LECTURE 13 (I)

GRA 6035
MATHEMATICS

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Review:

1. Matrix methods
2. Unconstrained optimization

Exam problems:

Exam 2013/12 Q.1-2
(final exam Dec' 2013)

Final exam:

Same structure as final exam 12/2013:

12 problems à 6p
1 bonus problem 6p

$100\% = 12 \cdot 6p = 72p$.
(max score 78p)

Grading scale 12/2013:

A: 100% - 92%
B: 92% - 77%
C: 77% - 58%
D: 58% - 46%
E: 46% - 40%

① Matrix methods

Basic techniques: $\begin{cases} \text{a) Solving linear systems (Gaussian elimination)} \\ \text{b) Determinants} \end{cases}$

i) Linear independence of vectors

$\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$: m-vectors $\rightsquigarrow$ $A = (\underline{v}_1 | \underline{v}_2 | \ldots \underline{v}_n)$

Are the vectors linearly independent?

Fact: If $m=n$, then $\boxed{\begin{array}{l} |A| \neq 0 \iff \text{linearly independent} \\ |A| = 0 \iff \text{linearly dependent} \end{array}}$

Fact: $A\underline{x} = \underline{0}$ ← Solve

$\boxed{\text{Only the trivial solution } \underline{x} = \underline{0} \iff \text{linearly independent}}$

no free variables

Ex: $A = \begin{pmatrix} 1 & t & -2 \\ 2 & 4 & -t \\ -t & -4 & -4 \end{pmatrix}$ $\quad \underline{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -t \end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix} t \\ 4 \\ -4 \end{pmatrix}$ $\underline{v}_3 = \begin{pmatrix} -2 \\ -t \\ -4 \end{pmatrix}$

$$\begin{vmatrix} 1 & t & -2 \\ 2 & 4 & -t \\ -t & -4 & -4 \end{vmatrix} = 1 \cdot (-16-4t) - 2(-4t-8) - t(-t^2+8)$$

$$= \cancel{-16} - 4t + \cancel{8t} + \cancel{16} + t^3 - \cancel{8t}$$

$$= t^3 - 4t = t(t^2-4) = t(t-2)(t+2)$$

$t = 0, 2, -2$: $|A| = 0 \iff$ vectors are lin. dependent

$t \neq 0, 2, -2$: $|A| \neq 0 \iff$ vectors are lin. independent

ii) Rank

A $m \times n$-matrix: $rk(A) =$ max number of linearly independent column vectors in A

Fact I:

If $m = n$, then $\begin{cases} rk\,A = n \iff |A| \neq 0 \\ rk\,A < n \iff |A| = 0 \end{cases}$

Fact 2:

$rk(A) = \#$ pivot positions in A

Ex: $A = \begin{pmatrix} 1 & t & -2 \\ 2 & 4 & -t \\ -t & -4 & -4 \end{pmatrix}$

$|A| = \underline{t^3 - 4t} \Rightarrow \begin{cases} |A| = 0 \iff t = 0, 2, -2 \to rk\,A < 3 \\ |A| \neq 0 \iff t \neq 0, 2, -2 \to rk\,A = 3 \end{cases}$

If $\underline{t = -2}$: $A = \begin{pmatrix} \boxed{\begin{matrix} 1 & -2 \\ 2 & 4 \end{matrix}} & \begin{matrix} -2 \\ 2 \end{matrix} \\ \begin{matrix} 2 & -4 \end{matrix} & -4 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -2 \\ 0 & 8 & 6 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad$ $\underline{rk\,A = 2 \text{ when } t = -2}$

$\begin{vmatrix} 1 & -2 \\ 2 & 4 \end{vmatrix} = 4 + 4 = 8 \neq 0$ $\quad$ $rk\,A = 2$ when $t = -2$

2-minor

$$\boxed{rk\,A = \text{maximal order of a non-zero minor of } A}$$

iii) Eigenvalues and eigenvectors

A $n \times n$-matrix: If $A\underline{x} = \lambda \underline{x}$ with $\underline{x} \neq \underline{0}$ then

- the number λ is a eigenvalue
- the vector $\underline{x}$ is a eigenvector

Fact 1: The eigenvalues are the solutions of the characteristic equation

$$\boxed{\det(A - \lambda I) = 0} \qquad I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Fact 2: The eigenvectors for A with eigenvalue λ^* are the solutions of

$$\boxed{(A - \lambda^* I)\underline{x} = \underline{0}}$$

Fact 3: There is a diagonal matrix D and a invertible matrix P such that

$$\boxed{P^{-1}AP = D} \quad \text{(diagonalization)}$$

if and only if
i) there are n eigenvalues (when you count with multiplicities) $\lambda_1, \lambda_2, \ldots, \lambda_n$
ii) there are n linearly independent eigenvectors $\underline{v_1}, \ldots, \underline{v_n}$

If this is the case,

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{pmatrix} \qquad P = \left(\underline{v_1} \,\vdots\, \underline{v_2} \,\vdots\, \cdots \,\vdots\, \underline{v_n} \right)$$

Ex: $A = \begin{pmatrix} 1 & -2 & -2 \\ 2 & 4 & 2 \\ 2 & -4 & -4 \end{pmatrix}$

Eigenvalues: $|A-\lambda I| = \begin{vmatrix} 1-\lambda & -2 & -2 \\ 2 & 4-\lambda & 2 \\ 2 & -4 & -4-\lambda \end{vmatrix} = 0$

$$(1-\lambda)\cdot\big((4-\lambda)(-4-\lambda)+8\big) - 2\cdot\underbrace{(\cancel{8}+2\lambda-\cancel{8})}_{\substack{+8(1-\lambda)\\ -8(\lambda-1)}} + 2\cdot\underbrace{(-4+8-2\lambda)}_{\substack{\lambda^2-\lambda^3\\ =\lambda^2(1-\lambda)}}$$

$$= (1-\lambda)(4-\lambda)(-4-\lambda) + 8(1-\lambda) - 8\lambda + 8$$

$$= (1-\lambda)\cdot\left[(4-\lambda)(-4-\lambda)+8+8\right]$$

$$= (1-\lambda)\cdot(\lambda^2) = \lambda^2(1-\lambda) = 0$$

$\underline{\lambda_1=0}$ $\underline{\lambda_2=0}$ $\underline{\lambda_1=1}$ $\rightsquigarrow D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Eigenvectors

$P = \left(\underline{v_1} \,\middle|\, \underline{v_2} \,\middle|\, \underline{v_3}\right)$

v_1: $\begin{pmatrix} 1/2 \\ -3/4 \\ 1 \end{pmatrix}$ $\quad$ v_2: ?

$\underline{\lambda=0}$: $\begin{array}{c} \\ -2 \\ -2 \end{array} \begin{pmatrix} 1 & -2 & -2 \\ 2 & 4 & 2 \\ 2 & -4 & -4 \end{pmatrix} \underline{x} = \underline{0}$

$$\downarrow$$

$\begin{pmatrix} 1 & -2 & -2 \\ 0 & 8 & 6 \\ 0 & 0 & 0 \end{pmatrix} \underline{x} = \underline{0}$

one free variable

$\underline{x} - 2y - 2z = 0$

$8\underline{y} + 6z = 0$

z free

$y = -\frac{6}{8}z = -\frac{3}{4}z$

$x = 2y + 2z$

$= 2z - \frac{3}{2}z$

$= \frac{1}{2}z$

Eigenvect. for $\lambda=0$:

$$\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/2\, z \\ -3/4\, z \\ z \end{pmatrix} = z\cdot\begin{pmatrix} 1/2 \\ -3/4 \\ 1 \end{pmatrix}$$

A is **not** diagonalizable.

If λ has multiplicity m, then $(A-\lambda I)\underline{x}=\underline{0}$ must have m free variables for A to be diagonalizable.

<u>If A is symmetric, then it is diagonalizable</u>

<u>Fact:</u> If A (n×n-matrix) has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then

$$\lambda_1+\lambda_2+\ldots+\lambda_n = tr(A) = a_{11}+a_{22}+\ldots+a_{nn}$$
$$\lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n = \det(A)$$

V) <u>Definiteness:</u> What is the definiteness of A

A symmetric n×n-matrix $\Bigg\}$ Leading principal minors: $D_1, D_2, \ldots, D_n$
Principal minors: $\Delta_1, \Delta_2, \ldots, \Delta_n$
(several of each order)

<u>Fact:</u>

i) $D_1>0, D_2>0, \ldots, D_n>0$: pos. definite
ii) $D_1<0, D_2>0, D_3<0, \ldots$: neg. definite
iii) $D_1\geq 0, D_2\geq 0, \ldots, D_n\geq 0$: pos. semidefinite or indefinite
iv) $D_1\leq 0, D_2\geq 0, D_3\leq 0, \ldots$: neg. semidefinite or indefinite
v) All other cases: indefinite

In case 3) and 4):

$\Delta_1, \Delta_2, \ldots, \Delta_n \geq 0 \iff A$ positive semidefinite

$\Delta_1 \leq 0, \Delta_2 \geq 0, \Delta_3 \leq 0, \ldots \iff A$ negative semidefinite

Δ_i means all principal minors of order i.

Ex: $A = \begin{pmatrix} 3 & 1 & 9 \\ 1 & 2 & 8 \\ 9 & 8 & 42 \end{pmatrix}$ symm.

$D_1 = 3$

$D_2 = 5$

$D_3 = 3\cdot(84-64) - 1\cdot(42-72) + 9\cdot(8-18) = 60 + 30 - 90 = 0$

Concl: A may be positive semidefinite

All principal minors

choose one row and the same column

$\Delta_1 = 3, \ 2, \ 42 \geq 0$ (1,1) (2,2) (3,3)

$\Delta_2 = 5, \ \begin{vmatrix} 2 & 8 \\ 8 & 42 \end{vmatrix} = 20, \ \begin{vmatrix} 3 & 9 \\ 9 & 42 \end{vmatrix} = 126 - 81 = 45 \geq 0$

$\Delta_3 = 0 \geq 0$

chose two rows and the same two col's: $\binom{12}{12}$ $\binom{13}{13}$ $\binom{23}{23}$

A is positive semidefinite

vi) Markov chains

A $n \times n$-matrix (transition matrix) $\begin{cases} \text{all entries } a_{ij} \in [0,1] \\ \text{each column has sum} = 1 \end{cases}$

if $a_{ij} > 0$ then it is called regular

$$\underline{x}_{n+1} = A\underline{x}_n \longrightarrow \underbrace{\lim_{n\to\infty} \underline{x}_n = \lim_{n\to\infty} A^n \cdot \underline{x}_0}_{\substack{\text{long run equilibrium} \\ \text{if the limit exists}}}$$

Fact: A regular Markov chain has a long run equilibrium state $\underline{x}$, and $\underline{x}$ is the unique eigenvector with $\lambda = 1$ with $x_1 + x_2 + \dots + x_n = 1$.

Exam 12/2013, Q2:

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

a)

$$A+I = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

echelon form

$$A-I = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

echelon form

$\mathrm{rk}(A+I) = \underline{2}$ $\quad \mathrm{rk}(A-I) = \underline{2}$

b) Since $A^T = A$, the matrix is symmetric and therefore diagonalizable.

<u>Alternative:</u>

Eigenvalues: $|A - \lambda I| = 0$ ← Notices that $\lambda = 1$ and $\lambda = -1$ are solutions.

⇓

$\lambda = 1$ and $\lambda = -1$ are eigenvalues, each with multiplicity 2

$m \geq$ # free var's ≥ 1

multiplicity

$\lambda = \pm 1$

2

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$P = \Big(\underbrace{v_1 \,|\, v_2}_{\lambda=1} \,\Big|\, \underbrace{v_3 \,|\, v_4}_{\lambda=-1} \Big)$$

← So this is possible; A is <u>diag.</u>

c) $\underline{\lambda = -1}$: $A - \lambda I = A - (-1)I = A + I$

$$(A+I)\underline{x} = \underline{0}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

z, w free

$x + w = 0$

$y - z = 0$

z free

w free

$x = -w$

$y = z$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -w \\ z \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ z \\ z \\ 0 \end{pmatrix} + \begin{pmatrix} -w \\ 0 \\ 0 \\ w \end{pmatrix} = z \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + w \cdot \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$\underline{v_3}$ $\underline{v_4}$

② Unconstrained optimization

Basic techniques:

$\boxed{\begin{array}{l} \text{a) Compute derivatives} \\ \text{b) Find Hessian matrices} \end{array}}$

a) Stationary pts:

Fact 1:

A stationary pt for f is a pt such that $\boxed{f'_{x_1} = f'_{x_2} = \dots = f'_{x_n} = 0}$

Fact 2:

If $\underline{x}^*$ is a local/global max/min, then $\underline{x}^*$ is a stationary pt.

Fact 3:

A stationary pt $\underline{x}^*$ can be classified as local max, local min or saddle pt using Hessian:

$H(f)(x^*)$ ↗ Hessian of f at $\underline{x}^*$

positive definite $\Rightarrow$ x^* local min

negative definite $\Rightarrow$ x^* local max

indefinite $\Rightarrow$ $\underline{x}^*$ saddle pt

other cases : no conclusion

Ex: $f(x,y,z) = x^2 + y^2 + z^2 + z^3 + 2yz - 2x + 12y$

Stationary pts:

$$f'_x = 2x - 2 = 0 \qquad \underline{x = 1}$$

$$f'_y = 2y + 2z + 12 = 0$$

$$f'_z = 2z + 3z^2 + 2y = 0$$

$y + z + 6 = 0 \Rightarrow y = \underline{-6 - z}$

$2z + 3z^2 + 2 \cdot (-6 - z) = 0 \qquad 3z^2 - 12 = 0$

$z^2 = 4$

$z = \pm 2$

$z = 2,\ y = -8,\ x = 1$

$z = -2,\ y = -4,\ x = 1$

Stat. pts: $(x,y,z) = (1,-8,2)$
$(1,-4,-2)$

Classification:

$$H(f) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2+6z \end{pmatrix}$$

$(1,-8,2)$: $H(f)(1,-8,2) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 14 \end{pmatrix}$

$D_1 = 2$
$D_2 = 4$
$D_3 = 2 \cdot 24 = 48$

positive defn. $\Rightarrow$ local min at $(1,-8,2)$

$(1,-4,-2)$: $H(f)(1,-4,-2)$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & -10 \end{pmatrix}$$

$D_1 = 2$
$D_2 = 4$
$D_3 = 2(-20-4) = -48$

indefinite $\Rightarrow$ saddle pt at $(1,-4,-2)$

Exam 12/2013, Q1:

$f = xw - yz$

a) $f'_x = \underline{w}$ $\quad f'_y = \underline{-z}$ $\quad f'_z = \underline{-y}$ $\quad f'_w = \underline{x}$

$$\underline{H(f) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}}$$

$f''_{xx} = 0 \quad f''_{xy} = 0 \ldots$

b) Stationary pts:

$$\left.\begin{array}{ll} f'_x = w = 0 & w = 0 \\ f'_y = -z = 0 & z = 0 \\ f'_z = -y = 0 & y = 0 \\ f'_w = x = 0 & x = 0 \end{array}\right\}$$

Stationary pts:
$(x, y, z, w) = \underline{\underline{(0, 0, 0, 0)}}$

$$H(f)(0,0,0,0) = \begin{pmatrix} 0 & \boxed{\begin{matrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{matrix}} \\ 0 & \\ 0 & \\ 1 & 0 \quad 0 \quad 0 \end{pmatrix}$$

$D_1 = 0$

$D_2 = 0$

$D_3 = 0$

$D_4 = -1 \cdot \begin{vmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{vmatrix}$

$= -1 \cdot (-1) \cdot 1 = \underline{1}$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$\Delta_1 = 0, 0, 0, 0$

$\Delta_2 = 0, 0, -1, \ldots$

Δ_1: 1 2 3 4

Δ_2: 12 13 14

23 24 34

$\Delta_2 < 0 \Rightarrow$ indefinite

$\Rightarrow (0,0,0,0)$ saddle point

pos. semidef: no

$\Delta_1 \geq 0, \Delta_2 \geq 0, \ldots$

neg. semidef: no

$\Delta_1 \leq 0, \Delta_2 \geq 0, \ldots$

c) If f has a global max, it is also a local max. But f does not have any local max (the only stationary pt is a saddle point).

ii) Convex / concave functions and global max/min

Fact 1:

f is convex $\Leftrightarrow$ $H(f)$ is positive semidefinite for all $\underline{x}$

f is concave $\Leftrightarrow$ $H(f)$ is negative semi definite — " —

Fact 2:

If f is concave, then any stationary pt is global max,

— " — convex ——— " ——— global min.

Ex: $f(x,y,z) = x^2 + y^2 + y^4 + yz - 1$

$$f'_x = 2x$$

$$f'_y = 2y + 4y^3 + z \qquad H(f) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2+12y^2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$f'_z = y$$

$$D_1 = 2 > 0$$

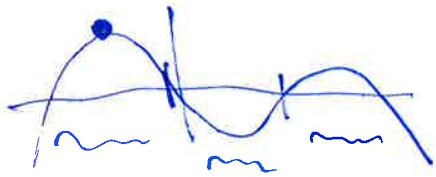
$$D_2 = 2(2+12y^2) = 4 + 24y^2 > 0 \qquad \text{for all } (x,y,z)$$

$$D_3 = 2 \cdot (0-1) = -2 < 0$$

$H(f)$ indefinite for all (x,y,z).

f is not convex, not concave

Even if f is not convex and not concave, it could of course still have global max/min.



iii) Envelope theorem

$f(x_1, .., x_n; a) = f(\underline{x}; a)$ function with parameter a

Consider the unconstrained optimization problem

$$\boxed{\max/\min \ f(\underline{x}; a)}$$

Assume that it has solution $\underline{x}^*(a)$ depending on a, and let $f^*(a) = f(\underline{x}^*(a))$ be the max/min value.

$$\boxed{\underline{\text{Envelope thm:}} \quad \frac{df^*(a)}{da} = \frac{\partial f}{\partial a}(\underline{x}^*(a))}$$

this tells you how changing a will change the max/min value

this is how compute it.

Ex: $\min f(x,y;h) = hx^4 + y^4 + 4x^2 - (6+h)xy + 4y^2 - 3h$

i) For which values of h is f convex?

$f'_x = 4hx^3 + 8x - (6+h)y$

$f'_y = 4y^3 - (6+h)x + 8y$

$$H(f) = \begin{pmatrix} 12hx^2 + 8 & -(6+h) \\ -(6+h) & 12y^2 + 8 \end{pmatrix}$$

$D_1 = 12hx^2 + 8 \quad \leftarrow$ When $h \geq 0$, $D_1 \geq 0$ for all (x,y)

$D_2 = (12hx^2+8)(12y^2+8) - (6+h)^2$

$= 144h\,x^2 + 96h\,x^2 + 96y^2 + [64 - (6+h)^2] \quad \leftarrow$ When $h \leq 2$, $D_2 \geq 0$ for all (x,y)

$\Delta_1 = 12y^2 + 8 > 0$ for all $(x,y) \quad \leftarrow$ Check the other principal minor of order 1 since $D_2 = 0$ at $(0,0)$ when $h=2$

Conclusion: When $0 \leq h \leq 2$ f is convex

ii) Find $x^*(h), y^*(h)$ when $h=0$:

$h=0 \Rightarrow f$ convex, so any stationary pt is global min.

Stationary pts: $h=0$

$8x - 6y = 0$

$4y^3 - 6x + 8y = 0$

$x = \frac{6y}{8} = 3y/4$

$4y^3 - 6\cdot(3y/4) + 8y = 0$

$4y^3 - \frac{18}{4}y + 8y = 0 \quad | \cdot 2$

$8y^3 - 9y + 16y = 0$

$8y^3 + 7y = 0$

$y(8y^2 + 7) = 0$

$y = 0$ or $y^2 = -7/8$ (no sol'n)

$y = 0 \Rightarrow x = 0$

Stat. pts: $(x,y) = (0,0)$

This is global min for $h=0$, so $(x^*(0), y^*(0)) = \underline{(0,0)}$

iii) If h increases from $h=0$, what happens with $f^*(h)$?

$f^*(0) = f(0,0) = -3h \quad \leftarrow$ min. value when $h=0$

$$\frac{df^*(h)}{dh} = \frac{\partial f}{\partial h}(x^*(h), y^*(h)) = (x^4 - xy - 3)\Big|_{x = x^*(h),\, y = y^*(h)}$$

$$= x^*(h)^4 - x^*(h)y^*(h) - 3$$

$$\frac{df^*(h)}{dh}\Big|_{h=0} = x^*(0)^4 - x^*(0)y^*(0) - 3 = 0^4 - 0\cdot 0 - 3 = \underline{-3}$$

rate of change at $h=0$

The minimum value will decrease when h increases from $h=0$

$\underline{\text{Ex}}$: $f(x,y;h) = hx^4 + y^4 + 4x^2 - (6+h)xy + 4y^2 - 3h$

h: parameter

$f'_x = 4hx^3 + 8x - (6+h)y$

$f'_y = 4y^3 - (6+h)x + 8y$

Let's solve the max-problem when $h=0$

$f'_x = 8x - 6y = 0$ $\quad x = \frac{6y}{8} = \frac{3}{4}y$

$f'_y = 4y^3 - 6x + 8y = 0$

$2 \cdot \mid 4y^3 - 6 \cdot \frac{3}{4}y + 8y = 0$

$8y^3 - 9y + 16y = 0$

$8y^3 + 7y = 0$

$y \cdot (8y^2 + 7) = 0$

$\underline{y=0}$ or $\cancel{8y^2+7=0}$

<u>Stationary pts:</u>

$$\begin{cases} x^*(0) = 0 \\ y^*(0) = 0 \end{cases}$$

$$H(f) = \begin{pmatrix} 8 & -6 \\ -6 & 12y^2+8 \end{pmatrix}$$

$D_1 = 8 > 0$

$D_2 = 8(12y^2+8) - 36$

$= \underbrace{96y^2}_{\geq 0} + \underbrace{64-36}_{>0} \geq 0$

f is convex when $h=0$

$\Downarrow$

$(x^*(0), y^*(0)) = (0,0)$ is global min (for $h=0$)

$\underline{f^*(0) = 0}$

What happens if h increases to $h>0$?

Envelope thm:

$$\frac{d f^*(h)}{dh} = \frac{\partial f}{\partial h}\left(x^*(0), y^*(0); 0\right)$$

$$= \left(x^4 - xy - 3\right)(0, 0; 0) = \underline{-3}$$

Interpretation:

h incr. from 0 to 0.1

$$\rightsquigarrow f^*(0.1) \simeq \underset{\substack{\parallel \\ 0}}{f^*(0)} + \underset{\substack{\parallel \\ 0.1\cdot(-3)}}{h \cdot \frac{d f^*(h)}{dh}} = \underline{-0.3}$$