

# LECTURE 5.B

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GRA 6035

MATHEMATICS

## PLAN:

- ① Markov chains (or processes)
- ② Quadratic forms
- ③ Definiteness of symmetric matrices

## Reading:

[ME] 6.2 Ex 3, 23.1 Ex 23.4,  
23.6, 13.1-13.5,  
16.1-16.4, 23.8

## Markov process

Ex: Families are classified as U (urban), S (suburban) and R (rural). At time  $t=n$  (after  $n$  years), the share of families in these groups can be described by the state vector

$$\underline{V}_n = \begin{pmatrix} U_n \\ S_n \\ R_n \end{pmatrix} \quad \begin{cases} U_n \geq 0 \\ S_n \geq 0 \\ R_n \geq 0 \end{cases}, \quad U_n + S_n + R_n = 1$$

Ex:

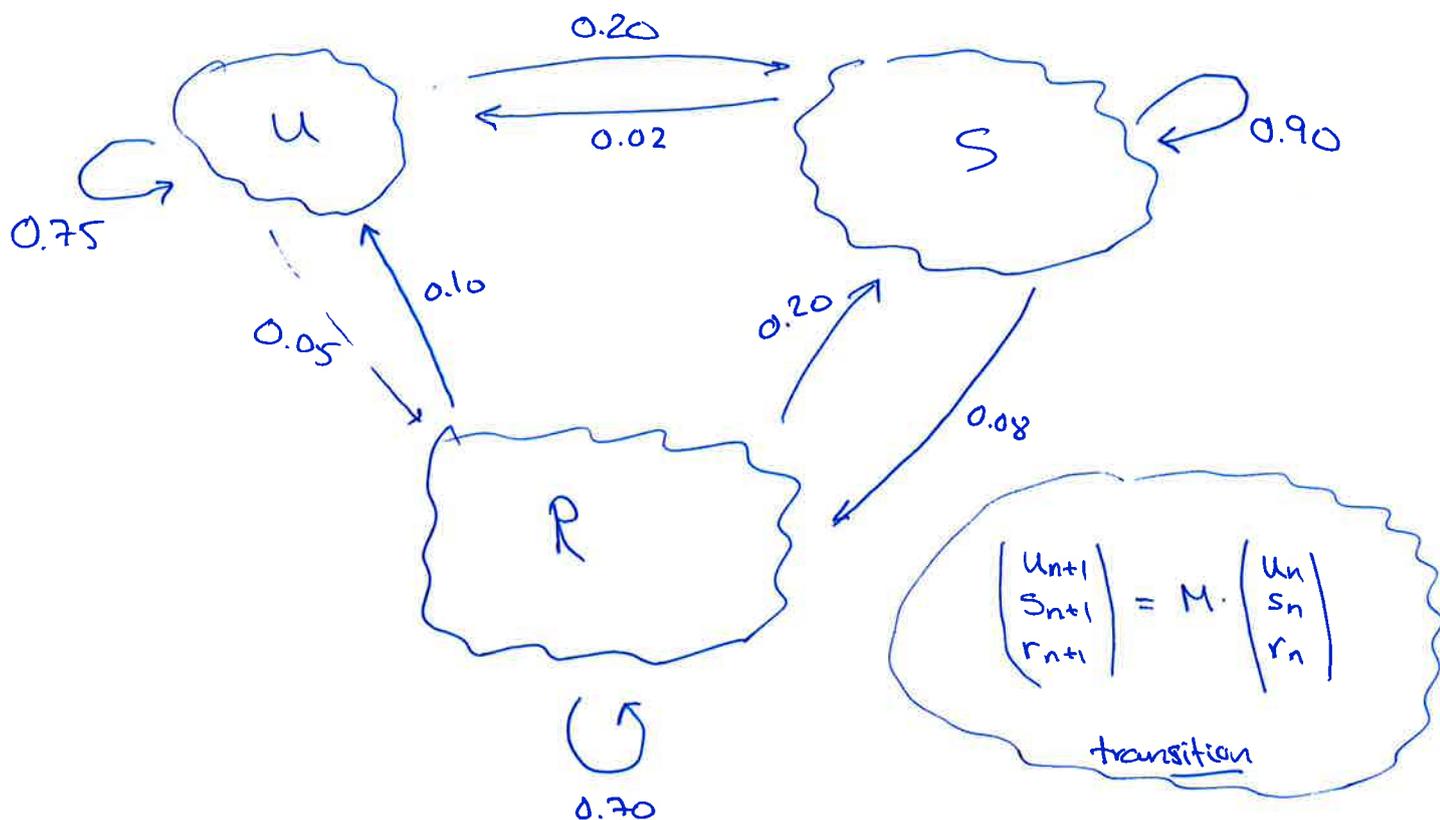
$$\underline{V} = \begin{pmatrix} 0.8 \\ 0.1 \\ 0.1 \end{pmatrix}$$

From year  $n$  to year  $n+1$ , the change in the shares are given by a transition matrix or Markov matrix

$$M = \begin{pmatrix} 0.75 & 0.02 & 0.10 \\ 0.20 & 0.90 & 0.20 \\ 0.05 & 0.08 & 0.70 \end{pmatrix}$$

$$\begin{cases} m_{ij} \geq 0 \\ \text{each column has sum } 1 \end{cases}$$

It can be described graphically as follows:



Markov process:

$$\underline{V}_0 = \begin{pmatrix} U_0 \\ S_0 \\ R_0 \end{pmatrix} \xrightarrow{\text{start}} \underline{V}_1 = M \cdot \underline{V}_0 \xrightarrow{\text{---}} \underline{V}_2 = M \underline{V}_1 = M^2 \underline{V}_0 \xrightarrow{\text{---}} \dots \xrightarrow{\text{---}} \underline{V}_n = M^n \cdot \underline{V}_0$$

The Markov process is regular if  $m_{ij} > 0$  for all  $ij$ . We assume that this the case. The following holds for all regular Markov processes:

Fact: i)  $\lambda=1$  is an eigenvalue of  $M$ , and there is a unique eigenvector  $\underline{v}$  with eigenvalue  $\lambda=1$  that is a state vector (i.e.  $\underline{v}=(v_i)$  with  $v_i \geq 0, v_1 + \dots + v_k = 1$ )

ii)  $\lim_{n \rightarrow \infty} M^n \cdot \underline{v}_0 = \underline{v}$  and  $\lim_{n \rightarrow \infty} M^n = (\underline{v} | \underline{v} | \dots | \underline{v})$

Ex:  $M = \begin{pmatrix} 0.75 & 0.02 & 0.10 \\ 0.20 & 0.90 & 0.20 \\ 0.05 & 0.08 & 0.70 \end{pmatrix}$

$\lambda=1$ :  $\begin{pmatrix} -0.25 & 0.02 & 0.10 \\ 0.20 & -0.10 & 0.20 \\ 0.05 & 0.08 & -0.30 \end{pmatrix} \rightarrow \begin{pmatrix} 5 & 8 & -30 \\ -25 & 2 & 10 \\ 20 & -10 & 20 \end{pmatrix} \rightarrow \begin{pmatrix} 5 & 8 & -30 \\ 0 & 42 & -140 \\ 0 & -42 & 140 \end{pmatrix}$

$\rightarrow \begin{pmatrix} \textcircled{5} & 8 & -30 \\ 0 & \textcircled{42} & -140 \\ 0 & 0 & 0 \end{pmatrix}$

$5x + 8y - 30z = 0$   
 $42y - 140z = 0$   
 $z$  free

$y = \frac{140z}{42} = \frac{10}{3}z$

$5x = 30z - 8 \cdot \frac{10}{3}z = \frac{90 - 80}{3}z$

$x = \frac{2}{3}z$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{2}{3}z \\ \frac{10}{3}z \\ z \end{pmatrix} = \frac{z}{3} \cdot \begin{pmatrix} 2 \\ 10 \\ 3 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} 2/15 \\ 10/15 \\ 3/15 \end{pmatrix}$  (with  $z=1/5$ )

Conclusion: As  $n \rightarrow \infty$  (in the long run)  $u = 2/15 \approx 13.3\%$  of families are urban,  $s = 10/15 \approx 66.7\%$  are suburban, and  $r = 3/15 = 20\%$  are rural.

Check: Compute  $M^{10}, M^{50}, M^{100}$  using Wolfram Alpha or other software.

It is also possible to compute  $M^n$  as

$M^n = P \cdot \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}^n \cdot P^{-1} \approx P \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} P^{-1}$   
 since  $\lambda_1=1, \lambda_2, \lambda_3 < 1$

## ② Quadratic forms

Defn: A quadratic form in  $x_1, x_2, \dots, x_n$  is a polynomial function

$$f(\underline{x}) = f(x_1, x_2, \dots, x_n)$$

where each term has degree two.

Ex:

$$n=1: f(x) = ax^2$$

$$n=2: f(x, y) = ax^2 + bxy + cy^2$$

$$n=3: ~~f(x, y, z) = a_{11}x_1^2~~$$

$$f(x_1, x_2, x_3) = a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 \\ + a_{22}x_2^2 + a_{23}x_2x_3 \\ + a_{33}x_3^2$$

Fact: Any quadratic form  $f(\underline{x})$  in  $n$  variables can be written as

$$f(\underline{x}) = \underline{x}^T \cdot A \cdot \underline{x} = (x_1 \ x_2 \ \dots \ x_n) \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

for a unique symmetric  $n \times n$ -matrix  $A$ .

Ex:  $f(x,y) = 2x^2 + 6xy - 7y^2$

$$= \underbrace{(x \ y) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\text{matrix}} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= (ax + cy \quad bx + dy) \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= (ax + cy)x + (bx + dy)y$$

$$= ax^2 + cyx + bxy + dy^2$$

$$= ax^2 + (b+c)xy + dy^2$$

$$A = \underline{\underline{\begin{pmatrix} 2 & 3 \\ 3 & -7 \end{pmatrix}}}$$

Ex:  $f(x_1, x_2, x_3, x_4) = \underline{2x_1^2} - \underline{4x_1x_3} + \underline{7x_2^2} - \underline{14x_1x_4} + \underline{x_4^2}$

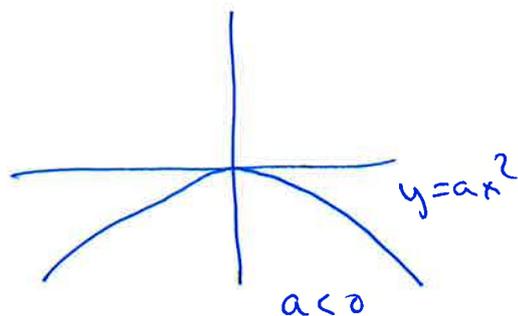
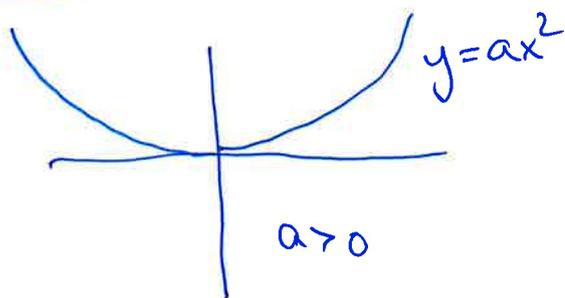
$$A = \begin{pmatrix} 2 & 0 & -2 & -7 \\ 0 & 7 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ -7 & 0 & 0 & 1 \end{pmatrix}$$

### Graph of a quadratic form

n=1:

$ax^2$

$A=(a)$



Defn:  $f$  quadratic form in  $n$  variables

$A$  the symmetric  $n \times n$ -matrix

Note:  $f(\underline{0}) = f(0, 0, \dots, 0) = 0$

$f$  is positive definite if  $f(\underline{x}) > 0$  for all  $\underline{x} \neq \underline{0}$   
negative definite if  $f(\underline{x}) < 0$  —||—

positive semidefinite if  $f(\underline{x}) \geq 0$  for all  $\underline{x}$   
negative semidefinite if  $f(\underline{x}) \leq 0$  —||—

indefinite if  $f(\underline{x})$  has both pos. and neg. values

Note: If  $f$  is positive definite, then  $\underline{x} = \underline{0}$  is a minimum.

If  $f$  is negative definite, then  $\underline{x} = \underline{0}$  is a maximum.

Ex:  $f(x, y, z) = x^2 + 2y^2 + 3z^2 \geq 0$  for all  $(x, y, z)$   
 $\Rightarrow f$  pos. semidefinite

$$x^2 + 2y^2 + 3z^2 = 0 \iff x = y = z = 0$$

$\Rightarrow f$  positive definite as well.

Ex:  $f(x, y, z) = x^2 - y^2 + z^2$  indefinite

$$\left. \begin{array}{l} f(1, 0, 0) = 1 \\ f(0, 1, 0) = -1 \end{array} \right\} \text{both pos. and neg. values}$$

In general,  $f(x_1, x_2, \dots, x_n) = c_1 x_1^2 + c_2 x_2^2 + \dots + c_n x_n^2$   
 has the following definiteness:

$c_1, c_2, \dots, c_n > 0$  : positive definite  
 $c_1, c_2, \dots, c_n < 0$  : negative definite  
 both pos. and neg.  $c_i$ 's : indefinite

Note:

$$A = \begin{pmatrix} c_1 & 0 & \dots & 0 \\ 0 & c_2 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & c_n \end{pmatrix}$$

$$\left. \begin{array}{l} \lambda_1 = c_1 \\ \lambda_2 = c_2 \\ \vdots \\ \lambda_n = c_n \end{array} \right\} \text{eigenvalues of } A.$$

What about  $f(x, y) = 4xy = (x+y)^2 - (x-y)^2$   
 $= (x^2 + 2xy + y^2) - (x^2 - 2xy + y^2)$

indefinite

Thm:  $f(x_1, \dots, x_n)$  quadratic form  
 A symmetric matrix with eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$

Then:

$f$ positive definite	$\iff$	$\lambda_1, \lambda_2, \dots, \lambda_n > 0$
$f$ positive semidefinite		$\lambda_1, \lambda_2, \dots, \lambda_n \geq 0$
$f$ negative definite		$\lambda_1, \lambda_2, \dots, \lambda_n < 0$
$f$ negative semidefinite		$\lambda_1, \lambda_2, \dots, \lambda_n \leq 0$
$f$ indefinite	$\iff$	both positive and negative eigenvalues

Ex:  $f(x, y, z) = x^2 - 6xz + 4y^2 + 2z^2$

$$A = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 4 & 0 \\ -3 & 0 & 2 \end{pmatrix} \quad \text{symm.}$$

$$\begin{vmatrix} 1-\lambda & 0 & -3 \\ 0 & 4-\lambda & 0 \\ -3 & 0 & 2-\lambda \end{vmatrix} = (4-\lambda) \cdot \begin{vmatrix} 1-\lambda & -3 \\ -3 & 2-\lambda \end{vmatrix}$$

$$= (4-\lambda) \cdot (\lambda^2 - 3\lambda - 7) = 0$$

$$\lambda = 4, \quad \lambda^2 - 3\lambda - 7 = 0$$

$$\lambda = \frac{3 \pm \sqrt{9 + 4 \cdot 7}}{2}$$

$$= \frac{3 \pm \sqrt{37}}{2}$$

$$\lambda_1 = 4$$

pos.

$$\lambda_2 = \frac{3 + \sqrt{37}}{2}$$

pos.

$$\lambda_3 = \frac{3 - \sqrt{37}}{2}$$

neg.

Conclusion:  $f$  is indefinite

$$\begin{aligned} f(x, y, z) &= x^2 - 6xz + 4y^2 + 2z^2 \\ &= (x - 3z)^2 + 4y^2 + 2z^2 - 9z^2 \\ &= (x - 3z)^2 + 4y^2 - 7z^2 \end{aligned}$$

Fact: Any quadratic form  $f(\underline{x})$  can be rewritten as

$$f(x_1, x_2, \dots, x_n) = \lambda_1 \cdot y_1^2 + \lambda_2 \cdot y_2^2 + \dots + \lambda_n \cdot y_n^2$$

Ex:  $f(x) = 2x_1^2 - 4x_1x_3 + 7x_2^2 - 14x_1x_4 + x_4^2$

$$A = \begin{pmatrix} 2 & 0 & -2 & -7 \\ 0 & 7 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ -7 & 0 & 0 & 1 \end{pmatrix}$$

Eigenvalues:

$$\begin{vmatrix} 2-\lambda & 0 & -2 & -7 \\ 0 & 7-\lambda & 0 & 0 \\ -2 & 0 & -\lambda & 0 \\ -7 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(7-\lambda) \cdot \begin{vmatrix} 2-\lambda & -2 & -7 \\ -2 & -\lambda & 0 \\ -7 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\lambda_1 = 7$  ,  $2 \cdot (-2(1-\lambda)) - \lambda \cdot (\lambda^2 - 3\lambda - 47) = 0$

$$-4(1-\lambda) - \lambda(\lambda^2 - 3\lambda - 47) = 0$$

$$-\lambda^3 + \dots + (-4)$$

$$(7-\lambda) \cdot (-\lambda^3 + \dots - 4) = \lambda^4 + \dots - 28$$

$$|A| = -28 = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \lambda_4$$

" " " "

7 " " "

} At least one of  $\lambda_2, \lambda_3, \lambda_4$  is negative

⇓

f is indefinite

# Method using principal minors

A symmetric  $n \times n$ -matrix

The leading principal minor of  $A$  of order  $i$  is a minor of  $A$  of order  $i$  obtained by keeping row  $1, 2, \dots, i$  and col.  $1, 2, \dots, i$ , and delete remaining rows and columns. It is denoted  $D_i$ .

Ex:

$$A = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 2 & 0 \\ -3 & 0 & 7 \end{pmatrix}$$

$$D_1 = 1$$
$$D_2 = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = \underline{2}$$

$$D_3 = \begin{vmatrix} 1 & 0 & -3 \\ 0 & 2 & 0 \\ -3 & 0 & 7 \end{vmatrix} = 2 \cdot (-2) = \underline{-4}$$

$$f(x, y, z) = x^2 - 6xz + 2y^2 + 7z^2$$

is indefinite.

Thm:  $f(x_1, \dots, x_n)$  quadratic form  
A symmetric matrix with leading principal minors  $D_1, D_2, \dots, D_n$ .

Then:

$f$  is pos. definite  $\iff D_1, D_2, \dots, D_n > 0$

$f$  is neg. definite  $\iff$   $D_1 < 0, D_2 > 0, D_3 < 0, \dots$   
 $(-1)^i D_i > 0$  for  $i = 1, 2, \dots, n$

$f$  is indefinite



The leading principle minors fails to follow the patterns above because at least one  $D_i$  has the wrong sign

Ex:  $f(x,y,z) = -x^2 - 2y^2 - 7z^2$

$$A = \begin{pmatrix} \boxed{-1} & 0 & 0 \\ 0 & \boxed{-2} & 0 \\ 0 & 0 & \boxed{-7} \end{pmatrix}$$

$$D_1 = -1$$

$$D_2 = (-1) \cdot (-2) = 2$$

$$D_3 = (-1) \cdot (-2) \cdot (-7) = -14$$

A principle minor of  $A$  of order  $i$  is a minor of order  $i$  obtained by deleting  $n-i$  rows and the same  $n-i$  columns. We write  $\Delta_i$  for any principle minor of order  $i$ .

$$A = \begin{pmatrix} \boxed{1} & 0 & \boxed{-3} \\ 0 & \boxed{2} & 0 \\ -3 & 0 & \boxed{2} \end{pmatrix}$$

$$\Delta_1 = \underline{1}, \quad \underline{2}, \quad \underline{2}$$

$$D: \begin{pmatrix} 2,3 \\ 2,3 \end{pmatrix} \quad \begin{pmatrix} 1,3 \\ 1,3 \end{pmatrix} \quad \begin{pmatrix} 1,2 \\ 1,2 \end{pmatrix}$$

$$\Delta_2 = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = \underline{2}, \quad \begin{vmatrix} 1 & -3 \\ -3 & 2 \end{vmatrix} = \underline{-7}, \quad \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = \underline{4}$$

$$D: \begin{pmatrix} 3 \\ 3 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Delta_3 = \begin{vmatrix} 1 & 0 & -3 \\ 0 & 2 & 0 \\ -3 & 0 & 2 \end{vmatrix} = \underline{-14}$$

Ex:  $f(x, y, z) = x^2 + 2xy + y^2 + z^2$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D_1 = 1$$

$$D_2 = 0$$

$$D_3 = 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

$$D_1 = 1 \quad D_2 = 0 \quad D_3 = 0$$

$$\Delta_1 = 1, 1, 1$$

$$\Delta_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0, \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$\Delta_3 = 0$$

f is positive semidefinite

Thm (continued)

$\Delta_1, \Delta_2, \dots, \Delta_n \geq 0$   
for all principle  
minor

$\iff$  f is positive  
semidefinite

$\Delta_i \leq 0$  for all  
principle minor  
of odd order,

$\Delta_i \geq 0$  for all  
principle minors  
of even degree

$\iff$  f is negative  
semidefinite

f is indefinite  $\iff$  the conditions  
for positive/negative  
definite/semidefinite  
are not satisfied

Ex:  $f(x_1, x_2, x_3, x_4) = 2x_1^2 - 4x_1x_3 + 7x_2^2 - 14x_1x_4 + x_4^2$

$$A = \begin{pmatrix} 2 & 0 & -2 & -7 \\ 0 & 7 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ -7 & 0 & 0 & 1 \end{pmatrix}$$

$f$  is indefinite

$$D_1 = 2$$

$$D_2 = 14$$

$$D_3 = \begin{vmatrix} 2 & 0 & -2 \\ 0 & 7 & 0 \\ -2 & 0 & 0 \end{vmatrix} = -2 \cdot (0 + 14) = -28$$

$$D_4 = -28 \quad \left( = 1 \cdot (-28) + 7 \cdot \begin{vmatrix} \dots & 0 \\ \dots & 0 \\ \dots & 0 \end{vmatrix} \right)$$

Ex:  $f(x, y, z) = x^2 + 2xy + y^2 - z^2$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$D_1 = 1$$

$$D_2 = 0$$

$$D_3 = -1 \cdot 0 = 0$$

$$\Delta_1 = 1, 1, -1$$

$$\Delta_2 = 0, -1, -1$$

$$\Delta_3 = 0$$

not pos. semidefn.  
not neg. semidefn.

$\Downarrow$   
indefinite