

LECTURE 9-A

EIVIND ERIKSEN

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GRA 6035

MATHEMATICS

PLAN:

- ① Envelope theorems
- ② Bordered Hessians

Reading: 19.2-19.3,
(19.4-19.6)

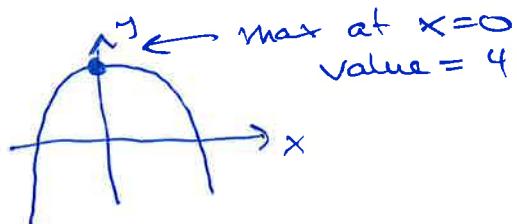
① Envelope theorems

Ex: $f(x; a) = -x^2 + 2ax + 4$

Find max for each fixed value of the parameter a .

$$f'_x(x; a) = -2x + 2a = 0 \\ \underline{x=a}$$

$a=0$: $f = -x^2 + 4$



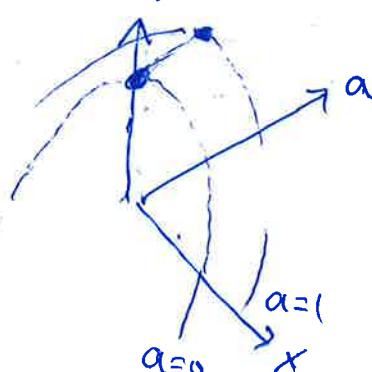
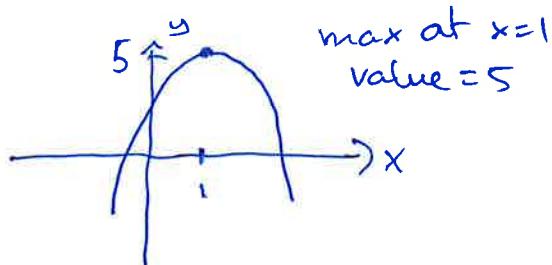
$$f''_{xx}(x; a) = -2 < 0 \\ \text{concave} \\ \Downarrow$$

$x^*(a) = a$ is max
for each a

Maximal value function

$$f^*(a) = f(x^*(a)) \\ = -a^2 + 2a^2 + 4 = \underline{\underline{a^2 + 4}}$$

$a=1$: $f = -x^2 + 2x + 4$



Envelope thm

optimization problem:

(unconstrained case)

$$\max/\min f(\underline{x}; a)$$

$$\underline{x} = x_1, x_2, \dots, x_n$$

a : parameter

Notation:

$$x^*(a) = \text{max/min point}$$

$$f^*(a) = f(x^*(a)) : \text{optimal value function}$$

Envelope thm:



$$\frac{df^*(a)}{da} = \frac{\partial f}{\partial a}(x^*(a); a)$$

this tells us
how $f^*(a)$
changes when
 a is changed

this gives us
a method for
computing $\frac{df^*(a)}{da}$.

Ex I: $f(x; a) = -x^2 + 2ax + 4$

Alt 1:

$$f^*(a) = a^2 + 4$$

||

$$\frac{df^*(a)}{da} = 2a$$

Alt 2: Envelope thm

$$\frac{df^*(a)}{da} = \frac{\partial f}{\partial a}(x^*(a), a)$$

$$= 2x^*(a) = 2a$$

$$x^*(a) : f' = -2x + 2a = 0$$

$$\underline{x} = a \text{ max}$$

$$x^*(a) = a$$

Ex 2: $I(x,y) = 13x + 9y$

$$C(x,y) = 500 + 4x + 2y + 0.04x^2 - 0.01xy + 0.01y^2$$

$$\begin{aligned} \pi(x,y) &= 13x + 9y - (500 + 4x + 2y + 0.04x^2 \\ &\quad - 0.01xy + 0.01y^2) \\ &= 9x + (9-2)y - 500 - 0.04x^2 + 0.01xy \\ &\quad - 0.01y^2 \end{aligned}$$

problem: $\max \pi(x,y; q)$

Step 1: Find $x^*(q), y^*(q)$ that maximizes profit.

Step 2: $\frac{d\pi^*(q)}{dq} = \frac{\partial \pi}{\partial q}(x^*(q), y^*(q); q)$

Envelope thm. $\rightarrow y \mid_{x=x^*(q), y=y^*(q)} = y^*(q)$
 $= \frac{800}{15}q - \frac{700}{15}$

Step 1: $\pi'_x = q - 0.08x + 0.01y = 0 \quad | \cdot 100$

$$\pi'_y = (q-2) + 0.01x - 0.02y = 0 \quad | \cdot 100$$

~~Max~~

$$\begin{aligned} 900 &= 8x + \frac{1}{5}y \quad]^2 \quad \downarrow 8. \\ (q-2)100 &= -x + 2y \quad \downarrow \end{aligned}$$

$$1800 + 100q - 200 = 15x$$

This is $\max \rightarrow x = \frac{1600 + 100q}{15}$

One stationary pt (x,y) for each value of q .

Use Hessian to check that this is max:

$$H(\pi) = \begin{pmatrix} -0.08 & 0.01 \\ 0.01 & -0.02 \end{pmatrix}$$

π is concave

$$D_1 = -0.08 < 0$$

$$\begin{aligned} D_2 &= 0.0016 - 0.0001 \\ &= 0.0015 > 0 \end{aligned}$$

Conclusion:

$$\frac{d\pi^*(q)}{dq} = y^*(q) = \frac{800}{15} \cdot q - \frac{700}{15}$$

$q = 7/8 : \frac{d\pi^*(q)}{dq} = 0$
 $q > 7/8 : \frac{d\pi^*(q)}{dq} > 0 \leftarrow \begin{matrix} \text{maximal profit} \\ \text{increases} \end{matrix} \text{ with inc. } q.$
 $q < 7/8 : \frac{d\pi^*(q)}{dq} < 0 \leftarrow \begin{matrix} \text{maximal profit} \\ \text{decreases} \end{matrix} \text{ with inc. } q. \\ (y^*(q) < 0 \text{ if } q < 7/8)$

Envelope theorem (constrained case)

max/min $f(\underline{x}; a)$ when

$$\begin{cases} g_1(\underline{x}; a) = 0 \\ g_2(\underline{x}; a) = 0 \\ \vdots \\ g_m(\underline{x}; a) = 0 \end{cases}$$

Lagrange problem

Let $\underline{x}^*(a)$ be the solution of the optimization problem.

$g_1(\underline{x}) = a_1$ is rewritten as $g_1(\underline{x}) - a_1 = 0$

Let $f^*(a) = f(\underline{x}^*(a))$ be the optimal value function.

Envelope theorem: (constrained case)

$$\frac{df^*(a)}{da} = \frac{\partial h}{\partial a} (\underline{x}^*(a); \lambda^*(a)) \quad \text{when } \underline{x}^*(a), \lambda^*(a) \text{ solves Foc+C.}$$

(Works in exactly the same way for Kuhn-Tucker problems).

Ex: $\max_{\substack{x \\ y \\ f(x,y) \\ g(x,y; a)}} x+3y$ when $x^2+y^2 \leq a$

$$f(x,y) = x+3y$$

$$g(x,y; a) = \underbrace{x^2+y^2-a}_{\leq 0}$$

$$\begin{aligned} \frac{df^*(a)}{da} &= \frac{\partial L}{\partial a} (x^*(a), y^*(a); \lambda^*(a)) \\ &= \lambda^*(a) = \sqrt{\frac{10}{4a}}, \quad a > 0 \end{aligned}$$

$$L = x+3y - \lambda \cdot (x^2+y^2-a)$$

$$\frac{\partial L}{\partial a} = \lambda$$

FOC: $1 - \lambda \cdot 2x = 0$
 $3 - \lambda \cdot 2y = 0$

~~$L(x,y)$~~ $x = 1/2\lambda$
 $y = 3/2\lambda$

C: $x^2+y^2 \leq a$
 $(\frac{1}{2\lambda})^2 + (\frac{3}{2\lambda})^2 \leq a$

$$\frac{10}{4\lambda^2} \leq a$$

CSC: (1) $x^2+y^2=a$
 $\lambda \geq 0$

$$\frac{10}{4\lambda^2} = a \quad 4\lambda^2 = \frac{10}{a}$$

$$\lambda^2 = \frac{5}{4a}$$

$$\lambda = \sqrt{\frac{10}{4a}}$$

$$x = \sqrt{\frac{a}{10}} \quad y = \frac{3\sqrt{\frac{a}{10}}}{2}$$

(2) $x^2+y^2 \leq a$
 $\lambda = 0$
no solution.

$\lambda^*(a) = \sqrt{\frac{a}{10}}$ $y^*(a) = 3 \cdot \sqrt{\frac{a}{10}}$
 $\lambda^*(a) = \sqrt{\frac{10}{4a}}$

max.

② Bordered Hessians

max/min $f(\underline{x})$ when

$$\begin{cases} g_1(\underline{x}) \leq a_1 \\ g_2(\underline{x}) \leq a_2 \\ \vdots \\ g_m(\underline{x}) \leq a_m \end{cases}$$

Lagrange problem.

Assume that $(\underline{x}^*, \lambda^*)$ is a solution to FOC+C.
 We can find out if it is a local max/min.
 using Bordered Hessian matrices.

$$B = \left(\begin{array}{c|c} 0 & \mathbb{I} \\ \hline \mathbb{I}^T & L'' \end{array} \right)_{m+n}$$

$$J = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} \end{pmatrix}$$

$$L'' = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ L_{m1} & L_{m2} & \cdots & L_{mn} \end{pmatrix}$$

Hessian matrix of h
 with respect to x_1, \dots, x_n .

Ex,

$$\boxed{\text{max } x+3y \text{ s.t. } x^2+y^2=10}$$

$$L = x+3y - \lambda \cdot (x^2+y^2)$$

$$\text{Candidate pt: } (x_1, y_1, \lambda) = (1, 3, 1/2)$$

$$J = (2x \ 2y)$$

$$H(L) = L'' = \begin{pmatrix} -2\lambda & 0 \\ 0 & -2\lambda \end{pmatrix}$$

$$B = \left(\begin{array}{c|cc} 0 & 2x & 2y \\ \hline 2x & -2\lambda & 0 \\ 2y & 0 & -2\lambda \end{array} \right) (1, 3, 1/2) = \begin{pmatrix} 0 & 2 & 6 \\ 2 & -1 & 0 \\ 6 & 0 & -1 \end{pmatrix}$$

Conditions for local max/min:

$(\underline{x}^*, \underline{\lambda}^*)$ candidate point
" " "

$x_1, x_2, \dots, x_n \geq 0, x_2, \dots, x_m$

$n = \# \text{vars}$ $m = \# \text{constraints}$

$B(\underline{x}^*; \underline{\lambda}^*)$ Bordered Hessian matrix at $\underline{x}^*; \underline{\lambda}^*$

$(n+m) \times (n+m)$

Method:

Compute the last $n-m$ leading principal minor

Result: ($n-m=1$ case)

$|B(\underline{x}^*; \underline{\lambda}^*)|$ same sign as $(-1)^n$: local max
 $|B(\underline{x}^*; \underline{\lambda}^*)|$ — + — $(-1)^m$: — + — min

Ex: $\max x+3y$ wth $x^2+y^2=10$
 $x^* = 1 \quad y^* = 3 \quad \lambda^* = 1/\sqrt{2}$

$$B(1, 3; 1/2) = \begin{vmatrix} 0 & 2 & 6 \\ 2 & -1 & 0 \\ 6 & 0 & -1 \end{vmatrix}$$

$$\begin{array}{ll} n=2 & (-1)^2 = +1 \\ m=1 & (-1)^1 = -1 \end{array}$$

$$\begin{aligned} D_3 &= |B| = -2 \cdot (-2) + 6 \cdot 6 \\ &= 40 > 0 \quad \leftarrow \text{same sign as } (-1)^n \\ &\Downarrow \\ \underline{(1, 3) \text{ is local max}} \end{aligned}$$

General case ($n > m$)

Compute $n-m$ last leading principal minors

- If the signs of the last $n-m$ leading principal minors are alternating, with the last one the same sign as $(-1)^n$

\Downarrow
local max

- * If the signs of the last $n-m$ leading principal minors all are the same sign as $(-1)^m$

\Downarrow
local min

What if you have a Kuhn-Tucker problem?

Candidate pt $x^*; \lambda^*$ that satisfy FOC + C + CSC.
Same conditions as in Lagrange problems, but the matrix

J

Should only contain the rows corresponding to binding constraints at $x^*; \lambda^*$.

Ex: max/min $x^2y^2z^2$ when $x^2+y^2+z^2=3$

Lagrange problem: $n=3$

$m=1$

A solution to Foc + c is

$$L = x^2y^2z^2 - \lambda \cdot (x^2 + y^2 + z^2)$$

$$\left\{ \begin{array}{l} x^* \\ x=1 \\ y^* \\ y=1 \\ z^* \\ z=1 \\ \lambda^* \\ \lambda=1 \end{array} \right.$$

$$B = \left(\begin{array}{c|ccc} 0 & 2x & 2y & 2z \\ \hline 2x & 2y^2z^2-2x & 4xy^2z^2 & 4xz^2y^2 \\ 2y & 4xy^2z^2 & 2x^2-2x & 4x^2y^2 \\ 2z & 4xz^2y^2 & 4x^2y^2 & 2y^2-2x \end{array} \right) \quad \begin{array}{l} (-1)^n = -1 \\ (-1)^m = -1 \end{array}$$

$$B(1,1,1;1) = \left(\begin{array}{c|cc|c} 0 & 2 & 2 & 2 \\ \hline 2 & 0 & 4 & 4 \\ 2 & 4 & 0 & 4 \\ \hline 2 & 4 & 4 & 0 \end{array} \right)$$

Compute D_3, D_4 : $\begin{cases} \text{local max: } D_3 > 0, D_4 < 0 \\ \text{local min: } D_3 < 0, D_4 < 0 \end{cases}$

$$\left. \begin{array}{l} D_3 = 32 \\ D_4 = -192 \end{array} \right\} \begin{array}{l} (x,y,z) = (1,1,1) \\ \text{is local max} \end{array}$$