

EXERCISE SESSION

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GKA 6035

MATHEMATICS

Problems

Workbook 5.5, 5.7, 5.11

6.9, 6.21, 6.27

Midterm V-2013, Problem 1-8

$$\underline{5.5} \quad A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad D_1 = a \quad D_1 = c$$

$$D_2 = ac - b^2$$

A pos. definite: $a > 0$, $ac - b^2 > 0$ ($ac > 0$)

A neg. definite: $a < 0$, $ac - b^2 > 0$

A pos. semidefinite: $a \geq 0$, $c \geq 0$, $ac - b^2 \geq 0$

A neg. semidefinite: $a \leq 0$, $c \leq 0$, $ac - b^2 \geq 0$

A indefinite: $ac - b^2 < 0$

These are all possibilities:

~~$ac - b^2 > 0$~~ $ac - b^2 > 0 : ac > b^2 \geq 0 \Rightarrow ac > 0$

$\Rightarrow \underline{a > 0, c > 0} \text{ or } \underline{a < 0, c < 0}$

~~$ac - b^2 = 0$~~ $ac - b^2 = 0 \Rightarrow ac = b^2 \geq 0 \Rightarrow ac \geq 0$

$\Rightarrow \underline{a \geq 0, c \geq 0} \text{ or } \underline{a \leq 0, c \leq 0}$

Indefinite

5.7 Prove that $A^T A$ is pos. defn. for any invertible matrix A .

Proof:

i) Show that $A^T A$ is positive semi-definite

$A^T A$ pos. semi-definite $\Leftrightarrow \underline{x}^T \cdot (A^T A) \cdot \underline{x} \geq 0$
for all vector \underline{x}

$$\underbrace{\underline{x}^T A^T A \underline{x}}_{(\underline{Ax})^T} = (\underline{Ax})^T \cdot (\underline{Ax}) = \underline{y}^T \underline{y}$$

write $y = Ax$

$$= (y_1, y_2, \dots, y_n) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}^T = y_1^2 + y_2^2 + \dots + y_n^2 \geq 0$$

So $\underline{x}^T A^T A \underline{x} \geq 0$ for all $\underline{x} \Leftrightarrow A^T A$ is pos. semi-definite

ii) Show that $A^T A$ is positive defn. when A is inv.

$A^T A$ is positive semi-definite $\Rightarrow \underbrace{\lambda_1, \lambda_2, \dots, \lambda_n}_{\text{eigenvalues of } A^T A} \geq 0$

$$|A| = 0 \Leftrightarrow |A|^2 = |A^T| \cdot |A| = |A^T A|$$

eigenvalues
of $A^T A$.

If $\lambda_i = 0$, then $\lambda_1 \lambda_2 \dots \lambda_n = 0$ and A is not invertible. This cannot happen since A is invertible by assumption. So $\lambda_1, \lambda_2, \dots, \lambda_n > 0$, and A is pos. definite.

$$5.11. \quad f(x_1, x_2, x_3) = \underline{x_1^2} + 6x_1x_2 + 3x_2^2 + \underline{2x_3^2}$$

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$H(f) = \begin{pmatrix} 2 & 6 & 0 \\ 6 & 6 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Symm. matrix of f

$$D_1 = 1$$

$$D_2 = 3 - 9 = \textcircled{-6}$$

$$D_3 = -12 \quad \downarrow$$

f indefinite
quadr. form.

Alt. 3

$$D_1 = 2$$

$$D_2 = \textcircled{-24}$$

$$D_3 = -96 \quad \downarrow$$

f indefinite
quadr. form / fn.

Alt. 3

$$6.9. \quad f(x, y) = -6x^2 + (2a+4)xy - y^2 + 4ay$$

(a is parameter)

For which values of a is f convex / concave?

$$f'_x = -12x + (2a+4)y$$

$$f'_y = (2a+4)x - 2y + 4a$$

$$\left. \begin{array}{l} f''_{xx} = -12 \\ f''_{xy} = 2a+4 \\ f''_{yy} = -2 \end{array} \right\}$$

$$H(f) = \begin{pmatrix} -12 & 2a+4 \\ 2a+4 & -2 \end{pmatrix}$$

$$H(f) = \begin{pmatrix} -12 & 2a+4 \\ 2a+4 & -2 \end{pmatrix}$$

f convex: $H(f)(x,y)$ pos. semidef. for all (x,y)
 f concave: $H(f)(x,y)$ neg. ——— ——— —

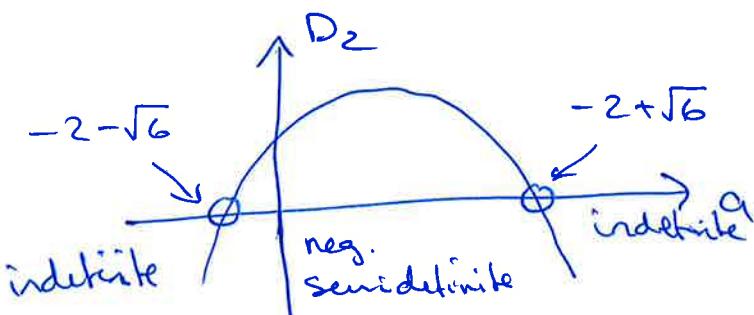
$$D_1 = -12 \quad \Delta_1 = -2$$

$$D_2 = 24 - (2a+4)^2$$

$$= 24 - (4a^2 + 16a + 16)$$

$$= -4a^2 - 16a + 8$$

$$\left. \begin{array}{l} D_2 \geq 0 : f \text{ concave} \\ D_2 < 0 : f \text{ not convex} \\ \text{not concave} \end{array} \right\}$$



$D_2 \geq 0$ when

$$-2 - \sqrt{6} \leq a \leq -2 + \sqrt{6}$$

Conclusion:

If $a \in [-2 - \sqrt{6}, -2 + \sqrt{6}]$
 then f is concave
 otherwise, f is
 neither convex nor
 concave

f convex: never

f concave: $-2 - \sqrt{6} \leq a \leq -2 + \sqrt{6}$

6.21

$$f(x,y,z) = x^3 + 3xy + 3xz + y^3 + 3yz + z^3$$

Find all local extremal pts (=local max/min) and classify their type.

$$f'_x = 3x^2 + 3y + 3z = 0$$

$$f'_y = 3x + 3y^2 + 3z = 0$$

$$f'_z = 3x + 3y + 3z^2 = 0$$

$$H(f) = \begin{pmatrix} 6x & 3 & 3 \\ 3 & 6y & 3 \\ 3 & 3 & 6z \end{pmatrix}$$

Stationary pts:

$$3x^2 + 3y + 3z = 0 \Rightarrow \boxed{\begin{array}{l} x^2 + y + z = 0 \\ x + y^2 + z = 0 \\ x + y + z^2 = 0 \end{array}}$$

$$z = -x^2 - y \Rightarrow x + y^2 + (-x^2 - y) = 0$$

$$x - y = x^2 - y^2 \quad \leftarrow \quad \boxed{x - x^2 = y - y^2}$$

$$\begin{aligned} (x-y) \cdot (1 - \frac{(x+y)}{(x+y)}) &= 0 \\ \underline{x-y=0} \quad \text{or} \quad \underline{1=x+y} \end{aligned} \quad \left\{ \begin{array}{l} x + y + (-x^2 - y)^2 = 0 \\ \boxed{x + y + x^4 - 2x^2y + y^2 = 0} \end{array} \right.$$

a) $x=y$: $x + x + (-x^2 - x)^2 = 0 \quad \underline{x=0} \quad \text{or} \quad x^3 + 2x^2 + x + 2 = 0$

$$2x + x^4 + 2x^3 + x^2 = 0 \quad x^2(x+2) + 1 \cdot (x+2) = 0$$
$$x^4 + 2x^3 + x^2 + 2x = 0 \quad (x^2 + 1)(x+2) = 0$$
$$x(x^3 + 2x^2 + x + 2) = 0 \quad x^2 + 1 = 0 \quad \text{or} \quad \underline{x=-2}$$

no soln.

$$\text{From a): } y = x, z = -x^2 - y$$

$$\begin{array}{lll} x=0 & y=0 & z=0 \\ x=-2 & y=-2 & z=-2 \end{array} \rightarrow \frac{(0,0,0)}{(-2,-2,-2)}$$

$$b) x+y=1 \Rightarrow y=1-x$$

$$\begin{aligned} x+y+(-x^2-y)^2 &= 0 & (-x^2-y)^2 &= -1 \\ 1+(x^2+1-x)^2 &= 0 & \text{impossible.} \end{aligned}$$

Concl: Stationary pts: $\begin{cases} (0,0,0) \\ (-2,-2,-2) \end{cases}$

$$H(t) = \begin{pmatrix} 6x & 3 & 3 \\ 3 & 6y & 3 \\ 3 & 3 & 6z \end{pmatrix}$$

$$\underline{(x,y,z)=(0,0,0)}: H(t)(0,0,0) = \begin{pmatrix} 0 & 3 & 3 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{pmatrix}$$

$$\begin{array}{l} D_1 = 0 \\ D_2 = -9 \end{array} \rightarrow \text{indefinite} \rightarrow \underline{\begin{array}{l} (0,0,0) \\ \text{saddle pt.} \end{array}}$$

$$\underline{(x,y,z)=(-2,-2,-2)}: H(t)(-2,-2,-2) = \begin{pmatrix} -12 & 3 & 3 \\ 3 & -12 & 3 \\ 3 & 3 & -12 \end{pmatrix}$$

neg. defn.

||

$(-2,-2,-2)$

is local max

$$\left\{ \begin{array}{l} D_1 = -12 < 0 \\ D_2 = 144 - 9 = 135 > 0 \\ D_3 = 3 \cdot (9 + 36) - 3 \cdot (-36 - 9) + (-12) \cdot D_2 \\ = 135 + 135 - 12 \cdot 135 = -10 \cdot 135 \\ = -1350 < 0 \end{array} \right.$$

$$6.27. \quad f(x,y,z,w) = x^5 + xy^2 - zw$$

a) Find all stationary pts.

$$f'_x = 5x^4 + y^2 = 0$$

$$f'_y = 2xy = 0 \implies x=0 \text{ or } y=0$$

$$f'_z = -w = 0 \implies w=0$$

$$f'_w = -z = 0 \implies z=0$$

$$\underline{x=0}: \quad 5x^4 + y^2 = 0 \\ y^2 = 0 \implies y=0$$

$$\underline{y=0}: \quad 5x^4 + y^2 = 0 \\ 5x^4 = 0 \implies x=0$$

||

Conclusion: $(0,0,0,0)$ only stat. point

$$b) H(f) = \begin{pmatrix} 20x^3 & 2y & 0 & 0 \\ 2y & 2x & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$H(f)(0,0,0,0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad D_1 = 0 \\ D_2 = 0 \\ D_3 = 0 \\ D_4 = 0$$

$$\Delta_1 = 0, 0, 0, 0 \\ \Delta_2 = 0, \dots, -1$$

indeterminate

$(0,0,0,0)$ is saddle pt.

Midterm exam: V-2013, Pr. 1-7

Not relevant
this time:
problem 8

- * bordered Hessians
- * Convex sets, bounded sets,
open/closed sets

1. Two free var. \Rightarrow Aut. C =

$$\underline{2.} \quad \underline{v}_1 = \begin{pmatrix} 0 \\ 2 \\ -3 \\ 1 \end{pmatrix} \quad \underline{v}_2 = \begin{pmatrix} -2 \\ 8 \\ -5 \\ -5 \end{pmatrix} \quad \underline{v}_3 = \begin{pmatrix} 1 \\ -1 \\ -2 \\ 4 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & -2 & 1 \\ 2 & 8 & -1 \\ -3 & -5 & -2 \\ 1 & -5 & 4 \end{pmatrix} \rightarrow \left(\begin{array}{ccc|c} 1 & -5 & 4 & 1 \\ 2 & 8 & -1 & 2 \\ -3 & -5 & -2 & 3 \\ 0 & -2 & 1 & 0 \end{array} \right)$$

$$\rightarrow \begin{pmatrix} 1 & -5 & 4 \\ 0 & 18 & -9 \\ 0 & -20 & 10 \\ 0 & -2 & 1 \end{pmatrix} \quad c_1 \cdot \underline{v}_1 + c_2 \cdot \underline{v}_2 + c_3 \cdot \underline{v}_3 = 0$$

two pivot positions,
in ~~the~~ col's 1, 2.

$\underline{v}_1, \underline{v}_2$ are lin. indep.
 \underline{v}_3 is a lin. comb.
of \underline{v}_1 and \underline{v}_2 .

AH. BS

$$\underline{c}_1 - 5\underline{c}_2 + 4\underline{c}_3 = 0$$

$$18\underline{c}_2 - 9\underline{c}_3 = 0$$

\uparrow
 c_3 free

$$\underline{c}_3 = 1: \quad c_2 = 1/2$$

$$c_1 = -3/2$$

$$-3/2 \underline{v}_1 + 1/2 \underline{v}_2 + 1 \cdot \underline{v}_3 = 0$$

$$\underline{v}_3 = \frac{3}{2} \cdot \underline{v}_1 - \frac{1}{2} \underline{v}_2.$$

$$3.) A = \begin{pmatrix} 0 & 2 & -3 & h & 4 \\ -2 & 8 & -5 & -5 & -2 \\ 1 & -1 & -2 & 4 & 7 \end{pmatrix} \xrightarrow{\quad}$$

$$\left(\begin{array}{ccccc} 1 & -1 & -2 & 4 & 7 \\ -2 & 8 & -5 & -5 & -2 \\ 0 & 2 & -3 & h & 4 \end{array} \right) \xrightarrow{\quad} \left[\begin{array}{ccccc} 1 & -1 & -2 & 4 & 7 \\ 0 & 6 & -9 & 3 & 12 \\ 0 & 2 & -3 & h & 4 \end{array} \right]_2$$

$$\left(\begin{array}{ccccc} 1 & -1 & -2 & 4 & 7 \\ 0 & 6 & -9 & 3 & 12 \\ 0 & 2 & -3 & h & 4 \end{array} \right) \xrightarrow{\quad} \left[\begin{array}{ccccc} 1 & -1 & -2 & 4 & 7 \\ 0 & 6 & -9 & 3 & 12 \\ 0 & 0 & 3 & h-1 & 0 \end{array} \right]_{-1/3}$$

$$\left(\begin{array}{ccccc} 1 & -1 & -2 & * & * \\ 0 & 6 & -9 & -* & * \\ 0 & 0 & 3 & (h-1) & 0 \end{array} \right)$$

$$h=1: \text{rk } A = 2$$

$$h \neq 1: \text{rk } A = 3$$

Auf B

$$4 \quad A = \begin{pmatrix} -1 & 3 \\ 4 & 0 \end{pmatrix}$$

$$\lambda^2 + \lambda - 12 = 0$$

$$\lambda = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-12)}}{2} = \frac{-1 \pm \sqrt{49}}{2} = \frac{-1 \pm 7}{2}$$

$$\lambda = -4, \quad \underline{\lambda = 3}$$

Auf A

$$5) \quad A = \begin{pmatrix} -1 & 3 \\ 4 & 0 \end{pmatrix} \quad \underline{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \underline{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A \cdot \underline{u} = \begin{pmatrix} -1 & 3 \\ 4 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \end{pmatrix} = \cancel{\underline{u}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\cancel{-4}$ ok

$$A \cdot \underline{v} = \begin{pmatrix} -1 & 3 \\ 4 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \neq \cancel{\underline{v}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Alt C

$$6. \quad Q(x_1, x_2) = h x_1^2 - 4 x_1 x_2 + 3 x_2^2$$

$$A = \begin{pmatrix} h & -2 \\ -2 & 3 \end{pmatrix} \quad D_1 = h \quad D_2 = 3h - 4$$

If $h < 4/3$ then $D_2 < 0 \Rightarrow$ indefinite

If $h \geq 4/3$ then $D_2 \geq 0 \quad \left. \begin{array}{l} D_1 > 0 \quad \Delta_1 = 3 > 0 \\ \end{array} \right\} \begin{array}{l} \text{pos.} \\ \text{semidef.} \end{array}$

Alt . D

7. $f(x,y) = x^4 + x^2 - 2xy + hy^2$

$$f'_x = 4x^3 + 2x - 2y$$

$$f'_y = -2x + 2hy$$

$$H(f) = \begin{pmatrix} 12x^2 + 2 & -2 \\ -2 & 2h \end{pmatrix}$$

$$D_1 = 12x^2 + 2 \geq 0$$

$$\Delta_1 = 2h$$

$$D_2 = (12x^2 + 2) \cdot 2h - 4$$

$$= 24hx^2 + 4h - 4$$

$$= 24h \cdot x^2 + (4h - 4)$$

If $h > 1$, then $24x^2 \cdot h + (4h - 4) > 0$ for all x, y

$\Rightarrow D_2 > 0$ for all x, y

$\Rightarrow f$ convex

If $h = 1$, then $D_2 = 24x^2 \geq 0$ and $\Delta_1 = 2h = 2$

$\Rightarrow f$ convex

If $h < 1$, then $D_2 < 0$ for $x = 0$

\Rightarrow not convex
not concave

Conclusion:

$\begin{cases} f \text{ convex if } h \geq 1 \\ f \text{ not convex if } h < 1 \\ \text{not concave} \end{cases}$

Alt C