LECTURE 2

GRA 6035

MATHEMATICS

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Plan:

1. Matrices and matrix algebra
2. Determinants
3. Minors, rank and linear systems

Readings:

[ME] 8.1-8.4 (8.5-8.6), 9.1-9.2 (9.3), 26.1-26.3 (26.4), 26.5

1 Matrices and matrix algebra

An $m \times n$-matrix A is a rectangular array of numbers with m rows, n columns.

Ex:

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 7 & -2 \end{pmatrix}$$

2x3-matrix

$a_{11} = 2$ $\quad a_{12} = 3$

entry in row 1 col. 2

In general:

$$A = (a_{ij})$$

Matrix operations:

1) Addition, subtraction: $A+B$, $A-B$

(when A and B have the same size)

Ex: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 3 & -1 \\ 7 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 10 & 6 \end{pmatrix}$

(position by position)

Ex: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 5 & 1 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} -4 & 1 \\ 3 & -3 \end{pmatrix}$

2) Scalar multiplication: scalar = number

Ex: $3 \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 9 & 12 \end{pmatrix}$

3) Matrix multiplication: $\underset{m\times n}{A} \cdot \underset{n\times p}{B} \rightsquigarrow$ matrix $m\times p$

$\boxed{\#\text{cols}(A) = \#\text{rows}(B)}$

Ex: $\underset{2\times 2}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}} \cdot \underset{2\times 3}{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}} = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 4 & 3 \end{pmatrix}$ $1\cdot 1 + 2\cdot 0$

Important:

i) $AB \neq BA$

ii) most other algebraic laws apply to matrices in the same way as numbers

Ex: $A\cdot(B+C) = A\cdot B + A\cdot C$

$$(A+B)^2 = (A+B)\cdot(A+B)$$
$$= A^2 + B\cdot A + A\cdot B + B^2$$

Ex: $\left.\begin{array}{l} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{array}\right\}$ example shows that $AB \neq BA$

Identity matrix:

$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ... I_n : $n\times n$ identity matrix

Property: $\boxed{\begin{array}{ll} A\cdot I = A \text{ and } & I\cdot A = A \\ A\cdot 0 = 0 & 0\cdot A = 0 \end{array}}$ (same role as 1 for numbers)

$0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$

zero matrix

these properties hold for all matrices A

4) Transpose: A^T

$\underset{m\times n}{A} \rightsquigarrow \underset{n\times m}{A^T}$

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 7 & 0 & 1 \\ 4 & 2 & 0 \end{pmatrix} \rightsquigarrow A^T = \begin{pmatrix} 1 & 7 & 4 \\ 2 & 0 & 2 \\ 3 & 1 & 0 \end{pmatrix}$

make the cols of A^T be the row of A

Defn: A is called a symmetric matrix if $A^T = A$ (i.e. $a_{ji} = a_{ij}$ for all i,j)

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 3 \end{pmatrix}$ $A^T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 3 \end{pmatrix}$

$A^T = A$ so A is symmetric.

Square matrices:

A matrix is called square if #rows = #cols.

The main diagonal in a square matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & & a_{nn} \end{pmatrix}$$

is $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$.

Powers of square matrices

$A^2 = A \cdot A$

$A^3 = A \cdot A \cdot A$

$\vdots$

Ex: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

Inverse matrices:

If A is a square matrix, an inverse of A is a matrix B such that

$$A \cdot B = I \quad \text{and} \quad B \cdot A = I$$

Fact:
- If there is an inverse matrix of A, it is unique, and it is called A^{-1}. $(A^{-1} = B)$
- If A^{-1} exists, then A is called invertible

$$\boxed{A \text{ invertible} \iff |A| \neq 0}$$

The $n=2$ case: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

1) $|A| = ad-bc$

2) If $ad-bc \neq 0$, then $A^{-1} = \frac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

If $ad-bc = 0$, then A^{-1} does not exist.

Think of $2^{-1} = \frac{1}{2}$. A number a is invertible if $a \neq 0$.

Ex: $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $|A| = 0$ A^{-1} does not exist.

Note: $A^{-1} \cdot B \neq B \cdot A^{-1}$ for a matrix B

Ex: Linear systems

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$\Updownarrow$

$A \cdot \underline{x} = \underline{b}$ (linear system in matrix form)

If A is invertible, then

$$A \cdot \underline{x} = b \implies A^{-1} \cdot (Ax) = A^{-1} \cdot \underline{b}$$

$$I \cdot \underline{x} = A^{-1} \cdot b \qquad \boxed{\underline{x} = A^{-1} \cdot \underline{b}}$$

Ex: $2x + 3y = 8$
$4x - 7y = 12$

$$\begin{pmatrix} 2 & 3 \\ 4 & -7 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 12 \end{pmatrix}$$ matrix form $A\underline{x} = \underline{b}$

Method II:

$$\left(\begin{array}{cc|c} 2 & 3 & 8 \\ 4 & -7 & 12 \end{array}\right)$$

Gaussian elimination

$\vdots$

$$\begin{pmatrix} 2 & 3 \\ 4 & -7 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2 & 3 \\ 4 & -7 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & -7 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 8 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & -7 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 8 \\ 12 \end{pmatrix}$$

$$= \underbrace{\frac{1}{-26} \begin{pmatrix} -7 & -3 \\ -4 & 2 \end{pmatrix}}_{A^{-1}} \cdot \begin{pmatrix} 8 \\ 12 \end{pmatrix}$$

$$= \frac{1}{-26} \cdot \begin{pmatrix} -56 - 36 \\ -32 + 24 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 92/26 \\ 8/26 \end{pmatrix}}}$$

Note: <u>One unique solution</u> when A is invertible
$(\underline{x} = A^{-1} \cdot \underline{b})$

② Determinants

A $\rightsquigarrow$ $\det(A) = |A|$

$n \times n$ matrix (square) — a number

> Note: We can only compute $|A|$ when A is square

Case $n=2$:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \underline{ad - bc}$$

Case $n=3$:

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \rightsquigarrow |A| = aei + bfg + cdh - ceg - afh - bdi$$

$$\left(\begin{array}{ccc|cc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array}\right.$$

This "method" does not work for $n > 3$.
The formula has $n!$ terms, each a product of n entries.

We will describe methods for finding $|A|$:

i) cofactor expansion
ii) Gaussian elimination

i) Cofactor expansion

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & & \vdots \end{pmatrix}$$

cofactor expansion along the first row

$$|A| = a_{11} \cdot C_{11} + a_{12} \cdot C_{12} + a_{13} \cdot C_{13} + \cdots + a_{1n} \cdot C_{1n}$$
$$= a_{11} \cdot (+1) \cdot M_{11} + a_{12} \cdot (-1) \cdot M_{12} + \cdots$$

C_{ij} = cofactor in position (i,j)

$= (-1)^{i+j} \cdot M_{ij}$

Signs:

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

M_{ij} = minor in position (i,j)

= determinant of the matrix you obtain by deleting $\begin{cases} \text{row } i \\ \text{col } j \end{cases}$

Ex:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 1 \cdot (+1) \cdot \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} + 1\,(-1) \cdot \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + \cancel{1(+1) \cdot |}$$
$$+ 1 \cdot (+1) \cdot \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 1 \cdot (18 - 12) - (9 - 4) + (3 - 2)$$
$$= 6 - 5 + 1 = \underline{\underline{2}}$$

Facts: i) You can compute the determinant of any square matrix using cofactor expansion.

ii) You can choose any row or any column; the result will be the same.

Ex:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 1\,(-1)\cdot\begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + 2\cdot(+1)\cdot\begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix}$$

$$+\ 3\cdot(-1)\cdot\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix}$$

$$= -(9-4) + 2\cdot(9-1)$$

$$-3(4-1) = -5+16-9$$

$$= \underline{2}$$

A square matrix is called diagonal is all entries outside the diagonal are zero.

Ex:

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad |A| = 3\cdot(+1)\cdot\begin{vmatrix} -1 & 0 \\ 0 & 2 \end{vmatrix} + 0 + 0$$

$$= 3\,((-1)\cdot(+2) - 0)$$

$$= 3\cdot(-1)\cdot(+2) = \underline{-6}$$

A square matrix is upper triangular if all entries below the diagonal are zero.

lower triangular is similar (all entries above the diagonal are zero)

Ex:

$$A = \begin{pmatrix} 1 & 17 & 8 \\ 0 & -3 & 84 \\ 0 & 0 & 2 \end{pmatrix}$$

Fact: * The determinant of an upper triangular matrix is the product of the entries on the diagonal: $|A| = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$

* All square matrices in echelon form are upper triangular.

$$\begin{vmatrix} 1 & 17 & 8 \\ 0 & -3 & 84 \\ 0 & 0 & 2 \end{vmatrix} = 1 \cdot \begin{vmatrix} -3 & 84 \\ 0 & 2 \end{vmatrix}$$

$$= 1 \cdot \left((-3) \cdot 2 - 0 \cdot 84 \right)$$

$$= 1 \cdot (-3) \cdot 2 = \underline{-6}$$

Method 2: Determinants using Gaussian elimination

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}}_{=A} \begin{matrix} \\ ]\cdot(-1) \\ ]\cdot(-1) \end{matrix} \rightarrow \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{pmatrix}}_{=B} \begin{matrix} \\ \\ ]\cdot(-2) \end{matrix} \rightarrow \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}}_{=E}$$

$|A| = 2$ $\qquad |B| = 1(8-6) = 2$ $\qquad |E| = 2$

Row operations: $A \rightarrow B$	**Effect:**
i) Add a multiple of one row to another row:	$|B| = |A|$
ii) Switch two rows:	$|B| = -|A|$
iii) Multiply a row with $c \neq 0$	$|B| = c \cdot |A|$

Properties of the determinants

i) $\det(A \cdot B) = \det(A) \cdot \det(B)$

ii) $\det(A^t) = \det(A)$

iii) $\det(A^{-1}) = 1/\det(A)$

Ex:

$$\begin{vmatrix} 4 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 6 & -2 & 0 \\ 1 & -1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{vmatrix} \quad -\frac{1}{4} \quad -\frac{1}{4}$$

$$= \begin{vmatrix} 4 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 6 & -2 & 0 \\ 0 & -1 & 2 & 1/4 & 1/4 \\ 0 & 1 & 0 & 1/4 & 1/4 \end{vmatrix}$$

$$= 4 \begin{vmatrix} 2 & 0 & 1 & -1 \\ 0 & 6 & -2 & 0 \\ -1 & 2 & 1/4 & 1/4 \\ 1 & 0 & 1/4 & 1/4 \end{vmatrix} \quad -2 \quad 1$$

$$= 4 \cdot \begin{vmatrix} 0 & 0 & 1/2 & -3/2 \\ 0 & 6 & -2 & 0 \\ 0 & 2 & 1/2 & 1/2 \\ 1 & 0 & 1/4 & 1/4 \end{vmatrix}$$

$$= 4 \cdot 1 \cdot (-1) \cdot \begin{vmatrix} 0 & 1/2 & -3/2 \\ 6 & -2 & 0 \\ 2 & 1/2 & 1/2 \end{vmatrix}$$

$$= -4 \cdot \left(-6 \cdot \left(1/4 + 3/4 \right) + 2(-3) \right)$$

$$= -4 \cdot (-6 - 6) = -4 \cdot (-12) = \underline{\underline{48}}$$

3 Minors and rank

A $m \times n$ matrix

A **minor of order r** is the determinat of an $r \times r$ submatrix of A, i.e. the determinat of a matrix obtained by delete $m-r$ rows, $n-r$ colums.

Ex: $A = \begin{pmatrix} 1 & 2 & 4 \\ 7 & -1 & 3 \end{pmatrix}$

2×3 matrix

Minors of order 2:

$$\begin{vmatrix} 1 & 2 \\ 7 & -1 \end{vmatrix} = -15$$

$$\begin{vmatrix} 1 & 4 \\ 7 & 3 \end{vmatrix} = -25$$

$$\begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix} = 10$$

Minors of order 1:

$$|1| \quad |2| \quad |4|$$
$$|7| \quad |-1| \quad |3|$$

or

$$\begin{matrix} 1 & 2 & 4 \\ 7 & -1 & 3 \end{matrix}$$

Fact: If A is an $m\times n$-matrix, the rank of A is the ~~order of a maximal~~ maximal order of a non-zero minor.

Ex:
$$A = \begin{pmatrix} 1 & 2 & 4 \\ 7 & -1 & 3 \end{pmatrix} \qquad \begin{vmatrix} 1 & 2 \\ 7 & -1 \end{vmatrix} = -15 \neq 0$$
$$\Downarrow$$
$$\text{rk}\, A = 2$$

Recall: The rank of A is also equal to # pivot positions in the echelon form of A

Start with minors of maximal order

Minors and linear systems:

Ex:
$$\begin{aligned} x + y + z + w &= 4 \\ x + 2y + 3z - w &= 2 \\ x + 3y - z &= 1 \end{aligned}$$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -2 & 3 & -1 \\ 1 & 3 & -1 & 0 \end{pmatrix}$$

maximal minors: order 3

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & 3 \\ 1 & 3 & -1 \end{vmatrix} = 1\cdot(2-9) - 1(-1-3) + 1\cdot(3+2)$$
$$= -7 + 4 + 5 = 2 \neq 0$$

$$\Rightarrow \text{rk}\, A = 3$$

Conclusion: Infinitely many solutions, w free variable.

$n=4$ $\text{rk}\,A = 3$ $\Rightarrow$ Degrees of freedom: $4-3=1$

$$x + y + z = 4 - w$$
$$x - 2y + 3z = 2 + w$$
$$x + 3y - z = 1$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 3 \\ 1 & 3 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4-w \\ 2+w \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 3 \\ 1 & 3 & -1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 4-w \\ 2+w \\ 1 \end{pmatrix}$$

Ex: $x + y = 2$ $\nearrow$ $y = 2 - x$, $\searrow$ $x = 2 - y$

When we solve a linear system using minors:

Find a maximal non-zero minor of order $r = \text{rk} A$. Look at the r equations that "go through" this minor (ignore the rest), solve for the variables that are "in" the minor (the rest are free).

Ex:

$$x + y + z = 3$$
$$x - y + 2z = 5$$
$$3x - y + 5z = 13$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & -1 & 5 \end{pmatrix} \qquad \hat{A} = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & 2 & 5 \\ 3 & -1 & 5 & 13 \end{array}\right)$$

$\underline{\mathrm{rk}\, A = \,?}$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & -1 & 5 \end{vmatrix} \begin{matrix} \\ \rfloor^{-1} \\ \rfloor^{-3} \end{matrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & -4 & 2 \end{vmatrix} = 1 \cdot \begin{vmatrix} -2 & 1 \\ -4 & 2 \end{vmatrix} = \underline{0}$$

$\mathrm{rk}\, A < 3$

use equation 1+2
z free

$$\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1 = -2 \neq 0 \Rightarrow \mathrm{rk}\, A = \underline{2}$$

(z free, solve for x and y from equation 1 and 2)

Know that A has two pivot positions, must check that there isn't a 3rd pivot in last column

$\Big\}$

no solutions

$\underline{\mathrm{rk}\, \hat{A} = 2}$

$$\begin{vmatrix} 1 & 1 & 3 \\ 1 & -1 & 5 \\ 3 & -1 & 13 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 1 & 3 \\ 1 & 2 & 5 \\ 3 & 5 & 13 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 1 & 3 \\ -1 & 2 & 5 \\ -1 & 5 & 13 \end{vmatrix} = 0$$

$$\mathrm{rk}\, \hat{A} < 3 \Rightarrow \mathrm{rk}\, \hat{A} = 2$$

$\Rightarrow$ Inf. many solutions, z free variable.

Skipped the computations of the last 3 minors, clearly the method with minors is not optimal in this example.