LECTURE 3 (B)

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GRA 6035 MATHEMATICS

Plan:

1. Vectors
2. Linear independence
3. Rank and linear independence

Readings:

[ME] 10.1 – 10.3 (10.4 – 10.7)
11.1

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1. Vectors

A **vector** is an $m \times 1$-matrix. Also called column vector, m-vector.

Ex:

$$\underline{v} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \quad \text{3-vector}$$

$$\underline{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix} \quad m\text{-vector}$$

Computing with vectors:

- add, subtract vectors (when they have the same size)
- multiply a vector with a scalar

Ex:

$$\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 6 \end{pmatrix} \qquad 3 \cdot \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 12 \end{pmatrix}$$

Linear combinations of vectors:

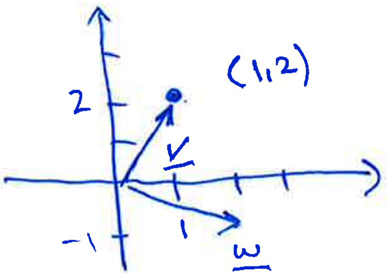
$\underline{v}_1, \underline{v}_2, \underline{v}_3, \ldots, \underline{v}_n$: m-vectors

A linear combination of these vectors:

$$\boxed{c_1 \cdot \underline{v}_1 + c_2 \cdot \underline{v}_2 + c_3 \underline{v}_3 + \ldots + c_n \cdot \underline{v}_n}$$

where $c_1, c_2, \ldots, c_n$ are given numbers.

Geometric interpretation of vectors



$\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

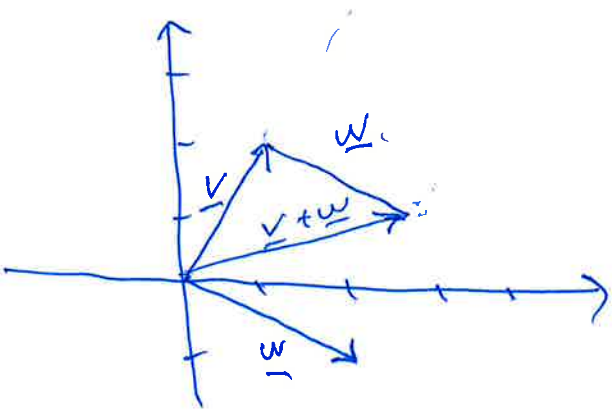
displacement:
"go 1 units along the first axis, then 2 units along (parallel with) the second axis"

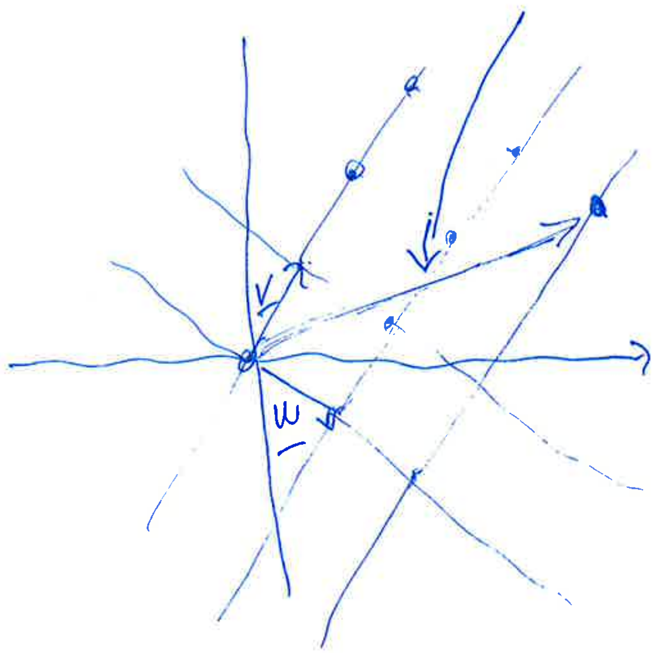
Linear combinations of $\underline{v}$ and $\underline{w}$:

$c_1 \cdot \underline{v} + c_2 \cdot \underline{w}$

$\underline{w} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

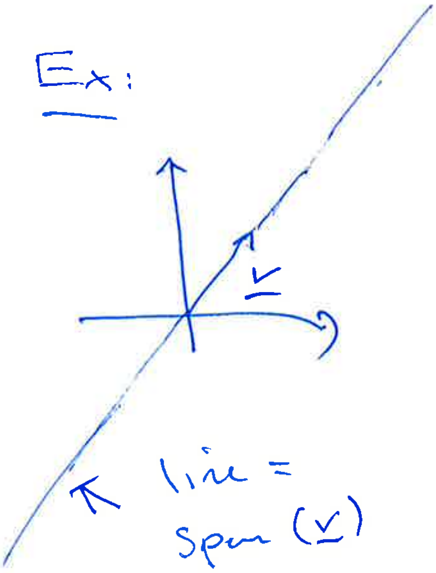
$3\underline{v} + 2\underline{w}$

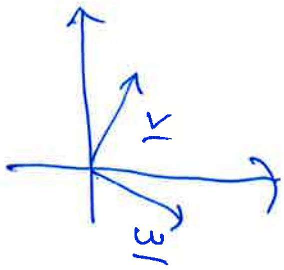




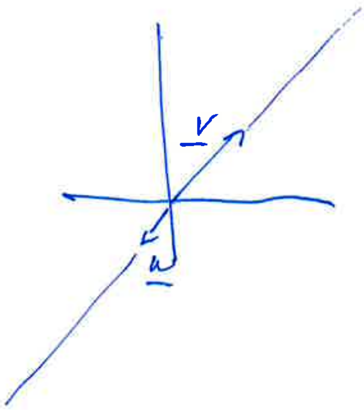
$\text{Span}(\underline{v_1}, \underline{v_2}, \dots, \underline{v_n})$ = all linear combinations $c_1\underline{v_1} + c_2\underline{v_2} + \dots + c_n\underline{v_n}$ of these vectors

Ex:





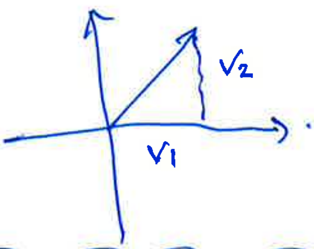
when $\underline{v}$ and $\underline{w}$ do not lie along the same line, Span($\underline{v}, \underline{w}$) = the plane spanned by the vectors



Span($\underline{v}, \underline{w}$) is a line if $\underline{v}$ and $\underline{w}$ have the same or opposite directions

Length of a vector:

$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$



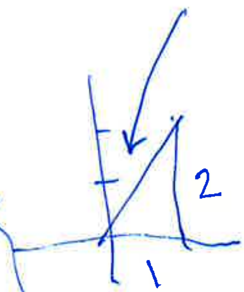
$|\underline{v}| = \sqrt{v_1^2 + v_2^2}$

the length of the vector $\underline{v}$

Ex: $\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

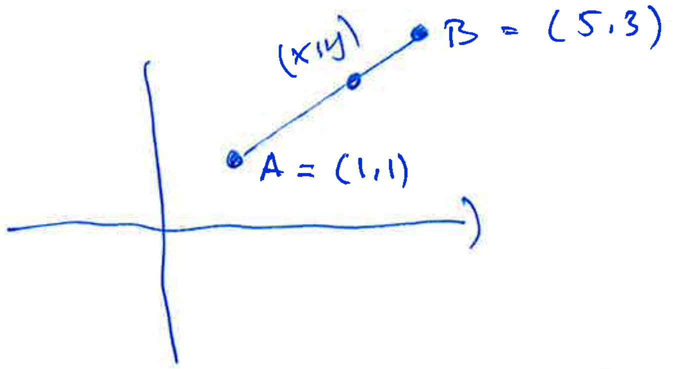
$|\underline{v}| = \sqrt{1^2 + 2^2}$

$= \sqrt{5} \simeq 2.2$



$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} \qquad |\underline{v}| = \sqrt{v_1^2 + v_2^2 + \dots + v_m^2}$$

Ex: Find an expression for the line segment [A,B] from A to B



$\underline{v}_A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\underline{v}_B = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

Vector from A to B:

$\underline{v}_B - \underline{v}_A = \begin{pmatrix} 5 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \underline{v}_A + t \cdot (\underline{v}_B - \underline{v}_A) \qquad t \in [0,1]$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + t \cdot \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 4t \\ 2t \end{pmatrix} = \begin{pmatrix} 1+4t \\ 1+2t \end{pmatrix}$$

Conclusion:

$$\left.\begin{aligned} x &= 1+4t \\ y &= 1+2t \end{aligned}\right\} t \in [0,1]$$

is a parametric description of the line segment [A,B].

$t = 1/2$: $x = 3$, $y = 2$

In general: $A, B \in \mathbb{R}^m$ (m-dim. space) the [A,B] is described by:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \underline{v}_A + t(\underline{v}_B - \underline{v}_A), \quad t \in [0,1]$$

Ex: Is $\begin{pmatrix}5\\3\end{pmatrix}$ a linear combination of $\underline{v_1} = \begin{pmatrix}1\\2\end{pmatrix}$ and $\underline{v_2} = \begin{pmatrix}3\\-1\end{pmatrix}$?

$$\begin{pmatrix}5\\3\end{pmatrix} = x_1 \cdot \begin{pmatrix}1\\2\end{pmatrix} + x_2 \cdot \begin{pmatrix}3\\-1\end{pmatrix}$$

vector equation

$$\begin{pmatrix}5\\3\end{pmatrix} = \begin{pmatrix}x_1\\2x_1\end{pmatrix} + \begin{pmatrix}3x_2\\-x_2\end{pmatrix} = \begin{pmatrix}x_1 + 3x_2\\2x_1 - x_2\end{pmatrix}$$

$$\boxed{\begin{aligned} x_1 + 3x_2 &= 5 \\ 2x_1 - x_2 &= 3 \end{aligned}}$$

linear system

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & -1 & 3 \end{array}\right) \rbrack \cdot(-2) \to \left(\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -7 & -7 \end{array}\right)$$

one solution
||
yes

$$x_1 + 3x_2 = 5$$
$$-7x_2 = -7$$

$$x_1 = 2$$
$$x_2 = 1$$

$$\begin{pmatrix}5\\3\end{pmatrix} = 2 \cdot \begin{pmatrix}1\\2\end{pmatrix} + 1 \cdot \begin{pmatrix}3\\-1\end{pmatrix}$$

Ex: $\underline{v}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix}1\\2\\3\end{pmatrix}$ $\underline{v}_3 = \begin{pmatrix}1\\4\\9\end{pmatrix}$

What are the linear comb. of $\underline{v}_1, \underline{v}_2, \underline{v}_3$?

Let us consider a general vector $\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$: is it a linear combination?

Vector eqn. with unknowns x_1, x_2, x_3 with parameters b_1, b_2, b_3

$$\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = x_1 \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} + x_2 \cdot \begin{pmatrix}1\\2\\3\end{pmatrix} + x_3 \cdot \begin{pmatrix}1\\4\\9\end{pmatrix}$$

$$\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = \begin{pmatrix}x_1 + x_2 + x_3\\ x_1 + 2x_2 + 4x_3\\ x_1 + 3x_2 + 9x_3\end{pmatrix}$$

Linear system with parameters:

Var: x_1, x_2, x_3

Par: b_1, b_2, b_3

Find the solutions (x_1, x_2, x_3) for each given value of (b_1, b_2, b_3)

$$\begin{aligned} x_1 + x_2 + x_3 &= b_1 \\ x_1 + 2x_2 + 4x_3 &= b_2 \\ x_1 + 3x_3 + 9x_3 &= b_3 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 1 & 2 & 4 & b_2 \\ 1 & 3 & 9 & b_3 \end{array}\right) \begin{array}{l} -1 \\ \\ -1 \end{array} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 1 & 3 & b_2 - b_1 \\ 0 & 2 & 8 & b_3 - b_1 \end{array}\right) \begin{array}{l} \\ -2 \\ \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 1 & 3 & b_2 - b_1 \\ 0 & 0 & 2 & b_3 - b_1 - 2(b_2 - b_1) \end{array}\right)$$

One solution

$\Downarrow$

all vectors $\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$ are lin. comb. of $\underline{v}_1, \underline{v}_2, \underline{v}_3$

$\Leftarrow$

Answer:

All 3-vectors

Ex: $\underline{v_1} = \begin{pmatrix}1\\1\\1\end{pmatrix}$ $\underline{v_2} = \begin{pmatrix}1\\2\\0\end{pmatrix}$ $\underline{v_3} = \begin{pmatrix}1\\4\\-2\end{pmatrix}$

What are the linear combinations?

$$x_1 \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} + x_2 \cdot \begin{pmatrix}1\\2\\0\end{pmatrix} + x_3 \cdot \begin{pmatrix}1\\4\\-2\end{pmatrix} = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$$

$$\begin{aligned} x_1 + x_2 + x_3 &= b_1 \\ x_1 + 2x_2 + 4x_3 &= b_2 \\ x_1 \qquad - 2x_3 &= b_3 \end{aligned} \longleftrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 1 & 2 & 4 & b_2 \\ 1 & 0 & -2 & b_3 \end{array}\right)$$

linear system

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 1 & 3 & b_2-b_1 \\ 0 & -1 & -3 & b_3-b_1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 1 & 3 & b_2-b_1 \\ 0 & 0 & 0 & b_3+b_2-2b_1 \end{array}\right)$$

$\boxed{b_3+b_2-2b_1=0}$: inf. many solution (x_3 free variable) (linear comb)

$\boxed{b_3+b_2-2b_1\neq 0}$: no solutions (not linear comb.)

Answer: The 3-vectors $\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$ such that $b_3+b_2-2b_1=0$ are the linear comb of $\underline{v_1}, \underline{v_2}, \underline{v_3}$.

$$\boxed{b_3 = -b_2 + 2b_1}$$

span$\{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ is a plane

$$b_3 = -b_2 + 2b_1 :$$

An example:

$b_1 = 1 \quad b_2 = 2 \quad b_3 = 0$

$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ is a linear comb. of $\underline{v_1}, \underline{v_2}, \underline{v_3}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

x_3 free

$$x_1 + x_2 + x_3 = 1$$
$$x_2 + 3x_3 = 1$$

$$x_2 = 1 - 3x_3$$
$$x_1 = 1 - x_2 - x_3 = 1 - (1-3x_3) - x_3$$
$$= 2x_3$$

$$x_1 = 2x_3$$
$$x_2 = 1 - 3x_3$$
x_3 : free

$x_3 = 1$: $\quad x_1 = 2, \ x_2 = -2, \ x_3 = 1$ $\qquad 2\underline{v_1} - 2\underline{v_2} + \underline{v_3} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

$x_3 = 3$: $\quad x_1 = 6, \ x_2 = -8, \ x_3 = 3$ $\qquad 6\underline{v_1} - 8\underline{v_2} + 3\underline{v_3} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

$$-4\underline{v_1} + 6\underline{v_2} - 2\underline{v_3} = \underline{0}$$

② Linear independence

Let $\underline{v_1}, \underline{v_2}, \underline{v_3}, \ldots, \underline{v_n}$ be given m-vectors.

The vectors $\{\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}\}$ are called <u>linear dependant</u> if one of the vectors (at least) can be written as a linear combination of the others.

<u>Ex:</u>
$$2\underline{v_1} - 2\underline{v_2} + \underline{v_3} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$
$$6\underline{v_1} - 8\underline{v_2} + 3\underline{v_3} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$
$$-4\underline{v_1} + 6\underline{v_2} - 2\underline{v_3} = \underline{0} \qquad \underline{v_1} = \frac{6}{4}\underline{v_2} - \frac{2}{4}\underline{v_3}$$

If $\{\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}\}$ are not linearly dependant, then they are called linearly independant.

At least one of the vectors $\{\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}\}$ is a linear combination of the others $\longrightarrow$ linearly dependant

None of the vectors are linear combinations of the others $\longrightarrow$ linear independant

Ex: $\underline{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ $\underline{v_3} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$ $\longrightarrow$ linearly dependant

$$\underline{v_1} = \frac{6}{4}\underline{v_2} - \frac{2}{4}\underline{v_3}$$

$$\text{Span}\{\underline{v_1}, \underline{v_2}, \underline{v_3}\} = \text{span}\{\underline{v_2}, \underline{v_3}\}$$

$$c_1' \underline{v_1} + c_2 \underline{v_2} + c_3 \underline{v_3} = c_1'\left(\frac{6}{4}\underline{v_2} - \frac{2}{4}\underline{v_3}\right) + c_2 \underline{v_2} + c_3 \underline{v_3}$$

Condition for linear independence:

$\{\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}\}$ linear independant $\Longleftrightarrow$ $x_1\underline{v_1} + x_2\underline{v_2} + \ldots + x_n\underline{v_n} = \underline{0}$ has only the trivial solution $x_1 = x_2 = \ldots = x_n = 0$

linear dependant $\Longleftrightarrow$ the eqn. above have non-trivial solutions

Ex: $\underline{v_1} = \begin{pmatrix}1\\1\\1\end{pmatrix}$ $\underline{v_2} = \begin{pmatrix}1\\2\\3\end{pmatrix}$ $\underline{v_3} = \begin{pmatrix}1\\4\\9\end{pmatrix}$

$$x_1 \cdot \underline{v_1} + x_2 \underline{v_2} + x_3 \underline{v_3} = \underline{0} \iff \left(\begin{array}{ccc|c}1&1&1&0\\1&2&4&0\\1&3&9&0\end{array}\right) \begin{matrix} \\ \cdot(-1) \\ \cdot(-1)\end{matrix}$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c}1&1&1&0\\0&1&3&0\\0&2&8&0\end{array}\right) \begin{matrix} \\ \\ \cdot(-2)\end{matrix}$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c}1&1&1&0\\0&1&3&0\\0&0&2&0\end{array}\right)$$

one solution
$\underline{x_1 = x_2 = x_3 = 0}$

$\{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ are linearly independent

Ex: $\underline{v_1} = \begin{pmatrix}1\\1\\1\end{pmatrix}$ $\underline{v_2} = \begin{pmatrix}1\\2\\0\end{pmatrix}$ $\underline{v_3} = \begin{pmatrix}1\\4\\-2\end{pmatrix}$

$$x_1 \underline{v_1} + x_2 \underline{v_2} + x_3 \underline{v_3} = \underline{0} \iff \left(\begin{array}{ccc|c}1&1&1&0\\1&2&4&0\\1&0&-2&0\end{array}\right) \begin{matrix} \\ \cdot(-1) \\ \cdot(-1)\end{matrix}$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c}1&1&1&0\\0&1&3&0\\0&-1&-3&0\end{array}\right) \begin{matrix} \\ \\ +1\end{matrix}$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c}1&1&1&0\\0&1&3&0\\0&0&0&0\end{array}\right)$$

$\{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ linearly dependent

x_3 free

$x_1 + x_2 + x_3 = 0$

$x_2 + 3x_3 = 0$

x_3 free

$\underline{x_3 = 1}$: $x_2 = -3$
$x_1 = 2$

$$2\underline{v_1} - 3\underline{v_2} + 1 \cdot \underline{v_3} = \underline{0}$$

$$\underline{v_3} = 3\underline{v_2} - 2\underline{v_1}$$

The case $m=n$:

$\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$ m-vectors

$$A = \left(\underline{v}_1 \,\vdots\, \underline{v}_2 \,\vdots\, \underline{v}_3 \,\vdots\, \cdots \,\vdots\, \underline{v}_n \right)$$

$n \times n$-matrix

$x_1 \underline{v}_1 + x_2 \underline{v}_2 + \ldots + x_n \underline{v}_n = \underline{0}$

Linear system with augmented matrix

$$\left(A \mid 0 \right)$$

Matrix form: $A \cdot \underline{x} = \underline{0}$

$|A| \neq 0 \Rightarrow A^{-1}$ exists

$\Downarrow$

$\underline{x} = A^{-1} \cdot \underline{0} = \underline{0}$

Fact:

$|A| \neq 0 \iff \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$ linearly independent

Ex: $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\underline{v}_3 = \begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix}$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 1 \cdot (18-12) - 1 \cdot (9-4) + 1\,(3-2) = 2 \neq 0$$

3 Rank and linear independence

$\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$: m-vectors $\rightsquigarrow$ $A = \left(\underline{v}_1 \mid \underline{v}_2 \mid \cdots \mid \underline{v}_n \right)$

$\operatorname{rk} A = \#$ pivot positions

$n - \operatorname{rk}(A) =$ degrees of freedom

Fact:

$\operatorname{rk} A = n \iff \{v_1, v_2, \ldots v_n\}$ linearly independent

$\operatorname{Rk}(A) =$ maximal number of linearly independent column vectors in A.

$= \dim \operatorname{span} \{\underline{v}_1 \ldots \underline{v}_n\}$

Ex: $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ $\underline{v}_3 = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$ $\rightsquigarrow$ $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 0 & -2 \end{pmatrix}$

pivot positions are marked ⊛

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 0 & -2 \end{pmatrix} \begin{matrix} \\ -1 \\ -1 \end{matrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{pmatrix} \begin{matrix} \\ \\ +1 \end{matrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

Conclusion: $\operatorname{rk} A = 2$ (since there are two pivot positions)

$\Downarrow$

there are ~~[illegible]~~ two linearly independent vectors among $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ (but not three).

We may choose $\{\underline{v}_1, \underline{v}_2\}$ as a set of linearly independent vectors (since pivots in col. 1+2)

Fact:

A: $n \times n$-matrix $\qquad |A| \neq 0 \iff \operatorname{rk} A = n$

Ex: Find the rank of $A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & t \\ 4 & 7-t & -6 \end{pmatrix}$ for all values of t.

$$|A| = \begin{vmatrix} 1 & 3 & 2 \\ 2 & 5 & t \\ 4 & 7-t & -6 \end{vmatrix} = 1 \cdot \left(-30 - t(7-t)\right) - 3 \cdot \left(-12 - 4t\right) + 2\left(2(7-t) - 20\right)$$

$$= -30 - 7t + t^2 + 36 + 12t + 28 - 4t - 40$$

$$= t^2 + t - 6$$

$|A| = 0$

$t^2 + t - 6 = 0$

$t = \dfrac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-6)}}{2}$

$t = \dfrac{-1 \pm \sqrt{25}}{2} = 2, -3$

$t \neq 2, -3$: $\quad |A| \neq 0 \Rightarrow \operatorname{rk} A = n = \underline{3}$

$t = 2$: $\quad |A| = 0 \Rightarrow \operatorname{rk} A < 3$

$\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = 5 - 6 = -1 \neq 0 \Rightarrow \operatorname{rk} A = \underline{2}$

$t = -3$: $\quad |A| = 0 \Rightarrow \operatorname{rk} A < 3$

$\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = -1 \neq 0 \Rightarrow \operatorname{rk} A = \underline{2}$