Lecture 10 (B)

GRA 6035

Eivind Eriksen OCT 30, 2014 MATHEMATICS

Plan:

① Differential equations

② First order differential equations

i) Seperable

ii) Linear

iii) Exact

Reading:

[HE] 24.1–24.2, (24.4 – 24.6),

+ Note on integration

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① Differential equations

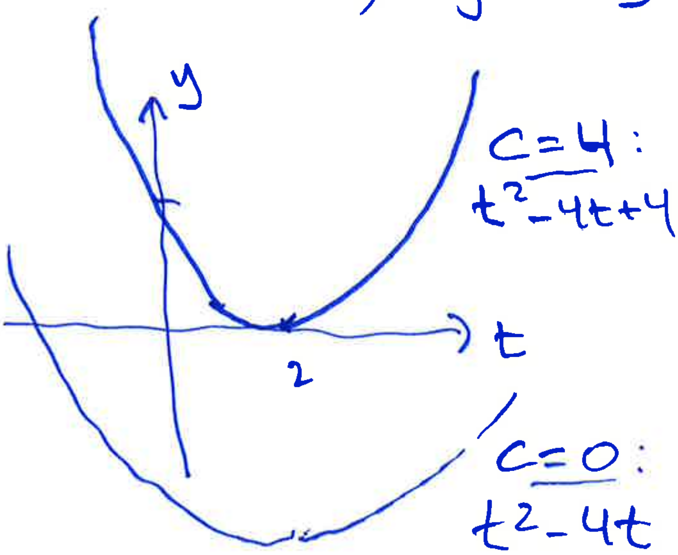
Ex: i) $y'(t) = 2t-4$

ii) $y'(t) = 2\cdot y(t)$

We want to find all functions $y = y(t)$ that satisfies the diff. eqn.

Notation: i) $y' = 2t-4$

ii) $y' = 2y$



i) $y' = 2t-4$

$y = \int 2t-4\,dt$

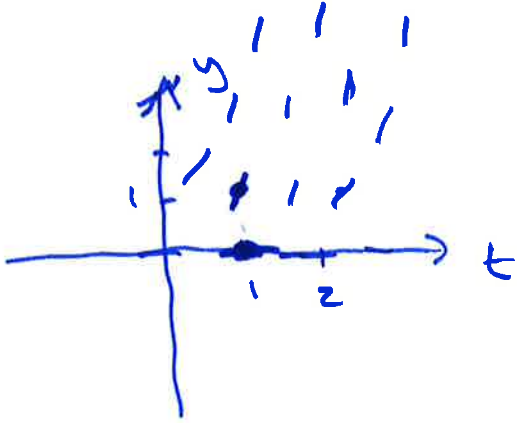
$= t^2 - 4t + C$

General solution:

$y = t^2 - 4t + C$

(C undetermined const.)

ii) $y' = 2y$



Think of this equation as

$y' = F(y,t) = 2y$

Can compute the slope at each pt. (t,y) using $F = 2y$

$(t,y) = (1,1)$

$F(1,1) = 2$

$(t,y) = (1,0)$

$F(1,0) = 0$

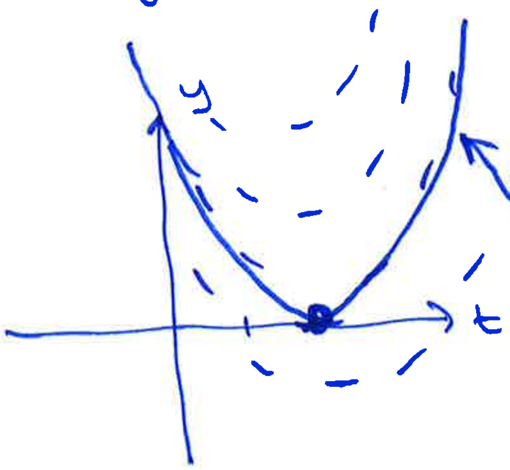
Initial value problem:

Diff. eqn + initial (start) condition

Ex: $y' = 2t - 4$, $\underbrace{y(2) = 0}_{t=2 \Rightarrow y=0}$ ← Initial cond.

General solution:

$y = t^2 - 4t + C$



Initial condition:

$0 = 2^2 - 4\cdot 2 + C$

$0 = 4 - 8 + C$

$C = 4$

Particular solution:

$y = t^2 - 4t + 4$

Definition:

A <u>differential equation</u> is an equation that relates the derivatives of a function $y = y(t)$ to y, t; that is, an equation involving the derivatives of y.

The <u>order</u> of a differential equation is the highest order of derivatives appearing in the equation.

<u>Ex</u>: $y' = 2y$ — first order

$y'' - 2y' + y = t$ — <u>second</u> order

<u>Note</u>: t is time in many applications

usual notation: $\dot{y} = y'$ $\ddot{y} = y''$

2 <u>First order differential equation</u>

$$y' = F(y, t) \leftarrow$$ not possible to solve this diff. eqn. by hand for most $F(y,t)$

$\boxed{\text{Autonomous: } F = F(y) \text{ only depends on } y.}$

i) <u>Separable diff. eqn.</u>

<u>Ex</u>: $y' = 2y = \underbrace{2}_{f(t)} \cdot \underbrace{y}_{g(y)}$ is separable

$y' = y + t$ is <u>not</u> separable

Defn: A first order diff. eqn. is <u>separable</u> if it can be written as

$$y' = f(t)\cdot g(y)$$

The reason for the name:

$$\frac{1}{g(y)}\cdot y' = f(t) \quad \leftarrow \text{diff. eqn. is } \underline{\text{separated}}$$

Explanation I:

$y' = \dfrac{dy}{dt}$:

$$\frac{1}{g(y)}\cdot\frac{dy}{dt} = f(t) \qquad |\cdot dt$$

$$\frac{1}{g(y)}\,dy = f(t)\,dt$$

$$\boxed{\int \frac{1}{g(y)}\,dy = \int f(t)\,dt}$$

Explanation 2:

$$\frac{1}{g(y)}\,y' = f(t)$$

$$\int \frac{1}{g(y)}\underbrace{y'\,dt}_{} = \int f(t)\,dt$$

$$\| \qquad \text{Substitution: } y = y(t),\ dy = y'\,dt$$

$$\int \frac{1}{g(y)}\,dy$$

$$\boxed{\int \frac{1}{g(y)}\,dy = \int f(t)\,dt}$$

Ex:

$$y' = 2y$$

$$\frac{1}{y}\,y' = 2$$

$$\int \frac{1}{y}\underbrace{y'\,dt}_{} = \int 2\,dt$$

$$\int \frac{1}{y}\,dy = \int 2\,dt$$

$$\ln|y| + C_1 = 2t + C_2$$

<u>implicit solution</u>

$$\ln|y| = 2t + C_2 - C_1$$

$$\ln|y| = 2t + C$$

$$|y| = e^{2t+C}$$

$$|y| = e^{2t}\cdot e^{C}$$

$$y = \pm e^{2t}\cdot e^{C} = K\cdot e^{2t}$$

Explicit solution: $y = \underline{K\cdot e^{2t}}$

We find the general solution of a separable diff. eqn. $y' = f(t)\cdot g(y)$ by computing the integrals in

$$\boxed{\int \frac{1}{g(y)}\,dy = \int f(t)\,dt}$$

and then solve the resulting <u>implicit eqn.</u> for y.

Ex: $y' = 3y^2 - 2ty^2 = (3-2t)\cdot y^2$

$$\frac{1}{y^2}\,y' = 3-2t$$

y^{-2}

$$\int \frac{1}{y^2}\,dy = \int 3-2t\,dt$$

$$-\frac{1}{y} = 3t - t^2 + C$$ ← implicit soln.

$$\frac{1}{y} = t^2 - 3t - C \quad \Big|\ (\ldots)^{-1}$$ (invert both sides)

$$y = \frac{1}{t^2-3t-C}$$ general (explicit) solution

b) Linear first order differential equations

<u>Defn:</u> A first order diff. equ. is <u>linear</u> if it can be written

$$\boxed{y' + a(t)\cdot y = b(t)} \Leftrightarrow y' = b(t) - a(t)y$$

<u>Ex:</u> $y' = y + t$ is <u>linear</u> $\left(y' - \underset{\uparrow}{y} = \underset{\uparrow}{t} \right)$

$a(t) = -1 \quad b(t) = t$

$ty' = y^2 + y^3 t$ is <u>not</u> linear

$y' + 3y = 2$ is <u>linear</u> and <u>autonomous</u> ($a(t) = 3$, $b(t) = 2$ are const.)

$\Updownarrow$

$y' = 2 - 3y = 1\cdot(2-3y)$ is <u>seperable also</u>

<u>Solution method:</u>

i) <u>Autonomous case:</u> $\left.\begin{array}{l} a(t) = a \\ b(t) = b \end{array}\right\}$ constants

<u>Ex:</u> $y' + 3y = 2 \quad | \cdot e^{3t} \qquad a = 3 \quad b = 2$

$$\underbrace{e^{3t}\cdot y' + 3e^{3t}\cdot y}_{} = 2e^{3t}$$

<u>In general:</u> e^{at}

$$\left(e^{3t}\cdot y\right)' = 2e^{3t}$$

$$e^{3t}\cdot y = \int 2e^{3t}\,dt = 2\cdot\tfrac{1}{3}e^{3t} + C$$

$$y = \frac{\frac{2}{3}e^{3t} + C}{e^{3t}} = \frac{2}{3} + C\cdot e^{-3t}$$

General formula: Autonomous case

$$y' + ay = b$$

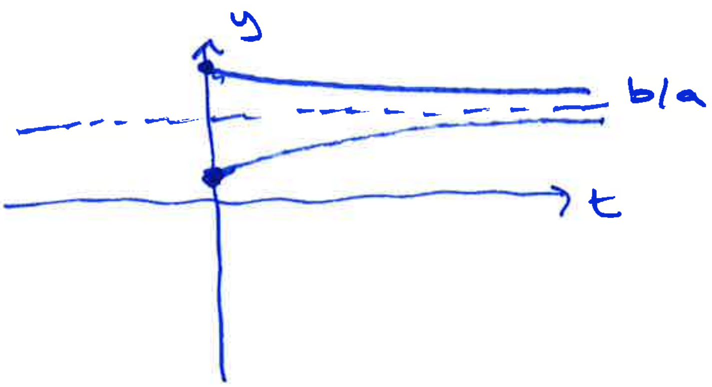
(same steps as before; int. factor is e^{at})

$$y = \frac{b}{a} + C \cdot e^{-at}$$

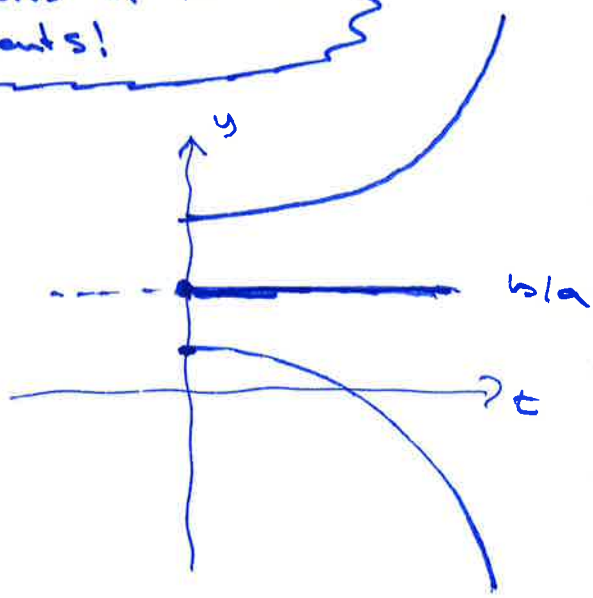
(general, explicit solution)

only works if a,b are constants!

Stability:



$a > 0$



$a < 0$

Globally asymptotically stable solution:

For all values of y_0 (that is, all values of C), we have:

$$\lim_{t\to\infty} y(t) = b/a$$

$\frac{b}{a}$: equilibrium value

unstable solution

(unless $C=0$)

No equilibrium value unless $C=0$ as

$$\lim_{t\to\infty} y(t) = \pm\infty$$

Ex: $y' + ty = 3t$

$a(t) = t$
$b(t) = 3t$
not auton.

$(y' + ty)\, e^{\frac{1}{2}t^2} = 3t\, e^{\frac{1}{2}t^2}$

Integrating factor: $e^{\int a(t)\,dt}$

$e^{\frac{1}{2}t^2} y' + t e^{\frac{1}{2}t^2} \cdot y = 3t e^{\frac{1}{2}t^2}$

$\left(e^{\frac{1}{2}t^2} \cdot y\right)' = 3t e^{\frac{1}{2}t^2}$

In this case: $e^{\int t\,dt} = e^{\frac{1}{2}t^2}$

$e^{\frac{1}{2}t^2} \cdot y = \int 3t\, e^{\frac{1}{2}t^2}\, dt$

$\boxed{\begin{aligned} u &= \tfrac{1}{2}t^2 \\ du &= t \cdot dt \end{aligned}}$

In general:

$\left(e^{\int a(t)\,dt} \cdot y\right)' = b(t) \cdot e^{\int a(t)\,dt}$

$= \int 3\, e^{u}\, du$

$= 3e^{u} + C$

$e^{\frac{1}{2}t^2} \cdot y = 3e^{\frac{1}{2}t^2} + C$

$$y = \frac{3 \cdot e^{\frac{1}{2}t^2} + C}{e^{\frac{1}{2}t^2}}$$

$$y = 3 + C \cdot e^{-\frac{1}{2}t^2}$$

general explicit solution.

$\lim_{t \to \infty} y(t) = 3 \Rightarrow$ globally asymptotically stable (equilibrium $= 3$)

General formula:

$$e^{\int a(t)\,dt} \cdot y = \int b(t)\, e^{\int a(t)\,dt}\, dt$$

$$\Downarrow$$

$$\boxed{y = \frac{1}{e^{\int a(t)\,dt}} \cdot \int b(t)\, e^{\int a(t)\,dt}\, dt}$$

Ex:

$$D = a - bp$$
$$S = \alpha + \beta p$$
$$p' = \lambda \cdot (D - S)$$

$$\boxed{\begin{array}{l} a, b > 0 \\ \alpha, \beta > 0 \\ \lambda > 0 \end{array}} \leftarrow \text{const.}$$

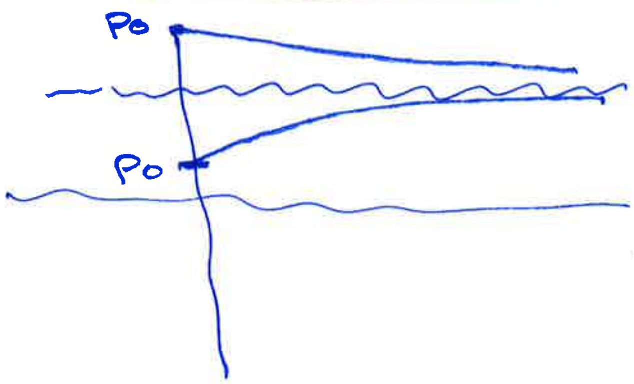
$$p' = \lambda \cdot \left[(a - bp) - (\alpha + \beta p)\right]$$

$$p' = \lambda \cdot (a - \alpha) + \lambda \cdot (-b - \beta)\, p$$

$$p' + \lambda \cdot (b + \beta)\, p = \lambda \cdot (a - \alpha) \leftarrow$$

first order linear auton. diff. eqn.

$$\boxed{p = \frac{\cancel{\lambda}(a - \alpha)}{\cancel{\lambda}(b + \beta)} + C \cdot e^{-\lambda(b+\beta)t}}$$



$$\frac{a - \alpha}{b + \beta}$$

globally asymptotically stable

General solution:

$$p = \frac{a-\alpha}{b+\beta} + C\cdot e^{-\lambda\cdot(b+\beta)t}$$

Numerical example:

$$\left.\begin{aligned} D &= 5000-4p \\ S &= 1000+6p \\ p' &= 0.5\cdot(D-S) \end{aligned}\right\}$$

$$p' = 0.5\left((5000-4p)-(1000+6p)\right)$$

$$= 0.5\cdot(4000-10p)$$

$$p' = \underline{2000-5p}$$

$$p' + 5p = 2000 \qquad \begin{cases} a = 5 \\ b = 2000 \end{cases}$$

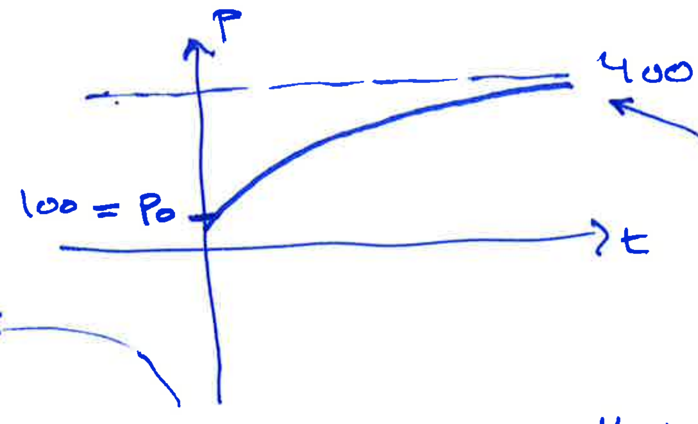
$$p = \frac{b}{a} + Ce^{-at}$$

$$= \frac{2000}{5} + C\cdot e^{-5t}$$

$$= \underline{\underline{400 + C\cdot e^{-5t}}}$$

$$p_0 = p(0) = 400 + C\cdot e^{0} = 400 + C$$

$$\Rightarrow C = p_0 - 400$$



$$p_0 = 100 \Rightarrow C = p_0 - 400 = 100 - 400 = -300$$

$$\Rightarrow p = \underline{\underline{400 - 300e^{-5t}}}$$

c) Exact differential equations

Ex: $\underline{1+ty^2} + \underline{t^2 y} \cdot y' = 0$

$$y' = \frac{-1-ty^2}{t^2 y}$$

not separable
not linear

Defn: A diff. eqn. (of order one) is exact if it can be written

$$p(y,t) + q(y,t) \cdot y' = 0$$

such that

$$\boxed{\frac{\partial p}{\partial y} = \frac{\partial q}{\partial t}}$$

Check: $p(y,t) = 1+ty^2$, $q(y,t) = t^2 y$

$$\left.\begin{array}{l} p(y,t) = 1+ty^2 \\ q(y,t) = t^2 y \end{array}\right\} \quad \frac{\partial p}{\partial y} = 2ty \;\; \| \;\; \frac{\partial q}{\partial t} = 2ty$$

Exact!

Solution method: $p(y,t) + q(y,t) \cdot y' = 0$ ← std. form

Find an expression $h(y,t)$ such that

$$\frac{\partial h}{\partial t} = p \quad \text{and} \quad \frac{\partial h}{\partial y} = q$$

This is possible if and only if the equation is exact.
If h satisfies these eqn's, then the (implicit) solution is $\boxed{h(y,t) = C}$

Ex: $1 + ty^2 + t^2 y \cdot y' = 0$

$$\boxed{\begin{aligned} p &= 1 + ty^2 = h'_t \\ q &= t^2 y = h'_y \end{aligned}}$$

these are the conditions h have to satisfy!

(1) $h'_t = 1 + ty^2 \implies h = \underline{t + \frac{1}{2} t^2 \cdot y^2 + \phi(y)}$

(2) $h'_y = \left(t + \frac{1}{2} t^2 y^2 + \phi(y) \right)'_y$

$$= \frac{1}{2} t^2 \cdot 2y + \phi'(y)$$

$$= t^2 y + \phi'(y) = t^2 y$$

$\Downarrow$

$\phi(y) = 0$ gives a solution: $\underline{h = t + \frac{1}{2} t^2 y^2}$

General solution: $h = C$

(implicit)

$$t + \frac{1}{2} t^2 y^2 = C$$

$$\left(\frac{1}{2} t^2\right) \cdot y^2 = C - t$$

$$y^2 = \frac{C - t}{\frac{1}{2} t^2} = \frac{2(C-t)}{t^2}$$

$$\underline{\underline{y = \pm \sqrt{\frac{2(C-t)}{t^2}}}}$$

(general explicit solution)

Note: Why is $h(t,y)=C$ the solution of the diff. eqn.?

Diff. eqn: $(1+ty^2) + (t^2y)\cdot y' = 0$

$$\underbrace{(1+ty^2)}_{=\,p} + \underbrace{(t^2y)}_{=\,q}\cdot y' = 0$$

When $h = t + \frac{1}{2}t^2y^2$ then $h'_t = p$ and $h'_y = q$

Notice that the total derivative of h (that is, the derivative w.r.t. t when we consider $y = y(t)$ as a function of t, not an independent variable):

$$\frac{dh}{dt} = \frac{d}{dt}\left(t + \frac{1}{2}t^2y^2\right) = 1 + \frac{1}{2}t^2\frac{d}{dt}(y^2) + \frac{1}{2}\frac{d}{dt}(t^2)\cdot y^2$$

$$= 1 + \frac{1}{2}t^2(2y\cdot y') + \frac{1}{2}\cdot 2t\cdot y^2$$

$$= 1 + \frac{1}{2}t^2\cdot 2y\cdot y' + ty^2$$

$$= \underbrace{1 + ty^2 + t^2y\cdot y'}$$

the right side of the diff. eqn.

$$\frac{dh}{dt} = 0$$

$$\Updownarrow$$

$$h = C$$

Conclusion:

When $\frac{\partial h}{\partial t} = p$ and $\frac{\partial h}{\partial y} = q$, then the total derivative

$$\frac{dh}{dt} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial y}\cdot\frac{dy}{dt} = 0$$

hence $h = C$.

Hint:

How to solve for h in an exact equation:

$$\begin{cases} \dfrac{\partial h}{\partial t} = p = 1 + ty^2 \\ \dfrac{\partial h}{\partial y} = q = t^2 y \end{cases}$$

① Solve for h in one of the eqn's (used the first here, if the other one is easier we could have started with that one):

$$\frac{\partial h}{\partial t} = 1 + ty^2 \Rightarrow h = \int \left(1 + ty^2\right) dt$$

consider y as a const. in this integral

$$= t + y^2 \cdot \tfrac{1}{2} t^2 + a(y)$$

the integration "constant" is a function in y, $a(y)$, since $a(y)'_t = 0$

② Insert the h you ~~found~~ found above in the other condition:

$$\frac{\partial h}{\partial y} = t^2 y$$

$$\frac{\partial}{\partial y}\left(t + y^2 \cdot \tfrac{1}{2} t^2 + a(y)\right) = t^2 y$$

$$0 + 2y \cdot \tfrac{1}{2} t^2 + a'(y) = t^2 y$$

$$y t^2 + a'(y) = t^2 y$$

$$\underline{a'(y) = 0}$$

we can choose $\underline{a(y) = 0}$ to get a solution.