LECTURE 1

GRA 6035
MATHEMATICS

Eivind Eriksen

AUG 21, 2014

Plan:

1. Overview of GRA 6035
2. Linear systems
3. Gaussian elimination
4. Rank of a matrix

Reading:

[ME] 6.1, (6.2), 7.1-7.4, (7.5)

[LSGE] 1-3

1 Overview

See Syllabus for short summary of the course.

② Linear systems:

Ex:
$$\begin{cases} x+y+z=3 \\ x+2y+4z=7 \\ x+3y+9z=13 \end{cases}$$

- 3 equations in 3 variables
- linear equations

A linear equation in $x_1, x_2, \dots, x_n$ has the form

$$a_1 \cdot x_1 + a_2 \cdot x_2 + \dots + a_n \cdot x_n = b$$

where $a_1, a_2, \dots, a_n$ and b are given numbers.

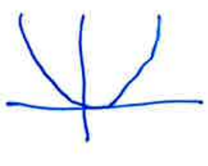
Ex: not linear

$x^2+y^2=1$ $\quad e^x = y-2$

$xy=1$

Fact: The graph of a linear equation is a straight line $(n=2)$, a plane $(n=3)$, hyperplane $(n>3)$.

Ex: $y=x^2$ not linear



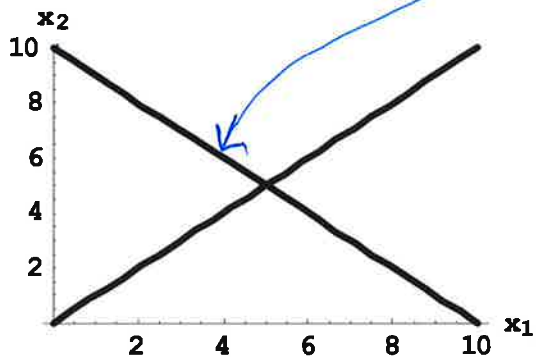
not a straight line

$(x_1+x_2=10)$ $(x_2=10-x_1)$

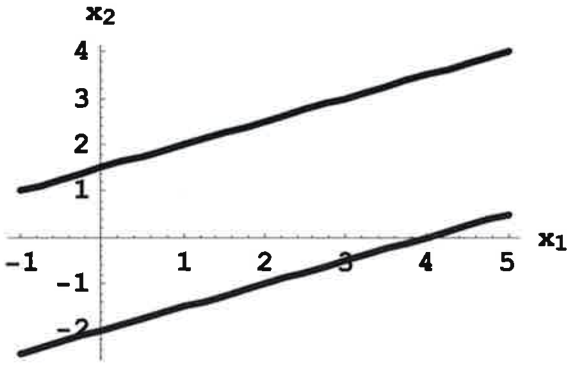
$x+y=10 \Rightarrow y=10-x$

EXAMPLE Two equations in two variables:

$$\begin{array}{rcrcr} x_1 & + & x_2 & = & 10 \\ -x_1 & + & x_2 & = & 0 \end{array} \qquad \begin{array}{rcrcr} x_1 & - & 2x_2 & = & -3 \\ 2x_1 & - & 4x_2 & = & 8 \end{array}$$

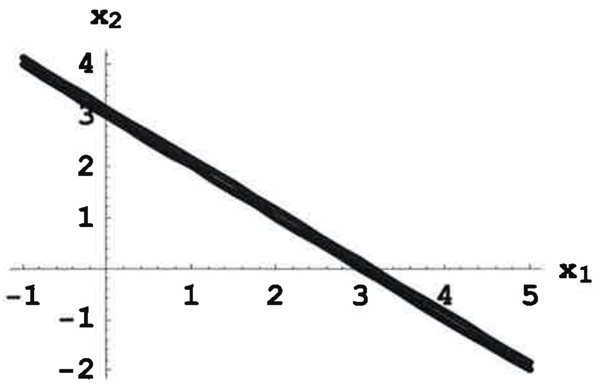


one unique solution



no solution

$$\begin{array}{rcrcr} x_1 & + & x_2 & = & 3 \\ -2x_1 & - & 2x_2 & = & -6 \end{array}$$



infinitely many solutions

BASIC FACT: A system of linear equations has either

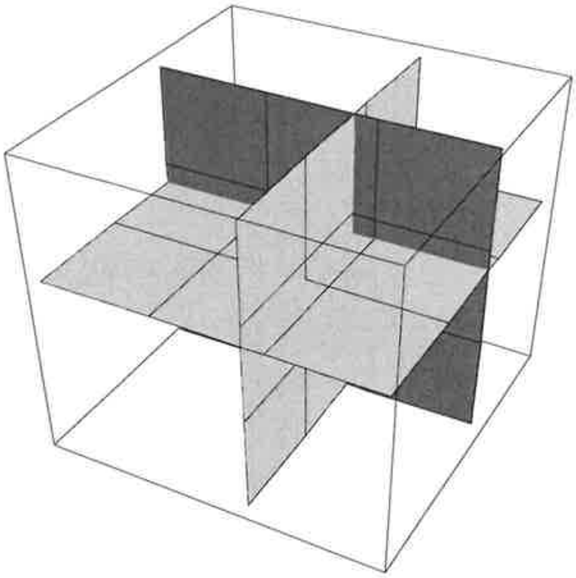
(i) exactly one solution *(consistent)* or

(ii) infinitely many solutions *(consistent)* or

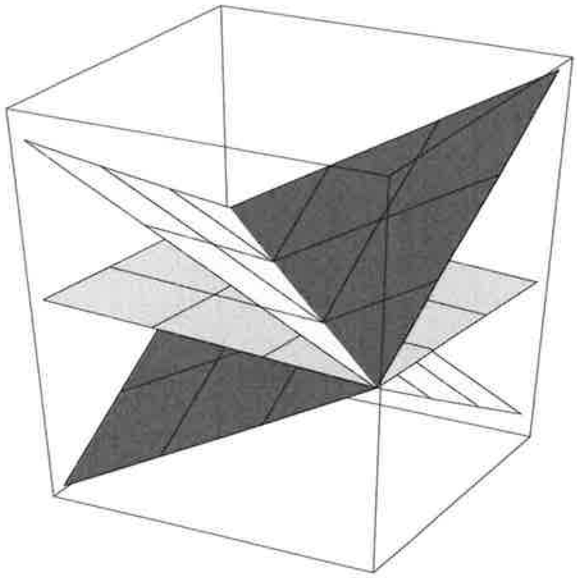
(iii) no solution *(inconsistent)*.

EXAMPLE: Three equations in three variables. Each equation determines a plane in 3-space.

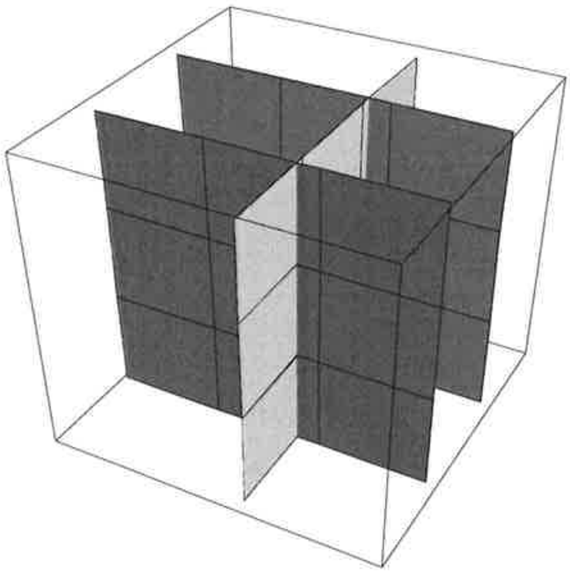
i) The planes intersect in one point. *(one solution)*



ii) The planes intersect in one line. *(infinitely many solutions)*



iii) There is not point in common to all three planes. *(no solution)*



Definition:

A <u>linear system</u> is a collection of m linear equations in n variables $(m \times n)$:

$$\text{m eqn's}\left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad\vdots \qquad\qquad\qquad\qquad\qquad\quad \vdots \\ \underbrace{a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n}_{n \text{ var's}} = b_m \end{array}\right.$$

Ex:

1) $$\begin{aligned} x+y &= 4 \\ x-y &= 2 \end{aligned}$$

$x=3, y=1$

2) $$\begin{aligned} x+y+z &= 3 \\ x+2y+4z &= 7 \\ x+3y+9z &= 13 \end{aligned}$$

$x=1, y=1, z=1$

A solution of a linear system is an n-tuple $(s_1, s_2, \dots, s_n)$ s.t. $x_1 = s_1, x_2 = s_2, \dots, x_n = s_n$ satisfies all m equations.

Result A:

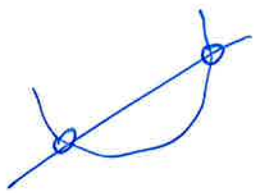
Any linear system has either

i) no solutions (inconsistent)

ii) one unique solution (consistent)

iii) infinitely many solutions (consistent, degrees of freedom)

Two solutions



this cannot happen in linear systems

Result (A):

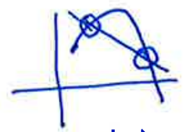
Any $m\times n$ linear system has either

i) one unique solution (consistent)

ii) no solutions (inconsistent)

iii) infinitely many solutions (consistent)

Non-linear case:



two solutions possible

Proof of A:

If there are two different solutions

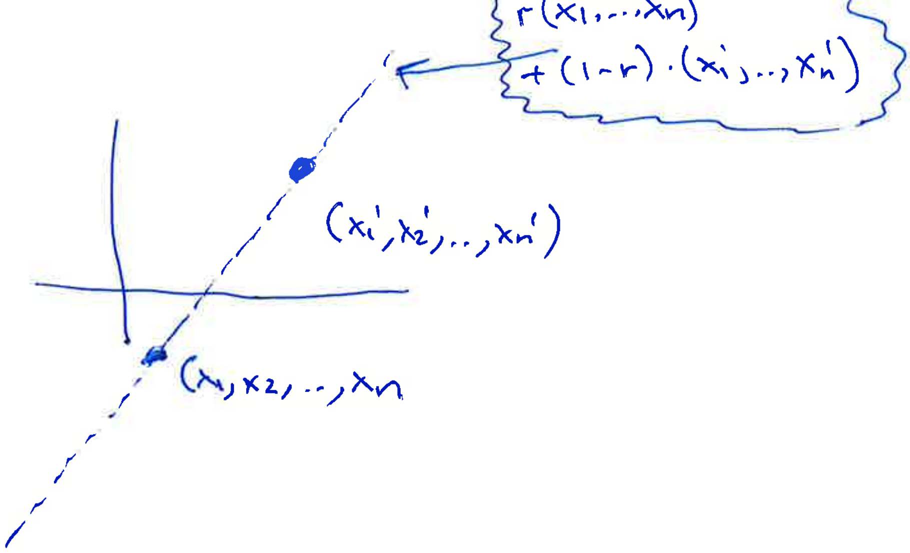
$$(x_1, x_2, \dots, x_n) \neq (x_1', x_2', \dots, x_n')$$

then

$$r(x_1, x_2, \dots, x_n) + (1-r)\cdot(x_1', x_2', \dots, x_n')$$
$$= (rx_1 + (1-r)x_1',\ rx_2 + (1-r)x_2',\ \dots,\ rx_n + (1-r)x_n')$$

is a solution for any number r. There are therefore infinitely many solutions

line through the two solutions.



This can be checked directly. Equation #i is satisfied since

$$a_{i1}\cdot(rx_1 + (1-r)x_1') + a_{i2}(rx_2 + (1-r)x_2') + \dots + a_{in}\cdot(rx_n + (1-r)x_n')$$
$$= r\cdot(a_{i1}x_1 + \dots + a_{in}x_n) + (1-r)\cdot(a_{i1}x_1' + \dots + a_{in}x_n')$$
$$= r\cdot b_i + (1-r)b_i = b_i$$

since $(x_1,\dots,x_n)$ and $(x_1',\dots,x_n')$ are solutions

So this equation holds for all i. Hence we have solution for all r. $\square$

How to solve linear systems:

Ex:

$$x + y = 4$$
$$x - y = 2$$

a) Substitution methods:

$$x + y = 4 \rightarrow \underline{y = 4 - x}$$
$$x - y = 2$$
$$x - (4 - x) = 2$$
$$2x - 4 = 2$$
$$\underline{x = 3}$$
$$\underline{y = 1}$$

b) Elimination methods:

$$\begin{array}{r} x + y = 4 \\ x - y = 2 \\ \hline 2x \quad = 6 \end{array}$$

$$\underline{x = 3} \rightarrow \underline{y = 1}$$

Gaussian elimination:

An elimination method that is
- efficient
- educational
- versatile

③ Gaussian elimination – method for solving linear systems

<u>Ex:</u>

$$\begin{aligned} x + y + z &= 3 \\ x + 2y + 4z &= 7 \\ x + 3y + 9z &= 13 \end{aligned} \quad \rightsquigarrow \quad \underbrace{\left(\begin{array}{ccc|c} \underbrace{\begin{matrix}1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9\end{matrix}}_{\text{coefficient matrix}} & \begin{matrix}3 \\ 7 \\ 13\end{matrix} \end{array}\right)}_{\text{augmented matrix}}$$

<u>Defn:</u> Two linear systems are <u>equivalent</u> if they have the same solutions.

<u>Elementary row operations</u>

i) Interchange two rows
ii) Multiply a row with $c \neq 0$
iii) Add a multiple of one row to another row

} do not change the solution

add (-1) · row #1 to row #2

<u>Ex:</u>

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{array}\right) \xrightarrow{-1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 1 & 3 & 9 & 13 \end{array}\right) \xrightarrow{-1}$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{array}\right) \xrightarrow{-2} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

<u>leading coefficient</u> = first non-zero entry in a row

<u>Defn:</u>

An <u>echelon form</u> is a matrix s.t.

i) All rows with only zeros are in the bottom of the matrix.

ii) Each leading coefficients is further to the right than the leading coeff. above.

When the augmented matrix is reduced to an echelon form (using elem. row operations) you see what the solutions are:

$$\begin{aligned} x + y + z &= 3 \\ y + 3z &= 4 \\ 2z &= 2 \end{aligned}$$

$\underline{x = 1}$

$\underline{y = 1}$

$\underline{z = 1}$

back substitution

<u>pivot</u>: leading coeff. in the echelon form.

<u>pivot positions</u>: position of pivots

<u>Gaussian elimination:</u>

linear system → augm. matrix $\xrightarrow[\text{operations}]{\text{elem. row}}$ echelon form $\xrightarrow{\text{back substitution}}$ Solutions

Cases with no solution / infinitely many sol's:

Ex:
$$\begin{aligned} x+y+z&=1 \\ x+2y+3z&=2 \\ 2x+3y+4z&=4 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 3 & 4 & 4 \end{array}\right) \begin{array}{l} -1 \\ -2 \end{array} \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 2 \end{array}\right) -1$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

echelon form

$0x+0y+0z=1$

no solutions

In general: no solutions $\Updownarrow$ pivot position in the last col.

Ex:
$$\begin{aligned} x+y+z&=3 \\ x+2y+4z&=7 \end{aligned}$$

no pivot in z-col.
$\Rightarrow$ z is a free var.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \end{array}\right) -1 \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \end{array}\right)$$

echelon form

$x+y+z=3$
$y+3z=4$

$y = 4-3z$

$x+(4-3z)+z=3 \Rightarrow x = 2z-1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \end{array}\right)$$

x, y: basic variables (pivot col's)

z: free variables (non-pivot col's)

degrees of freedom = # free var's (number of free var's)

$= 1$

It means that the solution set is one-dim.

Gauss-Jordan elimination:

$$\cdots \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \end{array}\right) \begin{matrix} \leftarrow \\ -1 \end{matrix} \to \left(\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 3 & 4 \end{array}\right)$$

echelon form

reduced echelon form
= echelon form with:
- all pivots = 1
- only zeros over each pivot.

$$x \quad -2z = -1$$
$$y + 3z = 4$$

no back substitution, see sol's directly

$$\boxed{\begin{aligned} x &= 2z - 1 \\ y &= -3z + 4 \\ z &= \text{free} \end{aligned}}$$

Ex:

$$\left(\begin{array}{ccc|c} 0 & 1 & 2 & 4 \\ 1 & 0 & 7 & 3 \\ \cdot & \cdot & \cdot & \cdot \end{array}\right) \updownarrow \to \left(\begin{array}{ccc|c} 1 & 0 & 7 & 3 \\ 0 & 1 & 2 & 4 \\ - & - & - & - \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 3 & 7 & 14 & -1 \\ 0 & 0 & -1 & 3 & 4 \\ 0 & 0 & 7 & 1 & -1 \end{array}\right)$$

Result B:

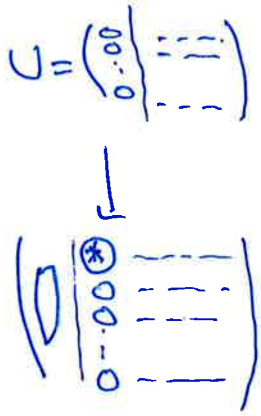
Any matrix can be reduced to echelon form using elementary row operations.

An echelon form is not unique.

The ~~pivots (and pivot~~. pivot positions and the reduced echelon form is unique.

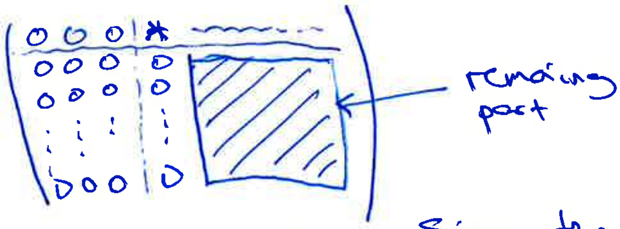
Proof of Result B and more details of Gauss/Gauss-Jordan elimination:

i) Any matrix can be reduced to an echelon form using Elementary row operations.



Start with any matrix U. Move to the right of any columns with only zeros, if any. Look at the first non-zero column, switch two rows in necessary to get a non-zero entry in the top corner. This is a pivot. Use it to get zeros under it.

Now look away from the first row, and look at the remaining part of the matrix.



Repeat the steps above. Since the new matrix is smaller than the original (one row less), we sooner or later get an echelon form this way.

Multiply each row with $\frac{1}{\text{pivot}}$ to set pivot $= 1$. Use each pivot to set zeros over it, starting from the rightmost. You set a reduced echelon form.

ii) If a matrix A can be reduced to reduced echelon forms U, V using elementary row operations, $U = V$.

We have $(A \,|\, 0) \dashrightarrow (U \,|\, 0)$ and $(A \,|\, 0) \dashrightarrow (V \,|\, 0)$, and elementary row operations do not change solutions of linear systems. So $U \cdot \underline{x} = \underline{0}$ and $V \cdot \underline{x} = \underline{0}$ have the same solutions.
Write $U = (c_1 | c_2 | \dots | c_n)$ and $V = (c_1' | c_2' | \dots | c_n')$ in terms of their columns. We have

$$c_i = x_1 c_1 + x_2 c_2 + \dots + x_{i-1} c_{i-1} \Leftrightarrow c_i \text{ non-pivot column in } U$$

$$\Updownarrow$$

$$c_i' = x_1 c_1' + x_2 c_2' + \dots + x_{i-1} c_{i-1}' \Leftrightarrow c_i' \text{ non-pivot column in } V$$

So U and V have the same pivot columns; They are in the positions, and the pivot columns are

$$\begin{array}{c}1\\ \\ \\ \\ \\ \end{array}\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix},\ \begin{array}{c} \\2\\ \\ \\ \end{array}\begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix},\ \begin{array}{c} \\ \\3\\ \\ \end{array}\begin{pmatrix}0\\0\\1\\\vdots\\0\end{pmatrix},\ \dots\ \begin{array}{c} \\ \\ \\ \\r\\ \\ \\ \end{array}\begin{pmatrix}0\\0\\0\\\vdots\\1\\\vdots\\0\end{pmatrix}$$

To show that $U=V$, we must show that non-pivot columns are equal. But each non-pivot column satisfy

$$\begin{cases} c_i = x_1 c_1 + \dots + \\ c_i' = x_1 c_1' + \dots + \end{cases}$$ (linear combination of pivot columns to the right)

and the pivot columns are equal, and the coeff's x_i are equal.
Hence $U=V$.

ii) You can always get from an echelon matrix to a reduced echelon matrix, using elementary row operations, without changing the pivot positions.

This follows from the last steps in i). $\square$

(4) Rank of a matrix

Defn: The rank of a matrix A is the number of pivots in the echelon form of A.

It is written: $rk(A)$.

Ex:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -2 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$rk\, A = 2$$

Rank and linear systems:

A $m \times n$ linear system

$$\left.\begin{array}{l} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{array}\right\} \qquad A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

coeff. matrix

Compute $rk\, A$, $rk\, \hat{A}$:

$$\hat{A} = \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right)$$

augmented matrix

Result C:

For an $m \times n$ linear system with coefficient matrix A and augmented matrix $\hat{A}$, we have

$\mathrm{rk}\, A < \mathrm{rk}\, \hat{A} \iff$ no solutions (pivots in last col.)

$\mathrm{rk}\, A = \mathrm{rk}\, \hat{A} \iff$ at least one solution (no pivots in last col.)

Moreover, if $\mathrm{rk}\, A = \mathrm{rk}\, \hat{A}$, then there are

$n - \mathrm{rk}(A)$ degrees of freedom $\begin{cases} n - \mathrm{rk}\, A = 0: & \text{one unique sol.} \\ n - \mathrm{rk}\, A > 0: & \text{free var's, infinitely many sol's} \end{cases}$

A linear system is <u>homogeneous</u> if $b_1 = b_2 = \dots = b_m = 0$:

$$\left.\begin{array}{l} a_{11}x_1 + \dots + a_{1n}x_n = 0 \\ \vdots \qquad\qquad\qquad \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = 0 \end{array}\right\} \text{homogeneous } m \times n \text{ linear system}$$

In this case, there is always at least one solution $x_1 = x_2 = \dots = x_n = 0$ (the <u>trivial</u> solution). That is, either this is the unique solution or there are inf. many sol's.

<u>Ex:</u> A 3×4 linear system that is homogeneous has <u>at least one free variable</u>:

$$\left(\begin{array}{cccc|c} * & \bullet & \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet & 0 & 0 \end{array}\right) \longleftarrow \text{maximum 3 pivots (one for each row)}$$

echelon form

but depending on coeff's in matrix

$\boxed{n - \mathrm{rk}\, A = 4 - \mathrm{rk}\, A \geq 1}$