

LECTURE 5-A

EIVIND ERIKSEN

SEP 19 2013

GRAGOSS

MATHEMATICS

PLAN:

- ① Applications of eigenvalues:
Markov processes
- ② Quadratic forms
- ③ Definiteness of symmetric matrices

Reading:

[NEJ] 6.2 Ex 3, 23.1 Ex 23.4, 23.6,
13.1 - 13.5, 16.1 - 16.4, 23.8

- ① Applications of eigenvalues: Markov chains/processes

A
nxn

→ Eigenvalues → Eigenvectors
 $\lambda_1, \lambda_2, \dots, \lambda_r$ v_1, v_2, \dots

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

$$P = \begin{pmatrix} v_1 & | & v_2 & | & \cdots \end{pmatrix}$$

A is diagonalized by P if P is an invertible matrix such that

$$P^{-1}AP = D \text{ is diagonal}$$

Let v_0 be an n-vector. Let us try to compute

$$v_0 \rightarrow A \cdot v_0 \rightarrow A^2 v_0 \rightarrow \dots \rightarrow A^m v_0 \rightarrow \dots$$

How can we compute A^m efficiently?

If $P^{-1}AP = D$ is a diagonalisation of A , this can be used to compute A^m when m is big:

$$P^{-1}AP = D \quad | P.$$

$$AP = PD \quad | \cdot P^{-1}$$

$$A = PDP^{-1}$$

$$A^m = A \cdot A \cdot \dots \cdot A$$

$$= (PDP^{-1}) \cdot (PDP^{-1}) \cdot (PDP^{-1}) \dots (PDP^{-1})$$

$$= P \cdot D^m \cdot P^{-1}$$

Fact:

$$A^m = P \cdot D^m \cdot P^{-1}$$

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

$$D^m = \begin{pmatrix} \lambda_1^m & 0 & \dots & 0 \\ 0 & \lambda_2^m & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^m \end{pmatrix}$$

$$D^2 = \begin{pmatrix} \lambda_1^2 & 0 & \dots & 0 \\ 0 & \lambda_2^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^2 \end{pmatrix}$$

Hence

$$A^m = P \cdot \begin{pmatrix} \lambda_1^m & 0 & \dots & 0 \\ 0 & \lambda_2^m & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^m \end{pmatrix} \cdot P^{-1}$$

Markov process

Ex: Families are classified as U (urban), S (suburban) and R (rural). At time $t=n$ (after n years), the share of families in these groups can be described by the state vector

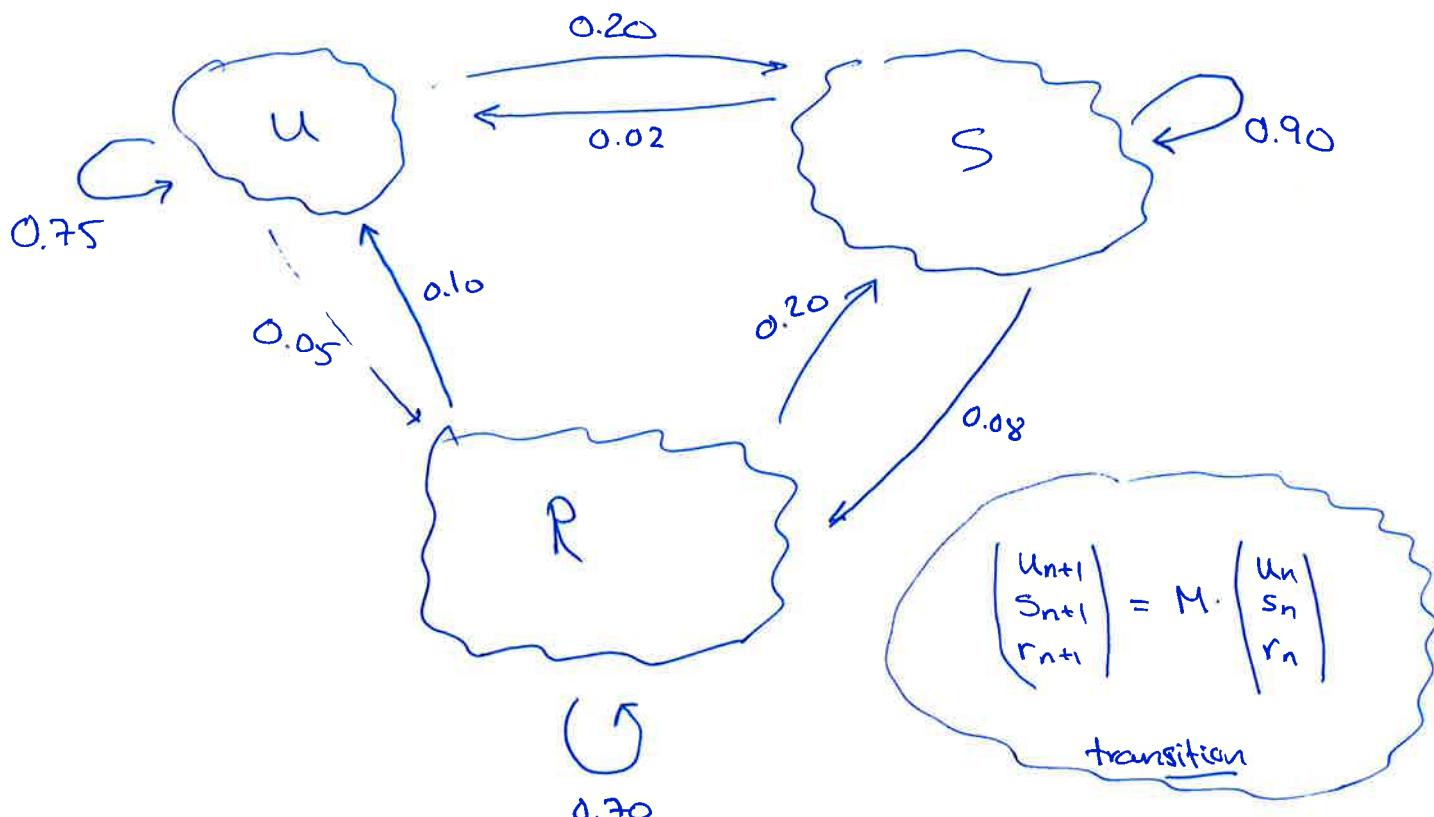
$$\underline{V}_n = \begin{pmatrix} U_n \\ S_n \\ R_n \end{pmatrix} \quad \left\{ \begin{array}{l} U_n \geq 0 \\ S_n \geq 0 \\ R_n \geq 0 \end{array} \right. , \quad U_n + S_n + R_n = 1$$

Ex:
 $\underline{V} = \begin{pmatrix} 0.8 \\ 0.1 \\ 0.1 \end{pmatrix}$

From year n to year $n+1$, the change in the shares are given by a transition matrix or Markov matrix

$$M = \begin{pmatrix} 0.75 & 0.02 & 0.10 \\ 0.20 & 0.90 & 0.20 \\ 0.05 & 0.08 & 0.70 \end{pmatrix} \quad \left\{ \begin{array}{l} m_{ij} \geq 0 \\ \text{each column has sum 1} \end{array} \right.$$

It can be described graphically as follows:



Markov process:

$$\underline{V}_0 = \begin{pmatrix} U_0 \\ S_0 \\ R_0 \end{pmatrix} \longrightarrow \underline{V}_1 = M \cdot \underline{V}_0 \longrightarrow \underline{V}_2 = M \cdot \underline{V}_1 = M^2 \cdot \underline{V}_0 \longrightarrow \dots \longrightarrow \underline{V}_n = M^n \cdot \underline{V}_0$$

The Markov process is regular if $m_{ij} > 0$ for all i, j . We assume that this is the case. The following holds for all regular Markov processes:

Fact: i) $\lambda=1$ is an eigenvalue of M , and there is a unique eigenvector \underline{v} with eigenvalue $\lambda=1$ that is a state vector (i.e. $\underline{v} = (v_i)$ with $v_i \geq 0$, $v_1 + \dots + v_k = 1$)

ii) $\lim_{n \rightarrow \infty} M^n \cdot \underline{v}_0 = \underline{v}$ and $\lim_{n \rightarrow \infty} M^n = (\underline{v} | \underline{v} | \dots | \underline{v})$

Ex: $M = \begin{pmatrix} 0.75 & 0.02 & 0.10 \\ 0.20 & 0.90 & 0.20 \\ 0.05 & 0.08 & 0.70 \end{pmatrix}$

$$\lambda=1: \begin{pmatrix} -0.25 & 0.02 & 0.10 \\ 0.20 & -0.10 & 0.20 \\ 0.05 & 0.08 & -0.30 \end{pmatrix} \rightarrow \begin{pmatrix} 5 & 8 & -30 \\ -25 & 2 & 10 \\ 20 & -10 & 20 \end{pmatrix} \rightarrow \begin{pmatrix} 5 & 8 & -30 \\ 0 & 42 & -140 \\ 0 & -42 & 140 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 5 & 8 & -30 \\ 0 & 42 & -140 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{aligned} 5x + 8y - 30z &= 0 \\ 42y - 140z &= 0 \end{aligned}$$

$$y = \frac{140z}{42} = \frac{10}{3}z$$

$$5x = 30 \cdot z - 8 \cdot \frac{10}{3}z = \frac{90 - 80}{3}z$$

$$x = \frac{2}{3}z$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2/3 \cdot z \\ 10/3 \cdot z \\ z \end{pmatrix} = \frac{z}{3} \cdot \begin{pmatrix} 2 \\ 10 \\ 3 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} 2/15 \\ 10/15 \\ 3/15 \end{pmatrix} \quad (\text{with } z=15)$$

Conclusion: As $n \rightarrow \infty$ (in the long run) $u = 2/15 \approx 13.3\%$ of families are urban, $s = 10/15 \approx 66.7\%$ are suburban, and $r = 3/15 = 20\%$ are rural.

Check: Compute M^{10}, M^{50}, M^{100} using Wolfram Alpha or other software.

It is also possible to compute M^n as

$$M^n = P \cdot \begin{pmatrix} 1 & & \\ \alpha_{10} & 0 & \\ 0 & \alpha_{20} & \\ 0 & 0 & \alpha_{30} \end{pmatrix}^n \cdot P^{-1} \approx P \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} P^{-1}$$

↑
Since $\alpha_1 = 1, \alpha_2, \alpha_3 < 1$

(2) Quadratic forms

Defn: A quadratic form is a polynomial function
 (in $x_1, x_2, \dots, x_n = \underline{x}$) where each term has
 degree two.

$$\begin{array}{ll} \underline{\text{Ex:}} & n=1 : f(x) = ax^2 \\ & n=2 : f(x,y) = ax^2 + bxy + cy^2 \\ & n=3 : \cancel{f(x,y,z)} = c_{11}x_1^2 + c_{12}x_1x_2 + c_{13}x_1x_3 \\ & \qquad\qquad\qquad f(x_1, x_2, x_3) \qquad\qquad\qquad + c_{22}x_2^2 + c_{23}x_2x_3 \\ & \qquad\qquad\qquad \qquad\qquad\qquad + c_{33}x_3^2 \end{array}$$

In general:

$$f(\underline{x}) = f(x_1, \dots, x_n) = c_{11}x_1^2 + c_{12}x_1x_2 + \dots + c_{1n}x_1x_n \\ + c_{22}x_2^2 + \dots + c_{2n}x_2x_n \\ \vdots \\ + c_{nn}x_n^2$$

Fact: Any quadratic form in n variables
 can be written in matrix form as

$$f(\underline{x}) = \underline{x}^T \cdot A \cdot \underline{x}$$

for a unique symmetric $n \times n$ -matrix A .

$$f(\underline{x}) = (x_1 \ x_2 \ \dots \ x_n) \cdot \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n \times 1}$$

$$\underline{\text{Ex:}} \quad f(x_1, x_2) = x_1^2 + 4x_1x_2 + 7x_2^2$$

$$= (x_1 \ x_2) \cdot \begin{pmatrix} 1 & 2 \\ 2 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\underline{\underline{A = \begin{pmatrix} 1 & 2 \\ 2 & 7 \end{pmatrix}}}$$

$$(x_1 \ x_2) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (ax_1 + cx_2 \ bx_1 + dx_2) \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= (ax_1 + cx_2)x_1 + (bx_1 + dx_2)x_2$$

$$= ax_1^2 + cx_2x_1 + bx_1x_2 + dx_2^2$$

$$= ax_1^2 + (b+c)x_1x_2 + dx_2^2$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

x_1^2	pos. (1,1)
x_1x_2	pos (1,2)
x_2^2	(2,1)
	pos (2,2)

}

$$\underline{\underline{A = \begin{pmatrix} 1 & 2 \\ 2 & 7 \end{pmatrix}}}$$

In general, if $f(x_1, \dots, x_n) = c_{11}x_1^2 + c_{12}x_1x_2 + \dots + c_{1n}x_1x_n$
 $+ c_{21}x_2^2 + \dots + c_{2n}x_2x_n$
 $\dots \dots + c_{nn}x_n^2$

then

$$A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \end{vmatrix}$$

is given by

$$a_{ii} = c_{ii}$$

$$a_{ij} = c_{ij}/z = a_{ji} \quad \text{when } i < j$$

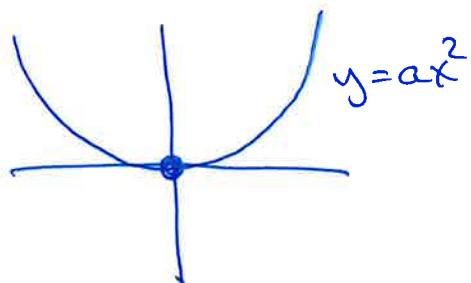
$$\text{Ex: } f(\underline{x}) = \underline{2x_1^2 - 4x_1x_3 + 7x_2^2} - \underline{14x_1x_4 + x_4^2}$$

$$A = \begin{pmatrix} 2 & 0 & -2 & -7 \\ 0 & 7 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ -7 & 0 & 0 & 1 \end{pmatrix}$$

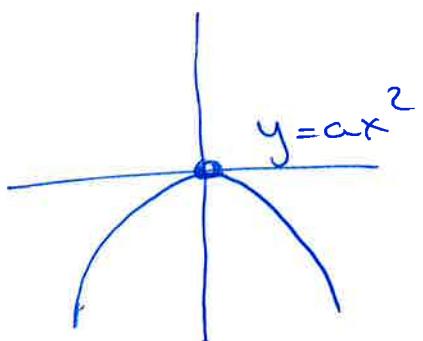
symm. matrix
of f

Graph of quadratic forms

$$\underline{n=1:} \\ f(x) = ax^2$$



$$a > 0$$



$$a < 0$$

Symm. matrix: $A = (a)$

Defn: Let f be quadratic form in n variables,
and let A be its symmetric $n \times n$ -matrix.
Note: $f(\underline{0}) = f(0, 0, \dots, 0) = 0$.

f is called positive semidefinite if

$$f(\underline{x}) \geq 0 \text{ for all } \underline{x}$$

f is called positive definite "

$$f(\underline{x}) > 0 \text{ for all } \underline{x} \neq \underline{0}$$

f is negative semidefinite "

$$f(\underline{x}) \leq 0 \text{ for all } \underline{x}$$

f is negative definite "

$$f(\underline{x}) < 0 \text{ for all } \underline{x} \neq \underline{0}$$

f is indefinite

" if f has both positive and negative values

Classify $f(\underline{x}) / A$: Find out which of the type of definiteness f / A has.

Note: $f(\underline{x})$ positive semidef. $\Rightarrow \underline{0}$ is minimum
 negative $\dashv \dashv$ $\underline{0}$ is maximum
 indefinite $\underline{0}$ is Saddle pt.

Ex: $f(x_1, x_2, x_3) = 2x_1^2 + 3x_2^2 + 7x_3^2$ $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix}$

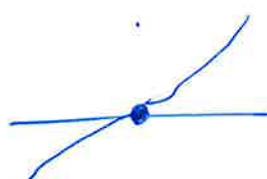
pos. semidef: $f(\underline{x}) \geq 0$ for all $\underline{x} = x_1, x_2, x_3$ ok.

It is actually positive definite.



$$f(x, y) = x^2 - y^2 \quad A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\left. \begin{array}{l} f(1, 0) = 1 > 0 \\ f(0, 1) = -1 < 0 \end{array} \right\} \text{indeterminate}$$



Ex: $f(x_1, x_2, \dots, x_n) = c_{11}x_1^2 + c_{22}x_2^2 + \dots + c_{nn}x_n^2$

$$A = \begin{pmatrix} c_{11} & 0 & \dots & 0 \\ 0 & c_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_{nn} \end{pmatrix}$$

$c_{11}, c_{22}, \dots, c_{nn} > 0$: positive definite

$c_{11}, c_{22}, \dots, c_{nn} \geq 0$: positive semidefinite

$c_{11}, c_{22}, \dots, c_{nn} < 0$: negative defn.

$c_{11}, c_{22}, \dots, c_{nn} \leq 0$: negative semidefinite

Both pos. and neg. c_{ii} 's. indistint

Note:

Eigenvalues
of $A =$

$$\lambda_1 = c_{11}$$

$$\lambda_2 = c_{22}$$

\vdots

$$\lambda_n = c_{nn}$$

Thm.

Let $f(\underline{x}) = f(x_1, \dots, x_n)$ be a quadratic form, with symmetric matrix A with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Then:

f positive definite $\iff \lambda_1, \lambda_2, \dots, \lambda_n > 0$

f positive semidefinite $\iff \lambda_1, \lambda_2, \dots, \lambda_n \geq 0$

f negative definite $\iff \lambda_1, \lambda_2, \dots, \lambda_n < 0$

f negative semidefinite $\iff \lambda_1, \lambda_2, \dots, \lambda_n \leq 0$

f indefinite \iff There are both positive and negative eigenvalues

Why?: $f(x_1, x_2, \dots, x_n) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2$,

where y_1, y_2, \dots, y_n are expressions in x_1, \dots, x_n .

$$\begin{aligned} \text{Ex: } f(x, y) &= 4xy = (x+y)^2 - (x-y)^2 = 1 \cdot z_1^2 - 1 \cdot z_2^2 \\ &= (x+y)^2 - (x-y)^2 \\ &= (x^2 + 2xy + y^2) - (x^2 - 2xy + y^2) \end{aligned}$$

$$\underline{\text{Ex:}} \quad f(x_1, y, z) = \underline{x^2} - 6xz + \underline{2y^2} + \underline{2z^2}$$

$$A = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 2 & 0 \\ -3 & 0 & 2 \end{pmatrix}$$

Eigenvalues of A:

$$\begin{vmatrix} 1-\lambda & 0 & -3 \\ 0 & 2-\lambda & 0 \\ -3 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda) \cdot \begin{vmatrix} 1-\lambda & -3 \\ -3 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda) \cdot (\lambda^2 - 3\lambda - 7) = 0$$

$$\lambda = 2 \quad \text{or} \quad \lambda^2 - 3\lambda - 7 = 0$$

$$\lambda = \frac{3 \pm \sqrt{9 + 28}}{2}$$

$$\lambda_1 = 2 \quad \lambda_2 = \frac{3 + \sqrt{37}}{2} \quad \lambda_3 = \frac{3 - \sqrt{37}}{2}$$

pos.

Pos.

neg.

$\Rightarrow f$ is indefinite

Method uses principal minors

A symm.
 $n \times n$ -matrix

A leading principal minor of order i is a minor of order i obtained by keeping

row $1, 2, 3, \dots, i$
col. $1, 2, 3, \dots, i$

and deleting the remaining rows/cols.

We call it D_i , $i=1, 2, \dots, n$.

Ex: $A = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 2 & 0 \\ -3 & 0 & 2 \end{pmatrix}$

$$D_1 = 1$$

$$D_2 = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2$$

$$D_3 = \begin{vmatrix} 1 & 0 & -3 \\ 0 & 2 & 0 \\ -3 & 0 & 2 \end{vmatrix} = 2 \cdot (2 - 9) = -14$$

Thm:

Let $f(\underline{x})$ be a quadratic form, let A be its symmetric matrix and let D_1, D_2, \dots, D_n be its leading principal minors, then:

$D_1, D_2, \dots, D_n > 0 \iff f$ is pos. definite

$D_1 < 0, D_2 > 0, \quad \iff f$ is neg. definite

$$D_3 < 0, \dots,$$

$$(-1)^i \cdot D_i > 0$$

If the leading principal minors fail to follow these patterns because at least one D_i has the wrong sign, then f is indefinite.

↑

i.e. $>$ instead of $<$ or
 $<$ instead of $>$

Ex: $f(x,y) = -x^2 - y^2$ negative definite

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad D_1 = -1$$

$$D_2 = (-1) \cdot (-1) = 1$$

$$f(x,y) = 7x^2 - 3y^2$$

$$A = \begin{pmatrix} 7 & 0 \\ 0 & -3 \end{pmatrix}$$

$$\begin{aligned} D_1 &= 7 \\ D_2 &= -21 \end{aligned}$$

Indefinite since

not $D_1 > 0, D_2 > 0$

not $D_1 < 0, D_2 > 0$

wrong sign in both cases

Principal minors: A symm $n \times n$ -matrix

A principle minor of order i is a minor of order i obtained by deleting the same rows and columns. The principal minors of order i are called Δ_i .

$$A = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 2 & 0 \\ -3 & 0 & 2 \end{pmatrix}$$

$$\Delta_1 = 1, 2, 2$$

$$D: \begin{pmatrix} \text{row } 2, 3 \\ \text{col } 2, 3 \end{pmatrix}, \begin{pmatrix} \text{row } 1, 3 \\ \text{col } 1, 3 \end{pmatrix}, \begin{pmatrix} \text{row } 1, 2 \\ \text{col } 1, 2 \end{pmatrix}$$

$$\Delta_2 = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}, \begin{vmatrix} 1 & -3 \\ -3 & 2 \end{vmatrix}, \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}$$

$$D: \begin{pmatrix} \text{row } 3 \\ \text{col } 3 \end{pmatrix}, \begin{pmatrix} \text{row } 2 \\ \text{col } 2 \end{pmatrix}, \begin{pmatrix} \text{row } 1 \\ \text{col } 1 \end{pmatrix}$$

$$\Delta_3: \begin{vmatrix} 1 & 0 & -3 \\ 0 & 2 & 0 \\ -3 & 0 & 2 \end{vmatrix} = -14$$

Thm (continued from last page)

f is positive semidefinite $\Leftrightarrow \Delta_i \geq 0$ for all principal minors

f is negative semidefinite $\Leftrightarrow \left\{ \begin{array}{l} \Delta_i \leq 0 \text{ for all principal minors of odd order} \\ \text{(and)} \\ \Delta_i \geq 0 \text{ for all principal minors of even order} \end{array} \right.$

$$\underline{\text{Ex: }} f(\underline{x}) = 2x_1^2 - 4x_1x_3 + 7x_2^2 - 14x_1x_4 + x_4^2$$

$$A = \begin{pmatrix} 2 & 0 & -2 & -7 \\ 0 & 7 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ -7 & 0 & 0 & 1 \end{pmatrix}$$

$$D_1 = 2$$

$$D_2 = \begin{vmatrix} 2 & 0 \\ 0 & 7 \end{vmatrix} = 14$$

$$D_3 = \begin{vmatrix} 2 & 0 & -2 \\ 0 & 7 & 0 \\ -2 & 0 & 0 \end{vmatrix} = -2 \cdot \begin{vmatrix} 0 & 7 \\ -2 & 0 \end{vmatrix} = -2(14) = -28$$

$$D_4 = \begin{vmatrix} 2 & 0 & -2 & -7 \\ 0 & 7 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ -7 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot D_3 - (-7) \cdot \begin{vmatrix} 0 & -2 & -7 \\ 7 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$$= D_3 + 7 \cdot 0 = -28$$

$\underbrace{\text{not positive detn. } (D_3 < 0)}$ } indefinite
 $\underbrace{\text{not neg. detn. } (D_1 > 0)}$ }

Would be difficult to compute eigenvalues
in this case. Try!