LECTURE 3 (F)

GRA 6035

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SEP. 8, 2014

MATHEMATICS

Plan:

1. Vectors
2. Linear independence
3. Rank and linear independence

Reading:

[ME] 10.1–10.3, (10.4–10.7), 11.1

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1 Vectors

A vector is an $m \times 1$ matrix (also called column vector, m-vector)

Ex:

$$\underline{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \underline{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix}$$

3-vector $\qquad$ m-vector

Computing with vectors:

- add/subtract vectors (of the same size)
- multiply a vector by a scalar

Ex:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 3 \end{pmatrix}$$

$$2 \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$$

Linear combination of vectors

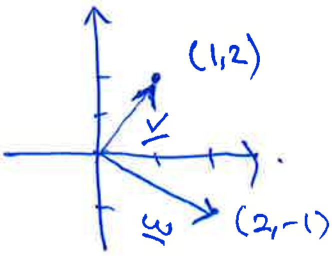
$\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$: m-vectors

A linear combination of $\{\underline{v}_1, \ldots, \underline{v}_n\}$ is an expression

$$c_1 \cdot \underline{v}_1 + c_2 \cdot \underline{v}_2 + \ldots + c_n \cdot \underline{v}_n$$

where $c_1, c_2, \ldots, c_n$ are given numbers.

Geometric interpretation of vectors:



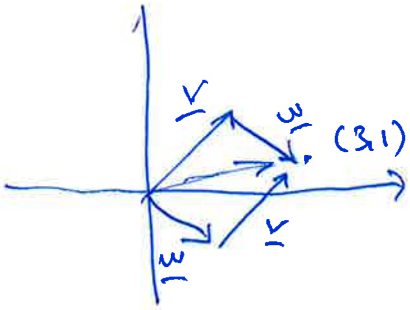
$\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

2-vector

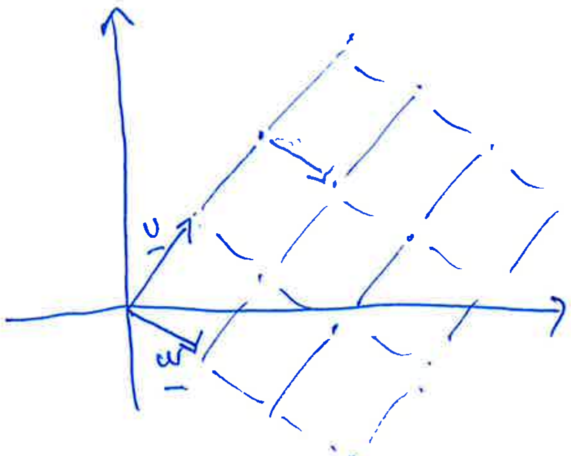
$\underline{w} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

we think of vectors as displacements

(drawn as arrows)



$\underline{v} + \underline{w} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$



$\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\underline{w} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

all vectors are linear comb. of $\underline{v}$ and $\underline{w}$

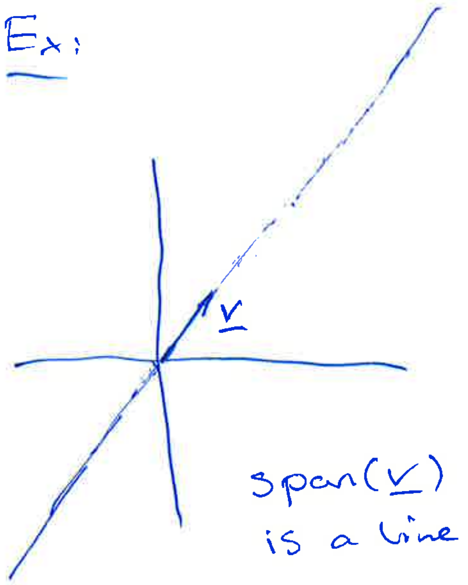
$c_1 \cdot \underline{v} + c_2 \cdot \underline{w}$

Ex: $2\underline{v} + \underline{w}$

$$= 2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1 \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

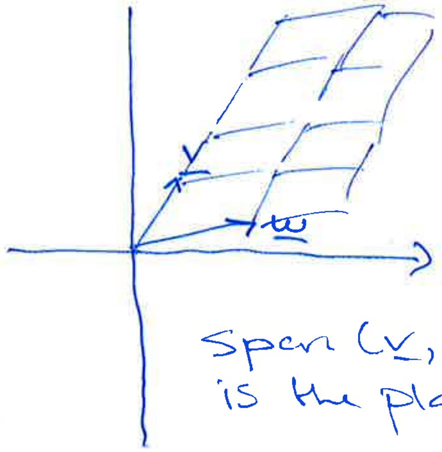
Defn: $\text{span}(\underline{v_1}, \underline{v_2}, \dots, \underline{v_n})$ = all linear combinations
$c_1 \underline{v_1} + c_2 \underline{v_2} + \dots + c_n \underline{v_n}$
of these vectors
($c_1, c_2, \dots, c_n$ numbers)

Ex:



$\text{span}(\underline{v})$
is a line

Ex:

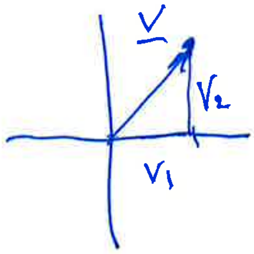


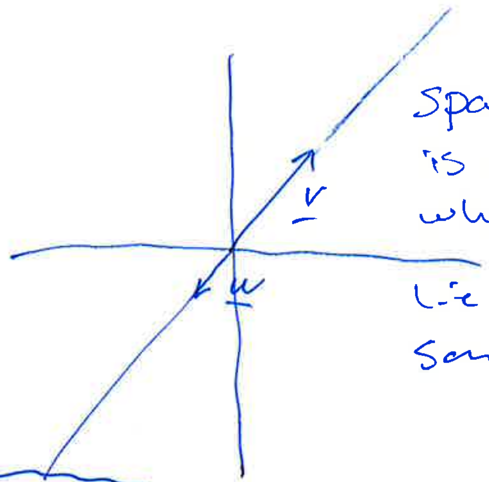
$\text{span}(\underline{v}, \underline{w})$
is the plane $\mathbb{R}^2$

($\mathbb{R}^n$ = n-dimensional
coordinate system)

Length:

$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ $\quad |\underline{v}| = \sqrt{v_1^2 + v_2^2}$

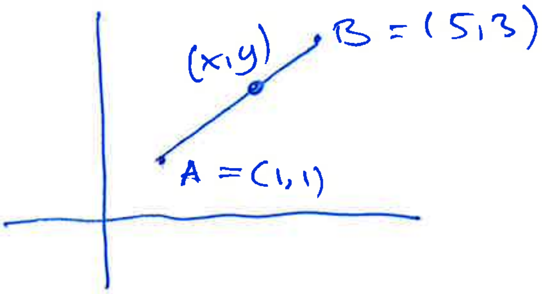




$\text{span}(\underline{v}, \underline{w})$
is a line
when $\underline{v}$ and $\underline{w}$
lie along the
same line

$\underline{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix}$ $\quad |\underline{w}| = \sqrt{w_1^2 + w_2^2 + \dots + w_n^2}$

Ex:



[A,B] = line segment from A to B

$\underline{v}_A = \vec{OA} = \begin{pmatrix}1\\1\end{pmatrix}$

$\underline{v}_B = \vec{OB} = \begin{pmatrix}5\\3\end{pmatrix}$

$$\begin{pmatrix}x\\y\end{pmatrix} = \underbrace{\begin{pmatrix}1\\1\end{pmatrix}}_{= v_A} + t\cdot\underbrace{\left[\begin{pmatrix}5\\3\end{pmatrix} - \begin{pmatrix}1\\1\end{pmatrix}\right]}_{= \underline{v}_B - \underline{v}_A = \vec{AB}} = \begin{pmatrix}1\\1\end{pmatrix} + t\cdot\begin{pmatrix}4\\2\end{pmatrix}, \quad t\in[0,1]$$

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} + t\cdot\begin{pmatrix}4\\2\end{pmatrix} = \begin{pmatrix}1+4t\\1+2t\end{pmatrix}$$

$$\boxed{\left.\begin{aligned} x &= 1+4t \\ y &= 1+2t \end{aligned}\right\} t\in[0,1]}$$

parametric description for the line segment

$$\left.\begin{aligned} x-1 &= 4t \\ y-1 &= 2t \end{aligned}\right\}$$

$$x-1 = 2\cdot(y-1)$$

$$x-1 = 2y-2$$

$$\boxed{x-2y=-1}$$

equation for the line

In general: the line segment [A,B] has parametric description

$$\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix} = \underline{v}_A + t\cdot(\underline{v}_B - \underline{v}_A), \quad t\in[0,1]$$

Ex: Is $\begin{pmatrix}5\\3\end{pmatrix}$ a linear combination of

$\underline{v}_1 = \begin{pmatrix}1\\2\end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix}3\\-1\end{pmatrix}$?

$$\begin{pmatrix}5\\3\end{pmatrix} = x_1 \cdot \begin{pmatrix}1\\2\end{pmatrix} + x_2 \cdot \begin{pmatrix}3\\-1\end{pmatrix} \longleftarrow \text{vector equation}$$

$$\begin{pmatrix}5\\3\end{pmatrix} = \begin{pmatrix}x_1\\2x_1\end{pmatrix} + \begin{pmatrix}3x_2\\-x_2\end{pmatrix} = \begin{pmatrix}x_1+3x_2\\2x_1-x_2\end{pmatrix}$$

$$\begin{aligned} x_1 + 3x_2 &= 5 \\ 2x_1 - x_2 &= 3 \end{aligned} \longleftarrow \text{linear system}$$

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & -1 & 3 \end{array}\right) \cdot(-2) \rightarrow \left(\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -7 & -7 \end{array}\right) \qquad \begin{aligned} x_1 &= 2 \\ x_2 &= 1 \end{aligned}$$

$$\begin{pmatrix}5\\3\end{pmatrix} = 2 \cdot \begin{pmatrix}1\\2\end{pmatrix} + 1 \cdot \begin{pmatrix}3\\-1\end{pmatrix} \qquad \underline{\text{Yes}}$$

Ex: $\underline{v}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix}1\\2\\3\end{pmatrix}$ $\underline{v}_3 = \begin{pmatrix}1\\4\\9\end{pmatrix}$

What are the linear combinations of $\underline{v}_1, \underline{v}_2, \underline{v}_3$?

Start with a general vector $\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$

Vector eqn (= lin.sys) with parameters

x_1, x_2, x_3: var's

b_1, b_2, b_3: parameters

Find the solution (x_1, x_2, x_3) for each given value of b_1, b_2, b_3

$$x_1 \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} + x_2 \cdot \begin{pmatrix}1\\2\\3\end{pmatrix} + x_3 \cdot \begin{pmatrix}1\\4\\9\end{pmatrix} = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$$

$$\begin{pmatrix}x_1 + x_2 + x_3\\x_1 + 2x_2 + 4x_3\\x_1 + 3x_2 + 9x_3\end{pmatrix} = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$$

$$\boxed{\begin{aligned} x_1 + x_2 + x_3 &= b_1 \\ x_1 + 2x_2 + 4x_3 &= b_2 \\ x_1 + 3x_2 + 9x_3 &= b_3 \end{aligned}}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 1 & 2 & 4 & b_2 \\ 1 & 3 & 9 & b_3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 1 & 3 & b_2 - b_1 \\ 0 & 2 & 8 & b_3 - b_1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 1 & 3 & b_2 - b_1 \\ 0 & 0 & 2 & b_3 - b_1 - 2(b_2 - b_1) \end{array}\right)$$

<u>one solution</u> for (x_1, x_2, x_3) for any b_1, b_2, b_3.

$\Downarrow$

All vectors $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ are linear comb. of $\underline{v}_1, \underline{v}_2, \underline{v}_3$

$\Leftarrow$

$$\text{Span}(\underline{v}_1, \underline{v}_2, \underline{v}_3) = \mathbb{R}^3$$

<u>Ex:</u> $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad \underline{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ $\quad \underline{v}_3 = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$

$$x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + x_3 \cdot \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 1 & 2 & 4 & b_2 \\ 1 & 0 & -2 & b_3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 1 & 3 & b_2 - b_1 \\ 0 & -1 & -3 & b_3 - b_1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 1 & 3 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 + b_2 - 2b_1 \end{array}\right)$$

i) <u>$b_3 + b_2 - 2b_1 = 0$</u>: x_3 free, inf. many solutions

ii) <u>$b_3 + b_2 - 2b_1 \neq 0$</u>: no solutions

$$\underline{\text{Span}}(\underline{v}_1, \underline{v}_2, \underline{v}_3) = \left\{ \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} : b_3 = 2b_1 - b_2 \right\}$$

2-dim. space

$b_3 = 2b_1 - b_2 \leftarrow$ vectors $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ in the span $(\underline{v_1}, \underline{v_2}, \underline{v_3})$

$$\left.\begin{array}{l} b_1 = 1 \\ b_2 = 2 \\ b_3 = 0 \end{array}\right\} \qquad \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = x_1 \cdot \underline{v_1} + x_2 \underline{v_2} + x_3 \underline{v_3}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \leftarrow b_1 \\ \leftarrow b_2 - b_1 \\ \leftarrow b_3 + b_2 - 2b_1 \end{array} \qquad \begin{array}{r} x_1 + x_2 + x_3 = 1 \\ x_2 + 3x_3 = 1 \end{array}$$

$x_3 = \underline{1}$ $\qquad$ $x_3 = \underline{2}$

$x_2 = 1 - 3x_3 = \underline{-2}$ $\qquad$ $x_2 = \underline{-5}$

$x_1 = 1 - x_2 - x_3 = \underline{2}$ $\qquad$ $x_1 = 4$

$$2\underline{v_1} - 2\underline{v_2} + 1 \cdot \underline{v_3} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$4\underline{v_1} - 5\underline{v_2} + 2\underline{v_3} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$-2\underline{v_1} + 3\underline{v_2} - \underline{v_3} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\boxed{\underline{v_3} = -2\underline{v_1} + 3\underline{v_2}}$$

$$\text{Span}\{\underline{v_1}, \underline{v_2}, \underline{v_3}\} = \text{span}\{\underline{v_1}, \underline{v_2}\}$$

$$c_1\underline{v_1} + c_2\underline{v_2} + c_3\underline{v_3}$$

$$= c_1\underline{v_1} + c_2\underline{v_2} + c_3 \cdot (-2\underline{v_1} + 3\underline{v_2})$$

(2) Linear independence

$\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}$: m-vectors

Defn:

$\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}\}$ is linearly dependent if at least one of the vectors is a linear comb. of the others

— " — is linearly independent otherwise.

Ex: $\underline{v_1} = \begin{pmatrix}1\\1\\1\end{pmatrix}$ $\underline{v_2} = \begin{pmatrix}1\\2\\3\end{pmatrix}$ $\underline{v_3} = \begin{pmatrix}1\\4\\9\end{pmatrix}$: linearly independent

Ex: $\underline{v_1} = \begin{pmatrix}1\\1\\1\end{pmatrix}$ $\underline{v_2} = \begin{pmatrix}1\\2\\0\end{pmatrix}$ $\underline{v_3} = \begin{pmatrix}1\\4\\-2\end{pmatrix}$: linearly dependent

Method to decide if $\{\underline{v_1}, \dots, \underline{v_n}\}$ are linearly independent:

$$x_1 \cdot \underline{v_1} + x_2 \cdot \underline{v_2} + \dots + x_n \cdot \underline{v_n} = \underline{0} \quad \longleftarrow \text{ Solve this linear system}$$

One solution

$x_1 = 0, x_2 = 0, \dots, x_n = 0$ (trivial solution)

$\Downarrow$

$\{\underline{v_1}, \dots, \underline{v_n}\}$ are linearly independent

infinitely many solutions

there are non-trivial solutions

$\Downarrow$

$\{\underline{v_1}, \dots, \underline{v_n}\}$ are linearly dependent

Ex: $\underline{v_3} = -2\underline{v_1} + 3\underline{v_2}$

$\underline{0} = -2\underline{v_1} + 3\underline{v_2} - \underline{v_3}$ $\quad x_1 = -2 \quad x_2 = 3 \quad x_3 = 1$

(non-trivial solution)

Ex: $\underline{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \underline{v_2} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad \underline{v_3} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$

$x_1 \cdot \underline{v_1} + x_2 \underline{v_2} + x_3 \underline{v_3} = \underline{0}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 0 \\ 1 & 0 & -2 & 0 \end{array}\right) \begin{matrix} \\ ]\cdot 1 \\ ]-1 \end{matrix}$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -1 & -3 & 0 \end{array}\right) \begin{matrix} \\ \\ ]\cdot 1 \end{matrix}$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

x_3 free $\Rightarrow$ inf. many solution

$\Downarrow$

non-trivial solutions

Conclusion:

$\{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ linearly dependent

$x_3 = 1$
$x_2 = -3$
$x_1 = 2$

(x_3 free)

$2\underline{v_1} - 3\underline{v_2} + \underline{v_3} = 0$

$\underline{v_3} = -2\underline{v_1} + 3\underline{v_2}$

The case $m=n$:

$\underline{A}\cdot\underline{x}=\underline{0}$ with A square

$|A|\neq 0$:

$A\underline{x}=\underline{0}$

$A^{-1}\cdot A\underline{x}=A^{-1}\cdot\underline{0}$

$\underline{x}=\underline{0}$

only trivial solution

$|A|=0$

$A\underline{x}=0$ has non-trivial solutions

$\underline{v_1},\underline{v_2},\ldots,\underline{v_n}$: m-vectors

$x_1\underline{v_1}+x_2\underline{v_2}+\ldots+x_n\underline{v_n}=\underline{0}$

$A=\left(\underline{v_1}\,|\,\underline{v_2}\,|\cdots|\,\underline{v_n}\right)$

Lin.sys.: $\left(A\,|\,\underline{0}\right)$

$A\cdot\underline{x}=\underline{0}$

Fact: $\underline{v_1},\ldots,\underline{v_n}$: n-vectors $\quad A=\left(\underline{v_1}\,|\,\underline{v_2}\,|\ldots|\,\underline{v_n}\right)$

$|A|\neq 0 \iff \{\underline{v_1},\ldots,\underline{v_n}\}$ linearly independent

Ex: $\begin{vmatrix}1&1&1\\1&2&4\\1&3&9\end{vmatrix}=2\neq 0 \quad \underline{v_1}=\begin{pmatrix}1\\1\\1\end{pmatrix} \quad \underline{v_2}=\begin{pmatrix}1\\2\\3\end{pmatrix} \quad \underline{v_3}=\begin{pmatrix}1\\4\\9\end{pmatrix}$

$\Rightarrow \{\underline{v_1},\underline{v_2},\underline{v_3}\}$ lin. independent

3 Rank and linear independence

$\{\underline{v_1},\ldots,\underline{v_n}\}$: m-vectors $\rightsquigarrow$ $A=\left(\underline{v_1}\,|\,\underline{v_2}\,|\ldots|\,\underline{v_n}\right)$

$x_1\underline{v_1}+\ldots+x_n\underline{v_n}=\underline{0}$ $\quad A\cdot\underline{x}=0$

m×n linear system

rk A

Compute rk A

Remember: Degrees of freedom $=$ $n-\text{rk}\,A$

$$m\left\{\left(\begin{array}{c|c} \quad A \quad & \begin{matrix}0\\0\\\vdots\\0\end{matrix}\end{array}\right)\right.$$
$\quad\; n$

$n-\text{rk}\,A=$ # cols without pivots

$=$ degrees of freedom

Fact: $\underline{v_1},\dots,\underline{v_n}$: m-vectors $\rightsquigarrow$ $A = (\underline{v_1} | \dots | \underline{v_n})$

$\operatorname{rk} A = n \iff \{\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}\}$ lin. independent

$\operatorname{rk}(A)$ = maximal number of linearly independent vectors among $\{\underline{v_1}, \dots, \underline{v_n}\}$

Ex: $\underline{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ $\underline{v_3} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$ $\rightsquigarrow$ $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 0 & -2 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 0 & -2 \end{pmatrix} \begin{matrix} \\ -1 \\ -1 \end{matrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{pmatrix} \begin{matrix} \\ \\ +1 \end{matrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$\operatorname{rk} A = \underline{2}$

maximal number of linearly independent vectors among $\{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ is $\underline{2}$

the vectors $\{\underline{v_1}, \underline{v_2}\}$ are linearly independent (pivots in cols $\underline{1,2}$)

(we know that from earlier $\underline{v_3} = -2\underline{v_1} + 3\underline{v_2}$)

$$\operatorname{rk} A = \left\{ \begin{array}{l} \text{dimension of} \\ \operatorname{span}\{\underline{v_1}, \dots, \underline{v_n}\} \end{array} \right.$$

Fact: A $n \times n$-matrix

$|A| \neq 0 \iff \operatorname{rk} A = n$

Ex: Find $\operatorname{rk} \begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & t \\ 4 & 7-t & -6 \end{pmatrix}$:

$$\begin{vmatrix} 1 & 3 & 2 \\ 2 & 5 & t \\ 4 & 7-t & -6 \end{vmatrix} = 1 \cdot (-30 - t(7-t)) - 3(-12 - 4t) + 2 \cdot (2(7-t) - 20)$$
$$= t^2 + t - 6$$

$\operatorname{rk} < 3$ when $t^2 + t - 6 = 0$
$\quad t = \underline{2, -3}$

$\operatorname{rk} = 3$ when $\underline{t \neq 2, -3}$

since $\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = 5 - 6 = -1 \neq 0$, $\operatorname{rk} \geq 2$

so

$$\operatorname{rk} = \begin{cases} 3, & t \neq 2, -3 \\ 2, & t = 2, -3 \end{cases}$$