semidefinite
5) All other cases: A is indefinite

In case 3) and 4):

$\Delta_1, \Delta_2, \dots, \Delta_n \geq 0 \iff A$ positive semidefinite

$\Delta_1 \leq 0, \Delta_2 \geq 0, \Delta_3 \leq 0, \dots \iff A$ negative semidefinite

$\Delta_i$ means **all** principal minors of order $i$.

Ex: $A = \begin{pmatrix} 3 & 1 & 9 \\ 1 & 2 & 8 \\ 9 & 8 & 42 \end{pmatrix}$ symm.

$D_1 = 3$

$D_2 = 5$

$D_3 = 3\cdot(84-64) - 1\cdot(42-72) + 9\cdot(8-18) = 60 + 30 - 90 = 0$

Concl: A may be positive semidefinite

All principal minors

$\Delta_1 = 3, 2, 42 \geq 0$

$\Delta_2 = 5, \begin{vmatrix} 2 & 8 \\ 8 & 42 \end{vmatrix} = 20, \begin{vmatrix} 3 & 9 \\ 9 & 42 \end{vmatrix} = 126 - 81 = 45 \geq 0$

$\Delta_3 = 0 \geq 0$

A is positive semidefinite

# ② Unconstrained optimization

Basic techniques: $\boxed{\begin{array}{l}\text{a) Compute derivatives} \\ \text{b) Find Hessian matrices}\end{array}}$

## a) Stationary pts:

Fact 1:

A stationary pt for $f$ is a pt such that $\boxed{f'_{x_1} = f'_{x_2} = \dots = f'_{x_n} = 0}$

Fact 2:

If $\underline{x}^*$ is a local/global max/min, then $\underline{x}^*$ is a stationary pt.

Fact 3:

A stationary pt $\underline{x}^*$ can be classified as local max, local min or saddle pt using Hessian:

$H(f)(\underline{x}^*)$ positive definite $\Rightarrow$ $x^*$ local min

negative definite $\Rightarrow$ $x^*$ local max

indefinite $\Rightarrow$ $\underline{x}^*$ saddle pt

Hessian of $f$ at $\underline{x}^*$

other cases : no conclusion

Ex: $f(x,y,z) = x^2 + y^2 + z^2 + z^3 + 2yz - 2x + 12y$

Stationary pts: $f'_x = 2x - 2 = 0$ $\qquad x = 1$

$f'_y = 2y + 2z + 12 = 0$

$f'_z = 2z + 3z^2 + 2y = 0$

$y + z + 6 = 0 \Rightarrow y = -6 - z$

$2z + 3z^2 + 2\cdot(-6-z) = 0 \qquad 3z^2 - 12 = 0$

$z^2 = 4$

$z = \pm 2$

$z = 2,\ y = -8,\ x = 1$

$z = -2,\ y = -4,\ x = 1$

Stat. pts: $(x,y,z) = (1,-8,2)$
$(1,-4,-2)$

Classification: $H(f) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2+6z \end{pmatrix}$

$(1,-8,2)$: $H(f)(1,-8,2) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 14 \end{pmatrix}$ $\quad D_1 = 2$, $D_2 = 4$, $D_3 = 2\cdot 24 = 48$

positive defn. $\Rightarrow$ local min at $(1,-8,2)$

$(1,-4,-2)$: $H(f)(1,-4,-2)$

$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & -10 \end{pmatrix}$ $\quad D_1 = 2$, $D_2 = 4$, $D_3 = 2(-20-4) = -48$

indefinite $\Rightarrow$ saddle pt at $(1,-4,-2)$

## ii) Convex / concave functions and global max/min

**Fact 1:**

$f$ is convex $\Longleftrightarrow$ $H(f)$ is positive semidefinite for all $\underline{x}$

$f$ is concave $\Longleftrightarrow$ $H(f)$ is negative semi definite — " —

**Fact 2:**

If $f$ is concave, then any stationary pt is global max,

— " — convex ——— " ——— global min.

Ex: $f(x,y,z) = x^2 + y^2 + y^4 + yz - 1$

$f'_x = 2x$

$f'_y = 2y + 4y^3 + z$

$f'_z = y$

$$H(f) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2+12y^2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$D_1 = 2 > 0$

$D_2 = 2(2+12y^2) = 4 + 24y^2 > 0$

$D_3 = 2 \cdot (0-1) = -2 < 0$

$H(f)$ indefinite for all $(x,y,z)$.

$f$ is not convex, not concave

Even if $f$ is not convex and not concave, it could of course still have global max/min.

iii) Envelope theorem

$f(x_1,..,x_n; a) = f(\underline{x}; a)$ function with parameter a

Consider the unconstrained optimization problem

$$\boxed{\max/\min \; f(\underline{x}; a)}$$

Assume that it has solution $\underline{x}^*(a)$ depending on a, and let $f^*(a) = f(\underline{x}^*(a))$ be the max/min value.

Envelope thm: $$\frac{df^*(a)}{da} = \frac{\partial f}{\partial a}(\underline{x}^*(a))$$

Ex: $\min f(x,y;h) = hx^4 + y^4 + 4x^2 - (6+h)xy + 4y^2 - 3h$

i) For which values of h is f convex?

$f'_x = 4hx^3 + 8x - (6+h)y$

$f'_y = 4y^3 - (6+h)x + 8y$

$$H(f) = \begin{pmatrix} 12hx^2 + 8 & -(6+h) \\ -(6+h) & 12y^2 + 8 \end{pmatrix}$$

$D_1 = 12hx^2 + 8$ ← When $h \geq 0$, $D_1 \geq 0$ for all $(x,y)$

$D_2 = (12hx^2+8)(12y^2+8) - (6+h)^2$

$= 144h\,x^2 + 96h\,x^2 + 96y^2 + [64 - (6+h)^2]$ ← When $h \leq 2$, $D_2 \geq 0$ for all $(x,y)$

$\Delta_1 = 12y^2 + 8 > 0$ for all $(x,y)$ ← Check the other principal minor of order 1 since $D_2 = 0$ at $(0,0)$ when $h=2$

Conclusion: When $0 \leq h \leq 2$ f is convex

ii) Find $x^*(h), y^*(h)$ when $h=0$:

$h=0 \Rightarrow f$ convex, so any stationary pt is global min.

Stationary pts: $h=0$

$8x - 6y = 0$

$4y^3 - 6x + 8y = 0$

$x = \frac{6y}{8} = 3y/4$

$4y^3 - 6\cdot(3y/4) + 8y = 0$

$4y^3 - \frac{18}{4}y + 8y = 0 \quad | \cdot 2$

$8y^3 - 9y + 16y = 0$

$8y^3 + 7y = 0$

$y(8y^2 + 7) = 0$

$y = 0$ or $y^2 = -7/8$ (no sol'n)

$y=0 \Rightarrow x=0$

Stat. pts: $(x,y) = (0,0)$

This is global min for $h=0$, so $(x^*(0), y^*(0)) = \underline{(0,0)}$

iii) If $h$ increases from $h=0$, what happens with $f^*(h)$?

$f^*(0) = f(0,0) = -3h \quad \leftarrow$ min. value when $h=0$

$$\frac{df^*(h)}{dh} = \frac{\partial f}{\partial h}(x^*(h), y^*(h)) = (x^4 - xy - 3)\Big|_{x=x^*(h),\, y=y^*(h)}$$

$$= x^*(h)^4 - x^*(h)y^*(h) - 3$$

$$\frac{df^*(h)}{dh}\Big|_{h=0} = x^*(0)^4 - x^*(0)y^*(0) - 3 = 0^4 - 0\cdot 0 - 3 = \underline{-3}$$

rate of change at $h=0$

The minimum value will decrease when $h$ increases from $h=0$

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|---|---|---|
| Examination date: | 13.12.2012 09:00 – 12:00 | Total no. of pages: 2 |
| Permitted examination support material: | A bilingual dictionary and BI-approved calculator TEXAS INSTRUMENTS BA II Plus | |
| Answer sheets: | Squares | |
| | Counts 80% of GRA 6035 | The subquestions are weighted equally |
| | | Responsible department: Economics |

## Question 1.

We consider the matrix $A$ given by

$$A = \begin{pmatrix} t & 1 & 1 \\ t & 2 & 1 \\ 4 & t & 2 \end{pmatrix}$$

(a) Compute the determinant and rank of $A$.
(b) Compute all eigenvalues of $A$ when $t = -2$. Is $A$ diagonalizable when $t = -2$?

## Question 2.

We consider the function $f$ with parameter $h$, given by $f(x, y, z; h) = 12 - x^4 - hx^2 - 3y^2 + 6xz - 6z^2 + h^2$. The function $f$ is defined for all points $(x, y, z) \in \mathbb{R}^3$.

(a) Compute the Hessian matrix of $f$, and show that $f$ is concave if and only if $h \geq H$ for a constant $H$. What is the value of $H$?
(b) Find the global maximum point $(x^*(h), y^*(h), z^*(h))$ of $f$ when $h \geq H$.
(c) Will the global maximum value $f^*(h)$ increase or decrease when the value of the parameter $h$ increases? We assume that the initial value of $h$ satisfies $h \geq H$.

## Question 3.

Find the general solution of the following differential equations:

(a) $y'' - 5y' + 6y = 10e^{-t}$
(b) $4te^{2t}y - (1 - 2t)e^{2t}y' = 0$ (when $t > 1/2$)

## Question 4.

We consider a model for housing prices, where $p_t$ is the price after $t$ years. The model is given by the difference equation

$$p_{t+2} - 2p_{t+1} + p_t = -15, \quad p_0 = 695,\ p_1 = 743$$

(a) Solve the difference equation.
(b) We define $d_t = p_{t+1} - p_t$ to be the change in housing prices. Show that $d_{t+1} - d_t$ is constant, and use this to determine when housing prices will increase and when housing prices will decrease.

## Question 5.

We consider the following optimization problem:

$$\max\ \ln(x^2 y) - x - y \text{ subject to } \begin{cases} x + y \geq 4 \\ x \geq 1 \\ y \geq 1 \end{cases}$$

Sketch the set of admissible points, and solve the optimization problem.

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QUESTION 1.

(a) We compute the determinant of $A$ using cofactor expansion along the first column, and find that

$$\det(A) = \begin{vmatrix} t & 1 & 1 \\ t & 2 & 1 \\ 4 & t & 2 \end{vmatrix} = t(4-t) - t(2-t) + 4\cdot(-1) = \mathbf{2t-4}$$

Since $\det(A) \neq 0$ for $t \neq 2$, and the minor $\left|\begin{smallmatrix} 1 & 1 \\ 2 & 1 \end{smallmatrix}\right| = -1$ of order two is non-zero, we have that

$$\text{rk}(A) = \begin{cases} \mathbf{3}, & t \neq 2 \leftarrow \text{since } |A| \neq 0 \\ \mathbf{2}, & t = 2 \leftarrow \text{since } |A| = 0 \text{ and } \left|\begin{smallmatrix} 1 & 1 \\ 2 & 1 \end{smallmatrix}\right| \neq 0 \end{cases}$$

(b) When $t = -2$, the characteristic equation of $A$ is given by

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 1 & 1 \\ -2 & 2-\lambda & 1 \\ 4 & -2 & 2-\lambda \end{vmatrix} = 0$$

Cofactor expansion along the first column gives

$$(-2-\lambda)((2-\lambda)^2+2) - (-2)(2-\lambda+2) + 4(1-(2-\lambda)) = 0$$

and we find that this reduces to

$$(-2-\lambda)(2-\lambda)^2 + 2(-2-\lambda) + 2(4-\lambda) + 4(\lambda-1) = (-2-\lambda)(2-\lambda)^2 = 0$$

since all terms except the first cancels

The eigenvalues are therefore $\lambda = \mathbf{-2}$ and $\lambda = \mathbf{2}$, where the last eigenvalue has multiplicity two. When $\lambda = 2$, the eigenvectors are given by $(A-2I)\mathbf{x} = \mathbf{0}$, and the matrix

$$A - 2I = \begin{pmatrix} -4 & 1 & 1 \\ -2 & 0 & 1 \\ 4 & -2 & 0 \end{pmatrix}$$

has rank two since $A - 2I$ has a non-zero minor $\left|\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right| = 1$ of order two — it cannot have rank three since $\lambda = 2$ is an eigenvalue. Therefore, the linear system has just one free variable while $\lambda = 2$ is an eigenvalue of multiplicity two. So $A$ is **not diagonalizable** when $t = -2$.

$|A-2I| = 0$ since 2 is eigenvalue

$\left|\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right| \neq 0 \Rightarrow rk(A-2I) = 2 \Rightarrow$ 2 pivots, 1 free var. $\Rightarrow$ only one lin. indep. eigenvector for $\lambda = 2$ $\Rightarrow$ not enough, not diag.

$D_1 = \underbrace{-12x^2}_{\leq 0 \text{ for all } x} \underbrace{-2h}_{\text{constant}, \leq 0 \text{ for } h \geq 0}$ $D_1 \leq 0$ for all $x,y,z$ when $h \geq 0$

$D_2 = -6 \cdot D_1 \geq 0$ for all $(x,y,z)$ when $h \geq 0$

QUESTION 2.

(a) We compute the partial derivatives and the Hessian matrix of $f$:

$$\begin{pmatrix} f'_x \\ f'_y \\ f'_z \end{pmatrix} = \begin{pmatrix} -4x^3 - 2hx + 6z \\ -6y \\ 6x - 12z \end{pmatrix}, \quad f'' = \begin{pmatrix} -12x^2 - 2h & 0 & 6 \\ 0 & -6 & 0 \\ 6 & 0 & -12 \end{pmatrix}$$

$D_3 = \underbrace{-6 \cdot 144x^2}_{\leq 0 \text{ for all } x} \underbrace{-6(24h-36)}_{\text{constant}, \leq 0 \text{ when } h \geq 1.5}$

$D_3 \leq 0$ for all $x,y,z$ when $h \geq 1.5$

We see that the leading principal minors are given by $D_1 = -12x^2 - 2h$, $D_2 = -6D_1$ and $D_3 = -6(144x^2 + 24h - 36)$. Hence $D_1 \leq 0$ for all $(x, y, z)$ if and only if $h \geq 0$, and if this is the case then $D_2 = -6D_1 \geq 0$. Moreover, $D_3 \leq 0$ for all $(x, y, z)$ if and only if $h \geq 3/2$. This means that $D_1 \leq 0$, $D_2 \geq 0$, $D_3 \leq 0$ if and only if $h \geq 3/2$, and the equalities are strict if $h > 3/2$. If $h = 3/2$, then $D_3 = 0$, and we compute the remaining principal minors. We find that $\Delta_1 = -6, -12 \leq 0$ and that $\Delta_2 = 144x^2, 72 \geq 0$. We conclude that $f$ is concave if and only if $h \geq 3/2$, and $H = \mathbf{3/2}$.

Since $D_1 = 0$ for $x=0$, $h=0$
$D_3 = 0$ for $x=0$, $h=1.5$
we must check $\Delta_i$

(b) We compute the stationary points, which are given by the equations

$$-4x^3 - 2hx + 6z = 0, \quad -6y = 0, \quad 6x - 12z = 0$$

The last two equations give $y = 0$ and $z = x/2$, and the first equations becomes

$$-4x^3 - 2hx + 3x = x(-4x^2 + 3 - 2h) = 0 \quad \Leftrightarrow \quad x = 0$$

since $x^2 = (3 - 2h)/4$ has no solutions when $h > 3/2$ and the solution $x = 0$ when $h = 3/2$. The stationary points are therefore given by $(x^*(h), y^*(h), z^*(h)) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$ when $h \geq 3/2$, and this is the global maximum since $f$ is concave.

(c) Let $h \geq 3/2$. By the Envelope Theorem, we have that

since $h \geq 1.5$

$$\frac{d}{dh} f^*(h) = \frac{\partial f}{\partial h}\Big|_{(x,y,z)=(0,0,0)} = (-x^2 + 2h)\Big|_{(x,y,z)=(0,0,0)} = 2h \geq 3$$

Since the derivative is positive, the maximal value will **increase** when $h$ increases. We could also compute $f^*(h) = f(0, 0, 0) = 12 + h^2$ explicitly for $h \geq 3/2$, and use this to see that $f^*(h)$ increases when $h$ increases.

this is the max for $h \geq 1.5$ this is $\frac{\partial f}{\partial h}$

QUESTION 3.

(a) The homogeneous equation $y'' - 5y' + 6y = 0$ has characteristic equation $r^2 - 5r + 6 = 0$, and therefore roots $r = 2, 3$. Hence the homogeneous solution is $y_h(t) = C_1e^{2t} + C_2e^{3t}$. To find a particular solution of $y'' - 5y' + 6y = 10e^{-t}$, we try $y = Ae^{-t}$. This gives $y' = -Ae^{-t}$ and $y'' = Ae^{-t}$, and substitution in the equation gives $(A + 5A + 6A)e^{-t} = 10e^{-t}$, or $12A = 10$. Hence $A = 5/6$ is a solution, and $y_p(t) = \frac{5}{6}e^{-t}$ is a particular solution. This gives general solution

$$y(t) = \mathbf{C_1e^{2t} + C_2e^{3t} + \frac{5}{6}e^{-t}}$$

(b) The differential equation $4te^{2t}y - (1 - 2t)e^{2t}y' = 0$ is exact if and only if there is a function $h(t, y)$ such that

$$\frac{\partial h}{\partial t} = 4te^{2t}y, \quad \frac{\partial h}{\partial y} = -(1 - 2t)e^{2t}$$

We see that $h(t, y) = -(1 - 2t)e^{2t}y$ is a solution to the last equation, and differentiation shows that it is a solution to the first equation as well. Therefore the solution of the exact differential equation is given by

$$h(t, y) = -(1 - 2t)e^{2t}y = C \quad \Rightarrow \quad y = \frac{\mathbf{Ce^{-2t}}}{\mathbf{2t - 1}} \quad (\text{when } t > 1/2)$$