

# LECTURE 8-B

EIVIND ERIKSEN

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GKA 6035

MATHEMATICS

PLAN:

- ① Lagrange multipliers
- ② Lagrange problems and second order conditions
- ③ Kuhn-Tucker problems and second order conditions

READING:

[ME] 19.1, 19.4

## ① Lagrange multipliers

Ex:  $\max f(x,y) = x + 3y$  when  $g(x,y) = x^2 + y^2 = 10$

Foc:  $L = f(x,y) - \lambda \cdot g(x,y)$

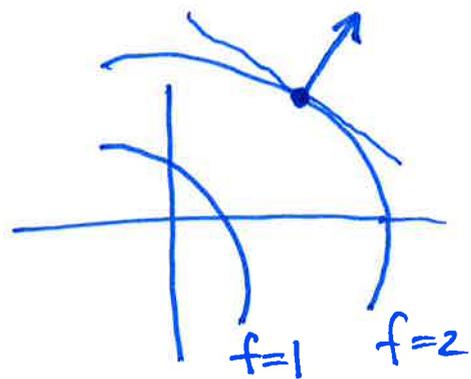
$$\begin{aligned} L'_x = 0 & \quad f'_x - \lambda \cdot g'_x = 0 \\ L'_y = 0 & \quad f'_y - \lambda \cdot g'_y = 0 \end{aligned}$$

$$\begin{pmatrix} f'_x \\ f'_y \end{pmatrix} = \lambda \cdot \begin{pmatrix} g'_x \\ g'_y \end{pmatrix}$$

Defn: The gradient of  $f$  is

$$\nabla f = \begin{pmatrix} f'_x \\ f'_y \end{pmatrix}$$

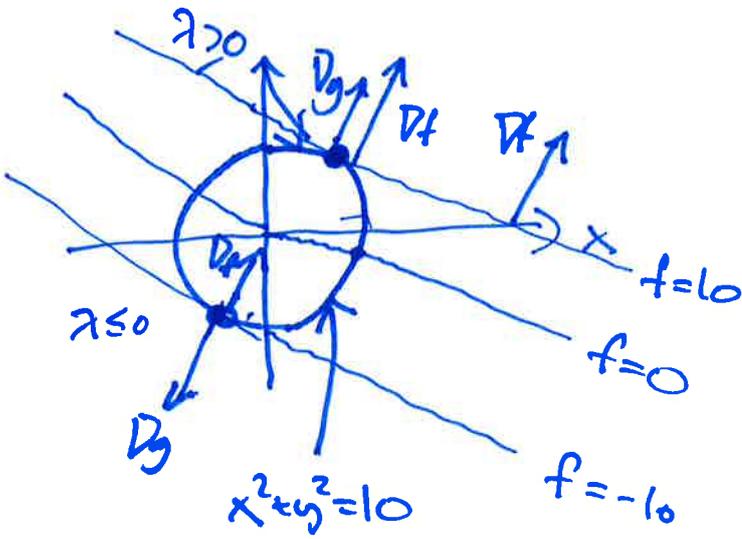
The gradient is the direction with the greatest rate of change.



( $f$  is some function)

Ex:  $f(x,y) = x + 3y$

$g(x,y) = x^2 + y^2 = 10$



$\nabla f = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$  where  $f$  increases most rapidly

$\nabla g = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$  where  $g$  increases most rapidly

Foc:  $\begin{pmatrix} f'_x \\ f'_y \end{pmatrix} = \lambda \cdot \begin{pmatrix} g'_x \\ g'_y \end{pmatrix}$

$\nabla f = \lambda \cdot \nabla g$

$\lambda \geq 0$

$\nabla f$  and  $\nabla g$  point in the same direction

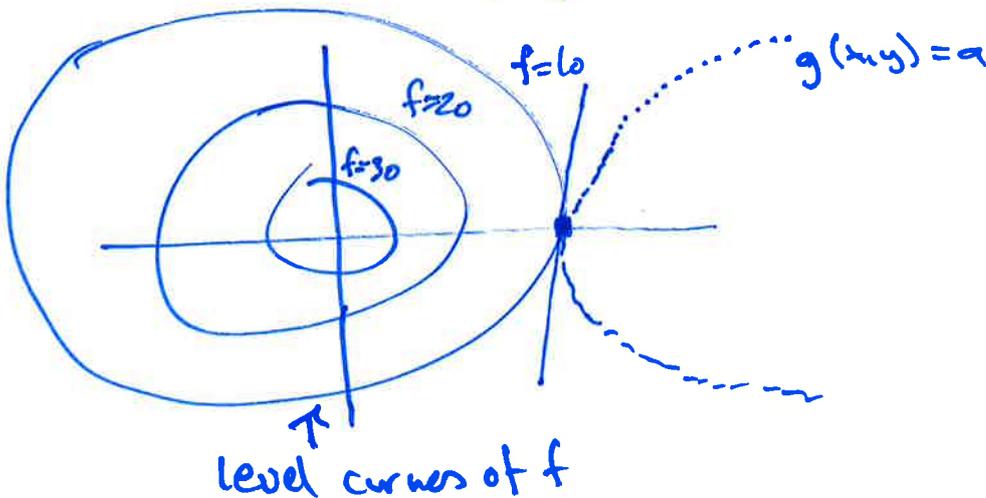
$\lambda \leq 0$

$\nabla f$  and  $\nabla g$  point in opposite directions

Conclusions:

1) FOC  $\Leftrightarrow \nabla f$  and  $\nabla g$  are parallel

The level curves of  $f$  and  $g$  ~~are tangent~~ have the same tangent line



2)  $\lambda \geq 0 \Leftrightarrow \nabla f$  and  $\nabla g$  have the same direction

$\lambda \leq 0 \Leftrightarrow \nabla f$  and  $\nabla g$  pt. in opposite direction

When problem is K.T. in std form,

$\lambda \geq 0$  max  
 $\lambda \leq 0$  min

## Interpretation of $\lambda$ :

Lagrange problem:  $\max/\min f(\underline{x})$  when  $\begin{cases} g_1(\underline{x}) = a_1 \\ g_2(\underline{x}) = a_2 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$

Let  $x_1^*(a), x_2^*(a), \dots, x_n^*(a)$  be the solution, and let  $f^*(a) = f(x_1^*(a), x_2^*(a), \dots, x_n^*(a))$  be the optimal value function.

If  $\lambda_1^*(a), \lambda_2^*(a), \dots, \lambda_m^*(a)$  are the Lagrange multipliers s.t.  $(\underline{x}^*(a), \lambda^*(a))$  solves FOC, then

$$\lambda_i^*(a) = \frac{\partial}{\partial a_i} f^*(a)$$

For example, if  $\lambda_1 = 3$  is part of the solution, then an increase of  $a_1$  with 1 unit will increase the max/min with  $\approx 3$ .

Kuhn-tucker problem: the exact same thing.

Ex:  $\max x+3y$  when  $x^2+y^2=a$ ,  $a>0$

$$L = x+3y - \lambda \cdot (x^2+y^2)$$

$$\begin{aligned} L'_x &= 1 - 2\lambda x = 0 \\ L'_y &= 3 - 2\lambda y = 0 \\ x^2 + y^2 &= a \end{aligned}$$

$$\begin{aligned} x &= 1/2\lambda \cdot \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{3}{2\lambda}\right)^2 = a \\ y &= 3/2\lambda \end{aligned}$$

$$\frac{10}{4\lambda^2} = a$$

$$\lambda^2 = \frac{10}{4a}$$

Max:  $\lambda^*(a) = \sqrt{\frac{10}{4a}}$   $x^*(a) = \sqrt{\frac{a}{10}}$   $y^*(a) = \sqrt{\frac{3a}{10}}$

$$\lambda = \pm \sqrt{\frac{10}{4a}}$$

If  $a > 0$ , then  $x^*(a) = \sqrt{\frac{a}{10}}$   $y^*(a) = \sqrt{\frac{9a}{10}}$   $\lambda^*(a) = \sqrt{\frac{10}{4a}}$

$a=10$ :  $x^*(10) = \underline{1}$ ,  $y^*(10) = \underline{3}$ ,  $\lambda^*(10) = \underline{1/2}$

Interpretation:

$a=10$ :  $x=1$ ,  $y=3$ ,  $\lambda = \underline{1/2}$

$a=11$ :

Exact solution for  $a=11$ :

$$\begin{aligned} f^*(a) &= x + 3y = \sqrt{\frac{a}{10}} + 3 \cdot \sqrt{\frac{9a}{10}} \\ &= 10 \cdot \sqrt{\frac{a}{10}} = \sqrt{10a} \end{aligned}$$

$$f^*(11) = \sqrt{10 \cdot 11} = \underline{\underline{\sqrt{110}}} \approx 10.488$$

change in  $a$   
 $= 11 - 10$

$f=10$  ↓

$$f \approx 10 + 1 \cdot 1/2 = \underline{10.5}$$

↑  
 $\lambda$

## ② Lagrange problems

$$\max/\min f(\underline{x}) \quad \text{whn} \quad \left\{ \begin{array}{l} g_1(\underline{x}) = a_1 \\ g_2(\underline{x}) = a_2 \\ \vdots \\ g_m(\underline{x}) = a_m \end{array} \right.$$

Method:

- ① Write down FOC + C and solve them.  
I get candidates for max/min.

$$h = f(\underline{x}) - \lambda_1 \cdot g_1(\underline{x}) - \dots - \lambda_m \cdot g_m(\underline{x})$$

$$\text{FOC: } \begin{array}{l} h'_{x_1} = 0 \\ h'_{x_2} = 0 \\ \vdots \\ h'_{x_n} = 0 \end{array}$$

$$C: \begin{array}{l} g_1(\underline{x}) = a_1 \\ g_2(\underline{x}) = a_2 \\ \vdots \\ g_m(\underline{x}) = a_m \end{array}$$

- ② Check if there are candidates  $\underline{x}$  where NDCQ fails; i.e. I write down NDCQ and find pts s.t. NDCQ fails + C. I get more candidates for max/min.

NDCQ:

$$\text{rk} \left( \frac{\partial g_i}{\partial x_j} \right) = m$$

NDCQ fails:

$$\text{rk} (-) < m$$

- ③ For each candidate from ① and ②, compute  $f$ .  
We conclude which is the best candidate.

④ Find out if the best candidate is actually max/min.

i) If the set  $D$  of adm. pts is bounded, then there is a max/min by Extreme Value Thm.

↓  
Best candidate is max/min

ii) Convexity/concavity:

Result:

If  $(x_1^*, \dots, x_n^*; \lambda_1^*, \dots, \lambda_m^*)$  satisfies FOC + C, look at the function

$$\underline{x} \mapsto L(\underline{x}; \lambda_1^*, \lambda_2^*, \dots, \lambda_m^*)$$

If this function is convex, the pt.  $(\underline{x}^*, \underline{\lambda}^*)$  is a min.

If this function is concave, the pt.  $(\underline{x}^*, \underline{\lambda}^*)$  is a max.

If you manage to use this criterion successfully, then you don't have to check NDCA.

iii) If i) and ii) does not work, try something else!

Ex:  $\max 2x^2 + y^2 + 3z^2$  s.t.  $\begin{cases} x - y + 2z = 3 \\ x + y = 3 \end{cases}$

① FOC + C:  $L = 2x^2 + y^2 + 3z^2 - \lambda_1(x - y + 2z) - \lambda_2(x + y)$

FOC: 
$$\begin{cases} L'_x = 4x - \lambda_1 - \lambda_2 = 0 \\ L'_y = 2y + \lambda_1 - \lambda_2 = 0 \\ L'_z = 6z - 2\lambda_1 = 0 \end{cases}$$

$x = \frac{\lambda_1 + \lambda_2}{4}$   
 $y = \frac{-\lambda_1 + \lambda_2}{2}$   
 $z = \frac{\lambda_1}{3}$

c: 
$$\begin{cases} 1) x - y + 2z = 3 \\ 2) x + y = 3 \end{cases}$$

1)  $\frac{\lambda_1 + \lambda_2}{4} - \frac{-\lambda_1 + \lambda_2}{2} + 2 \cdot \frac{\lambda_1}{3} = 3 \quad | \cdot 12$

$3(\lambda_1 + \lambda_2) - 6(-\lambda_1 + \lambda_2) + 2 \cdot 4\lambda_1 = 36$

1)  $17\lambda_1 - 3\lambda_2 = 36$

2)  $\frac{\lambda_1 + \lambda_2}{4} + \frac{-\lambda_1 + \lambda_2}{2} = 3 \quad | \cdot 4$

$\lambda_1 + \lambda_2 + 2(-\lambda_1 + \lambda_2) = 12$

2)  $- \lambda_1 + 3\lambda_2 = 12$

$16\lambda_1 = 48 \Rightarrow \lambda_1 = 3 \quad \lambda_2 = 5$

$x = 2 \quad y = 1 \quad z = 1$

One solution:  $x=2, y=1, z=1 \quad \lambda_1=3, \lambda_2=5$

Candidate for max:  $(x, y, z) = (2, 1, 1)$   $\lambda_1 = 3, \lambda_2 = 5$

$$L = 2x^2 + y^2 + 3z^2 - \lambda_1(x - y + 2z) - \lambda_2(x + y)$$

$$L(x, y, z; 3, 5) = 2x^2 + y^2 + 3z^2 - 3(x - y + 2z) - 5(x + y)$$

$$H(L(x, y, z; 3, 5)) = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$D_1 = 4 > 0$$

$$D_2 = 8 > 0$$

$$D_3 = 48 > 0$$

↓

$L(x, y, z; 3, 5)$

is convex

$$(x, y, z) = (2, 1, 1)$$

is min

$$f = 2 \cdot 2^2 + 1^2 + 3 \cdot 1^2 = \underline{12}$$

Conclusion:  $(x, y, z) = (2, 1, 1)$   $f = 12$  is min  
(there is no max)

Alt 1): Is D bounded?

$$x - y + 2z = 3$$

$$x + y = 3$$

↓

$$y = 3 - x$$

$$x - (3 - x) + 2z = 3$$

$$2x + 2z - 3 = 3$$

$$z = \frac{6 - 2x}{2} = \underline{3 - x}$$

That is:  $\begin{cases} y = 3 - x \\ z = 3 - x \\ x \text{ is free} \end{cases}$  constr.

Third Alt: Try to eliminate constr:

$$\begin{aligned} f &= 2x^2 + y^2 + 3z^2 \\ &= 2x^2 + (3-x)^2 + 3 \cdot (3-x)^2 \\ &= 2x^2 + 4(3-x)^2 \\ &= 2x^2 + 4(9 - 6x + x^2) \\ &= \underline{6x^2 - 24x + 36} \end{aligned}$$

$$f' = 12x - 24 = 0$$

$$x = 2$$

min.

$f''(x) = 12 > 0$  NO MAX

D is not bounded.  
(x can be as large as we want)

#### ④ Kuhn - Tucker problems:

$$\max f(x) \quad \text{when} \quad \begin{cases} g_1(x) \leq a_1 \\ g_2(x) \leq a_2 \\ \vdots \\ g_m(x) \leq a_m \end{cases} \quad (\text{std. form})$$

#### Method:

① Write down and solve Kuhn-Tucker conditions:

$$\text{FOC} + C + \text{CSC}$$

Get list of candidates for max.

② Check if there are admissible pts such that NDCQ fails

$$\boxed{\text{NDCQ fails}} + C$$

Get more candidates for max.

③ Compute  $f$  for each pt. from ① and ② and find best candidate for max.

FOC:

$$\boxed{\begin{aligned} L'_{x_1} &= 0 \\ L'_{x_2} &= 0 \\ \vdots \\ L'_{x_n} &= 0 \end{aligned}}$$

C:

$$\boxed{\begin{aligned} g_1(x) &\leq a_1 \\ \vdots \\ g_m(x) &\leq a_m \end{aligned}}$$

CSC:

$$\boxed{\begin{aligned} \lambda_1, \dots, \lambda_m &\geq 0 \\ \text{and} \\ \lambda_1 (g_1(x) - a_1) &= 0 \\ \lambda_2 (g_2(x) - a_2) &= 0 \\ \vdots \\ \lambda_m (g_m(x) - a_m) &= 0 \end{aligned}}$$

NDCQ:

$$\text{rk} \left( \frac{dg_i}{dx_j} \right) = \# \text{ binding constraints}$$

↑  
include only  $g$ 's  
corresponding to  
binding constraints

④ Check if best candidate is max:

i) If  $D =$  set of admissible pts is bounded, then there is a max by Extreme Value Thm.

⇓

Best candidate is max

ii) If  $(x_1^*, x_2^*, \dots, x_n; \lambda_1^*, \lambda_2^*, \dots, \lambda_m^*)$  is a candidate for max such that FOC + C + CSC hold, look at

$$\underline{x} \mapsto L(\underline{x}; \lambda_1^*, \lambda_2^*, \dots, \lambda_m^*)$$

If this is a concave function in  $\underline{x}$ , then  $(\underline{x}^*; \underline{\lambda}^*)$  is max. (No need to check NDCQ if this works)

iii) Try something else.

Ex 1  $\min 2x^2 + y^2 + 3z^2$  s.t.  $\begin{cases} x - y + 2z \geq 3 \\ x + y \geq 3 \end{cases}$

Std. form:  $\max \underbrace{-2x^2 - y^2 - 3z^2}_{f(x,y,z)} \text{ s.t. } \begin{cases} -x + y - 2z \leq -3 \\ -x - y \leq -3 \end{cases}$

① Foc + C + CSC:

$$L = -2x^2 - y^2 - 3z^2 - \lambda_1(-x + y - 2z) - \lambda_2(-x - y)$$

$$= -2x^2 - y^2 - 3z^2 + \lambda_1(x - y + 2z) + \lambda_2(x + y)$$

Foc:

$$\begin{cases} L'_x = -4x + \lambda_1 + \lambda_2 = 0 \\ L'_y = -2y - \lambda_1 + \lambda_2 = 0 \\ L'_z = -6z + 2\lambda_1 = 0 \end{cases}$$

C:

$$\begin{cases} x - y + 2z \geq 3 \\ x + y \geq 3 \end{cases}$$

Cases:

- a)  $\begin{cases} x - y + 2z = 3 \\ x + y = 3 \end{cases}$
- b)  $\begin{cases} x - y + 2z = 3 \\ x + y > 3 \end{cases}$
- c)  $\begin{cases} x - y + 2z > 3 \\ x + y = 3 \end{cases}$
- d)  $\begin{cases} x - y + 2z > 3 \\ x + y > 3 \end{cases}$

CSC:

a)	b)	c)	d)
$\lambda_1 \geq 0$	$\lambda_1 \geq 0$	$\lambda_1 = 0$	$\lambda_1 = 0$
$\lambda_2 \geq 0$	$\lambda_2 = 0$	$\lambda_2 \geq 0$	$\lambda_2 = 0$

$$\begin{aligned}
 \text{a)} \quad & -4x + \lambda_1 + \lambda_2 = 0 \\
 & -2y - \lambda_1 + \lambda_2 = 0 \\
 & -6z + 2\lambda_1 = 0 \\
 & x - y + 2z = 3 \\
 & x + y = 3
 \end{aligned}$$

$$\begin{aligned}
 \lambda_1 &\geq 0 \\
 \lambda_2 &\geq 0
 \end{aligned}$$

$$\boxed{
 \begin{aligned}
 x=2, y=1, z=1 & \text{ ok.} \\
 \lambda_1=3, \lambda_2=5 & \quad f = \underline{-12}
 \end{aligned}
 }$$

Best candidate for max (so far).

ok.

$$\begin{aligned}
 \text{b)} \quad & -4x + \lambda_1 + \lambda_2 = 0 \\
 & -2y - \lambda_1 + \lambda_2 = 0 \\
 & -6z + 2\lambda_1 = 0 \\
 & x - y + 2z = 3 \\
 & x + y > 3
 \end{aligned}$$

$$\begin{aligned}
 \lambda_1 &\geq 0 \\
 \lambda_2 &= 0
 \end{aligned}$$

$$\begin{aligned}
 x &= \lambda_1/4 \\
 y &= ~~\lambda_1/2~~ - \lambda_1/2 \\
 z &= \lambda_1/3
 \end{aligned}$$

$$\lambda_1/4 + \lambda_1/2 + 2\lambda_1/3 = 3$$

$$3\lambda_1 + 6\lambda_1 + 8\lambda_1 = 36$$

$$17\lambda_1 = 36$$

$$\lambda_1 = \underline{36/17}$$

$$x + y = \frac{9}{17} - \frac{18}{17} = \frac{-9}{17} < 3$$

no soln in b)

$$\begin{aligned}
 \text{c)} \quad & -4x_1 + \lambda_1 + \lambda_2 = 0 \\
 & -2y - \lambda_1 + \lambda_2 = 0 \\
 & -6z + 2\lambda_1 = 0
 \end{aligned}$$

$$\begin{aligned}
 x - y + 2z &> 3 \\
 x + y &= 3 \\
 \lambda_1 &= 0 \\
 \lambda_2 &\geq 0
 \end{aligned}$$

$$\begin{aligned}
 x_1 &= \lambda_2/4 \\
 y &= \lambda_2/2 \\
 z &= 0
 \end{aligned}$$

$$\frac{\lambda_2}{4} + \frac{\lambda_2}{2} = 3$$

$$\begin{aligned}
 \lambda_2 + 2\lambda_2 &= 12 \\
 \lambda_2 &= \underline{4}
 \end{aligned}$$

$$x - y + 2z = 1 - 2 + 2 \cdot 0 = -1 < 3$$

no soln in c)

$$\begin{aligned}
 \text{d)} \quad & \text{For} \\
 & x - y + 2z > 3 \\
 & x + y > 3 \\
 & \lambda_1 = 0 \quad \lambda_2 = 0
 \end{aligned}$$

$$\begin{aligned}
 x = y = z &= 0 \\
 \lambda_1 = \lambda_2 &= 0
 \end{aligned}$$

$$x + y = 0 < 3 \quad \text{no soln in d)}$$

Check concavity for  $x=2, y=1, z=1, \lambda_1=3, \lambda_2=5$

$$h = -2x^2 - y^2 - 3z^2 + 3(x-y+2z) + 5(x+y)$$

$$H(h) = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

$$D_1 = -4$$

$$D_2 = 8$$

$$D_3 = -48$$

neg. defn.

for all  $x, y, z$

$\Downarrow$

concave in

$(x, y, z)$

$\Downarrow$

$x=2, y=1, z=1$

is max

Not necessary to check NDCQ in this case.

If you want to check NDCQ:

$$g_1 = -x + y - 2z$$

$$g_2 = -x - y$$

a)  $\text{rk} \begin{pmatrix} -1 & 1 & -2 \\ -1 & -1 & 0 \end{pmatrix} = 2$  ok

b)  $\text{rk} \begin{pmatrix} -1 & 1 & -2 \end{pmatrix} = 1$  ok

c)  $\text{rk} \begin{pmatrix} -1 & -1 & 0 \end{pmatrix} = 1$  ok

d) no condition ok

No pts where NDCQ fails.

Recall: a)  $\frac{1}{2}$  } both binding  $\Rightarrow$  incl. row 1 and 2

b) 1 binding

- 1 1 - 1

c) 2 binding

- 1 1 - 2

d) none binding

$\Rightarrow$  no rows to include (nothing to check)