LECTURE 7 (B)

GRA 6035

MATHEMATICS

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Oct. 2, 2014

Plan:

1. Constrained optimization
2. Admissible sets
3. Lagrange problems
4. Kuhn-Tucker problems

Reading:

[ME] 18.1-18.7, (12.3-12.5), 21.1

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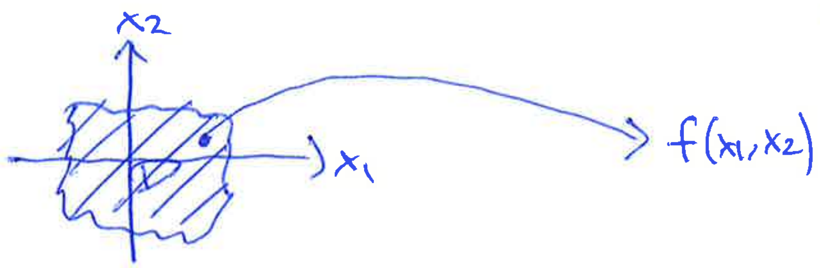
1 Constrained optimization

max/min $f(\underline{x}) = f(x_1, x_2, \ldots, x_n)$ when $\underline{x}$ satisfies certain constraints

$\mathbb{R}^2$



D = set of admissible points (points that satisfies all constraints)

subset of $\mathbb{R}^2$

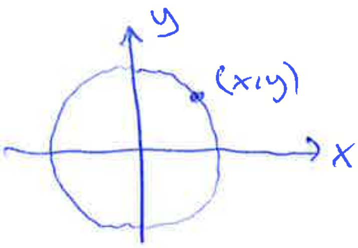
(example in two variables)

Ex: max/min $\underbrace{x+3y}_{f(x,y)}$ when $x^2+y^2=10$

Ex: max/min $\underbrace{x^2+3y^2}_{f(x,y)}$ when $\begin{cases} x \geq 0 \\ y \geq 0 \\ 2x+3y \leq 30 \end{cases}$

Ex: Constraint $x^2+y^2=10$

objective function: $f(x,y) = x+3y$



circle with radius $\sqrt{10}$ and center in $(0,0)$

In general, $(x-x_0)^2+(y-y_0)^2=r^2$ is a circle with radius r and center (x_0,y_0).

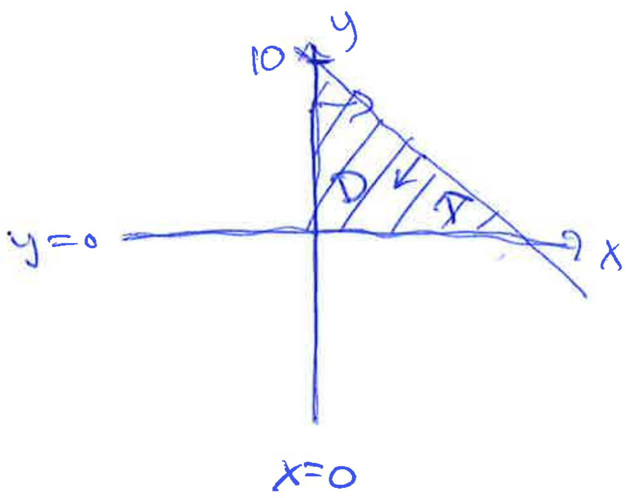
Ex: Constraints $\begin{cases} x\geq 0 \\ y \geq 0 \\ 2x+3y\leq 30 \end{cases}$ $\longrightarrow$ $x=0$

$y=0$

$2x+3y=30$

$3y=30-2x$

$y=10-\frac{2}{3}x$



a) Lagrange problems: Equality constraints $(=)$

b) Kuhn-Tucker problems: Closed inequality constraints ($\leq$ or $\geq$)

(2) The set of admissible points

$f(\underline{x}) = f(x_1, \dots, x_n)$: objective function in n variables

D : set of admissible points = all points that satisfies all constraints

D is a subset of $\mathbb{R}^n$.

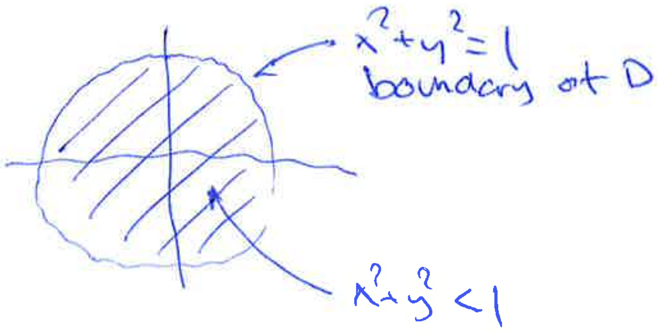
Defn:

The set D is <u>closed</u> if all boundary points are included in the set D.

The set D is <u>open</u> if no ~~all~~ boundary points are included in the set D.

A <u>boundary point</u> is a point such that any disc around the point contains both points in D and outside of D.

Ex: $D = \{(x,y) : x^2+y^2 \le 1\}$



boundary of D (= circle) is included in D

$\Downarrow$

D is <u>closed</u>

Note:

Boundary of D: replace inequalities by equality

Closed sets: Given by constraints with $=$ $\le$ $\ge$

Open sets: " " $<$ $>$

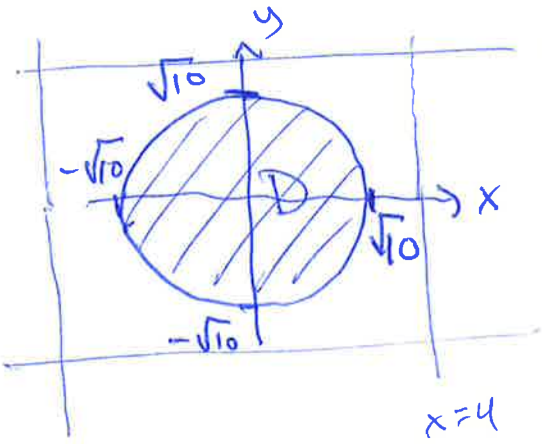
The set D is <u>bounded</u> if there is a finite "box" that contains all points in D, i.e. that there are (finite) numbers

$$a_1 \le b_1$$
$$a_2 \le b_2$$
$$\vdots$$
$$a_n \le b_n$$

such that $a_1 \le x_1 \le b_1,\ a_2 \le x_2 \le b_2, \dots, a_n \le x_n \le b_n$ for all points $(x_1, \dots, x_n)$ in D.

<u>Ex:</u>

$D = \{(x,y) : x^2+y^2 \le 10\}$

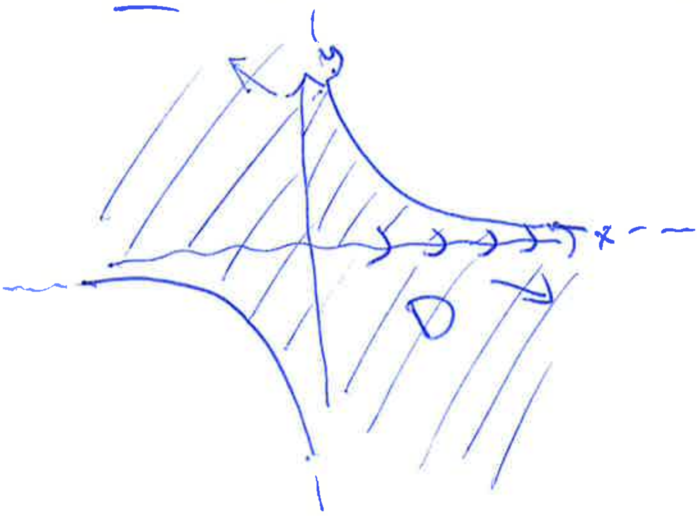


$$\boxed{\begin{matrix} -4 \le x \le 4 \\ -4 \le y \le 4 \end{matrix}}$$

rectangle = finite box that contains D

<u>D is bounded.</u>

<u>Ex:</u> $D = \{(x,y) : xy \le 1\}$



Boundary: $xy = 1$
$y = 1/x$

$xy < 1$: $y < \frac{1}{x}$ if $x > 0$
$y > \frac{1}{x}$ if $x < 0$

<u>D is not bounded</u>

$\underbrace{x = a,\ y = 1/a}_{\text{in } D}$ for all a

Thm (Weierstrass' extreme value thm)

If f is a continuous function defined on a set D that is closed and bounded, then f has a global min and a global max.

In other words,

$$\max / \min f(\underline{x}) \quad \text{when } \underline{x} \text{ is in } D$$

has a solution if D is $\underbrace{\underline{\text{closed and bounded}}}_{\text{compact}}$

Note:

In a Lagrange / Kuhn-Tucker problem, the set D of admissible points is closed.

To check if D is bounded:

* Sketch the set D
* Consider the equations / inequalities

} Is there a biggest and smallest value for each of the var's in the set D.

Ex: $x^2+y^2+z^2 \le 16$ $\qquad x^2+y^2-z^2 \le 16$ $\qquad x^3+y^3+z^3 \le 8$

$$\left.\begin{array}{l} -4 \le x \le 4 \\ -4 \le y \le 4 \\ -4 \le z \le 4 \end{array}\right\}$$ ~~bounded~~ bounded

$x=a, y=0, z=a$ is in D for all a

not bounded

$x=a, y=a, z=a$ is in D when $a<0$

not bounded

③ Lagrange problems

Std. form

$$\max/\min \; f(x_1, \ldots, x_n) \quad \text{when} \quad \begin{cases} g_1(x_1, \ldots, x_n) = a_1 \\ g_2(x_1, \ldots, x_n) = a_2 \\ \vdots \\ g_m(x_1, \ldots, x_n) = a_m \end{cases}$$

Ex: max/min $\underbrace{x + 3y}_{f(x,y)}$ when $\underbrace{x^2 + y^2}_{g(x,y)} = \underbrace{10}_{a}$

Can also write the constraint as $\underbrace{x^2+y^2-10}_{g(x,y)} = 0$

We form the <u>Lagrangian function</u>

$$\mathcal{L}(x, y, \lambda) = f(x,y) - \lambda \cdot g(x,y)$$

$$= x + 3y - \lambda \cdot (x^2 + y^2)$$

FOC:

$$\mathcal{L}'_x = \boxed{1 - \lambda \cdot 2x = 0} \qquad x = \frac{1}{2\lambda} \quad (\lambda \neq 0)$$

$$\mathcal{L}'_y = \boxed{3 - \lambda \cdot 2y = 0} \qquad y = \frac{3}{2\lambda}$$

C:

$$\boxed{x^2 + y^2 = 10}$$

$$\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{3}{2\lambda}\right)^2 = 10$$

$$\frac{1}{4\lambda^2} + \frac{9}{4\lambda^2} = \frac{10}{4\lambda^2} = 10 \quad | \cdot 4\lambda^2$$

$$10 = 10 \cdot 4\lambda^2 \quad | : 40$$

$$\lambda^2 = 1/4$$

$$\lambda = \pm 1/2$$

Candidates for max/min:

$\lambda = 1/2$: $x = 1, y = 3, \lambda = 1/2$ — $f = 10$ max?

$\lambda = -1/2$: $x = -1, y = -3, \lambda = -1/2$ — min? $f = -10$

General method for solving Lagrange problems:

<u>Problem</u>: max/min $f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) = a_1 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$

① Form the Lagrangian function

$$\mathcal{L}(\underline{x}\,;\underline{\lambda}) = f(\underline{x}) - \lambda_1 \cdot g_1(\underline{x}) - \lambda_2 \cdot g_2(\underline{x}) - \dots - \lambda_m \cdot g_m(x)$$

$$\parallel$$

$$\mathcal{L}(x_1, x_2, \dots, x_n; \lambda_1, \dots, \lambda_m)$$

and compute its partial derivatives w.r.t. $x_1, x_2, \dots, x_n$.

② Consider the <u>Lagrange conditions</u>: $\parallel$ FOC+C

<u>FOC</u>: $\boxed{\begin{array}{l} \mathcal{L}'_{x_1} = 0 \\ \mathcal{L}'_{x_2} = 0 \\ \vdots \\ \mathcal{L}'_{x_n} = 0 \end{array}}$ <u>C</u>: $\boxed{\begin{array}{l} g_1(\underline{x}) = a_1 \\ g_2(\underline{x}) = a_2 \\ \vdots \\ g_m(\underline{x}) = a_m \end{array}}$

③ <u>Solve the Lagrange conditions</u>:

$(n+m)$ equations in $(n+m)$ variables

Result:

If $\underline{x}^* = (x_1^*, x_2^*, \dots, x_n^*)$ is a solution the Lagrange problem (max/min), then either

i) there are unique Lagrange multipliers $\lambda_1^*, \dots, \lambda_m^*$ such that $(x_1^*, \dots, x_n^*; \lambda_1^*, \dots, \lambda_m^*)$ is a solution to FOC+C (Lagrange conditions).

or

ii) $\underline{x}^*$ fails to satisfy NDCQ (non-degenerate constraint qualification).

$$\boxed{\underline{x}^* \text{ max/min} \implies \begin{cases} (\underline{x}^*, \lambda^*) \text{ satisfies FOC+C} \\ \text{or} \\ \text{NDCQ fails at } \underline{x}^* \end{cases}}$$

Conclusion:

When you solve FOC+C, you get candidates for max/min.

Ex: max/min $x+3y$ when $x^2+y^2 = 10$

Solutions to FOC+C: $(1,3;\tfrac{1}{2})$, $(-1,-3;-\tfrac{1}{2})$
$f=10$ $\quad f=-10$

We know that D is bounded $\implies$ there is max/min.
NDCQ holds at all points in D $\implies$ max/min must be among pts with FOC+C.
$(1,3)$ is max $\quad (-1,-3)$ is min $\Leftarrow$

NDCQ:

$$\mathrm{rk}\begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \vdots & & & \\ \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_n} \end{pmatrix} = m$$

Ex: $x^2+y^2=10$ $\quad g(x,y)=x^2+y^2$

NDCQ: $\mathrm{rk}\begin{pmatrix} 2x & 2y \end{pmatrix} = 1$

NDCQ fails: $\mathrm{rk}\,(2x \;\; 2y) = 0$

$$\Downarrow$$

$$2x=0,\ 2y=0$$

$$\Downarrow$$

$x=0, y=0$ not admissible

So NDCQ holds for all pts in D.

Ex: ~~[illegible]~~

max $f(x,y) = y$ when $x^2+y^3=0$

$$\mathcal{L} = y - \lambda\cdot(x^2+y^3)$$

FOC: $\begin{cases} \mathcal{L}'_x = -\lambda\cdot 2x = 0 \\ \mathcal{L}'_y = 1-\lambda\cdot 3y^2 = 0 \end{cases}$

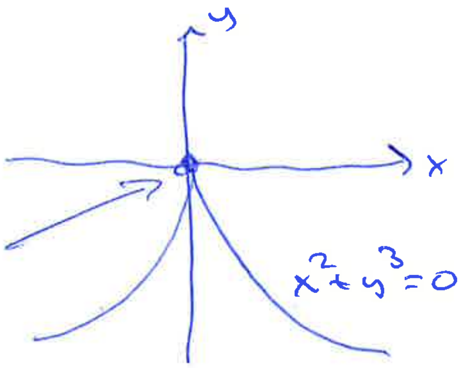
C: $x^2+y^3 = 0$

$\lambda = 0$ or $x=0$

$\lambda = 0$	$x=0$
$1=0$ not possible	$y=0$ $1=0$ not possible

no sol's to FOC+C

max



$\underline{x}^*$ max $\Rightarrow$ $\underline{x}^*$ satisfy FOC+C

or

$\underline{x}^*$ fails NDCQ.

NDCQ: $\operatorname{rk}\begin{pmatrix} 2x & 3y^2\end{pmatrix} = 1$

NDCQ fails: $\operatorname{rk}\begin{pmatrix} 2x & 3y^2\end{pmatrix} = 0$

$\Downarrow$

$2x = 0 \quad 3y^2=0$

$x=0, \quad y=0$

NDCQ fails at (0,0) $\leftarrow$ this is the max.

Lagrange method:

1. Write down and solve Lagrange conditions (FOC + C). These points are candidates for max/min.

2. Find all pts where NDCQ fails (if any). These points are also candidates for max/min.

3. Compute $f(\underline{x})$ for each candidate point from 1 and 2 $\longrightarrow$ find the best candidate for max and min.

④ Kuhn - Tucker problems

$$\max f(x_1, \ldots, x_n) \quad \text{when} \quad \begin{cases} g_1(x_1, \ldots, x_n) \leq a_1 \\ \vdots \\ g_m(x_1, \ldots, x_n) \leq a_m \end{cases}$$

std. form

Note:

* Any min-problem can be made into a max-problem by replacing f with $-f$

* Any constraint given by $g(\underline{x}) \geq a$ can be replaced by $-g(\underline{x}) \leq -a$

Ex: $\min x + 3y$ when $x^2 + y^2 \leq 10$

$= \max -(x+3y)$ when $x^2 + y^2 \leq 10$

Method for solving Kuhn-Tucker problem:

① Make sure that the problem is in std. form and form the Lagrangian

$$\mathcal{L}(\underline{x}; \underline{\lambda}) = f(\underline{x}) - \lambda_1 \cdot g_1(\underline{x}) - \lambda_2 g_2(\underline{x}) - \cdots - \lambda_m g_m(\underline{x})$$

② Consider the Kuhn-Tucker conditions:

FOC: $\boxed{\begin{matrix} \mathcal{L}'_{x_1} = 0 \\ \mathcal{L}'_{x_2} = 0 \\ \vdots \\ \mathcal{L}'_{x_n} = 0 \end{matrix}}$

C: $\boxed{\begin{matrix} g_1(\underline{x}) \leq a_1 \\ g_2(\underline{x}) \leq a_2 \\ \vdots \\ g_m(\underline{x}) \leq a_m \end{matrix}}$

CSC: Complementary Slackness Conditions

(see next page)

CSC:

$$\lambda_1 \geq 0 \quad \text{and} \quad \lambda_1 \cdot (g_1(\underline{x}) - a_1) = 0$$
$$\lambda_2 \geq 0 \qquad\qquad \lambda_2 \cdot (g_2(\underline{x}) - a_2) = 0$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$\lambda_m \geq 0 \qquad\qquad \lambda_m (g_m(\underline{x}) - a_m) = 0$$

(conditions based on the fact that the problem is in std. form)

All $\lambda_i \geq 0$, and when $\lambda_i > 0$ then $g_i(x) = a_i$ must hold by equality.

Result:

If $\underline{x}^*$ is max in a Kuhn-Tucker problem, then either

i) $(\underline{x}^*; \underline{\lambda}^*)$ satisfy the Kuhn-Tucker conditions FOC + C + CSC for a unique $\underline{\lambda}^*$

or

ii) NDCQ fails at $\underline{x}^*$

NDCQ for Kuhn-Tucker problems:

A point $\underline{x}$ in D satisfies the constraints

$g_i(\underline{x}) \leq a \iff g_i(\underline{x}) = a$ or $g_i(\underline{x}) < a$

constraint i is <u>binding</u> (for $g_i(\underline{x}) = a$)

constraint i is <u>non-binding</u> (for $g_i(\underline{x}) < a$)

<u>NDCQ:</u>

$$\text{rk}\begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} \end{pmatrix} = \#\text{ binding constraints at } \underline{x}$$

include only rows corresponding to constraints that are binding at $\underline{x}$

<u>Ex:</u> $x^2+y^2 \leq 10 \iff$ a) $x^2+y^2 = 10$ or b) $x^2+y^2 < 10$

<u>Case a)</u>: NDCQ: $\text{rk}\,(2x \quad 2y) = 1$ ok ✓

<u>Case b)</u>: NDCQ: no condition ok ✓

Ex: $\boxed{\max\ x+3y \quad \text{when} \quad x^2+y^2 \le 10}$ (std form)

List of candidates for max:

$\boxed{\text{Solutions of Kuhn-Tucker conditions}}$ + $\boxed{\text{NDCQ fails}}$

Kuhn-Tucker conditions:

$$\mathcal{L} = f(x,y) - \lambda \cdot g(x,y)$$
$$= x + 3y - \lambda \cdot (x^2+y^2)$$

FOC: $\mathcal{L}'_x = 0$ $\quad$ $\boxed{1 - \lambda \cdot 2x = 0}$

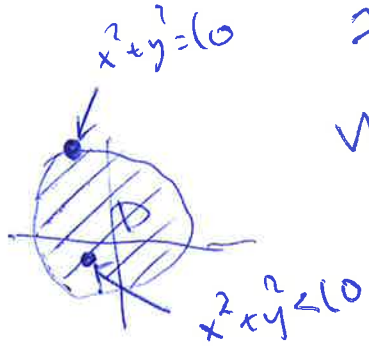
$\mathcal{L}'_y = 0$ $\quad$ $\boxed{3 - \lambda \cdot 2y = 0}$

C: $\boxed{x^2+y^2 \le 10}$

CSC: Complementary slackness conditions

$$\boxed{\lambda \ge 0 \quad \text{and} \quad \lambda \cdot (\underbrace{x^2+y^2-10}_{g(x,y)-a}) = 0}$$

$\lambda \ge 0$ and $(\lambda = 0$ if $x^2+y^2 < 10)$



We say that the constraint is binding if $x^2+y^2=10$ and non-binding if $x^2+y^2<10$.

Kuhn-Tucker Conditions: $\boxed{FOC} + \boxed{C} + \boxed{CSC}$

$\mathcal{L} = f - \lambda \cdot g$

$$FOC \begin{cases} 1 - \lambda \cdot 2x = 0 \\ 3 - \lambda \cdot 2y = 0 \end{cases}$$

$$C \ \{ \ x^2 + y^2 \leq 10$$

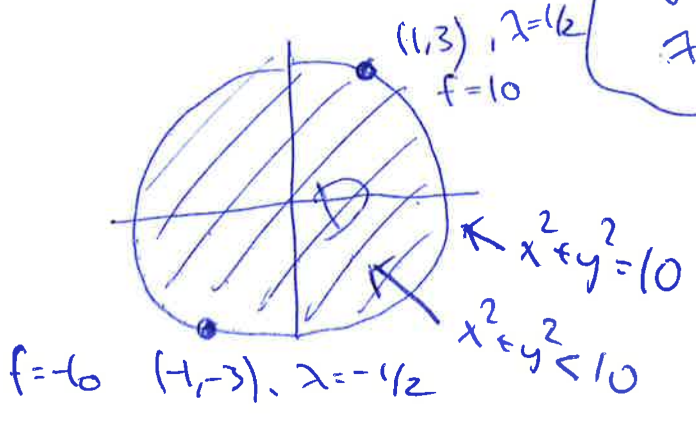
$$CSC \begin{cases} \lambda \geq 0 \\ \lambda = 0 \ \text{ if } x^2+y^2 < 10 \end{cases}$$

$\left.\begin{array}{l} f'_x = 0 \\ f'_y = 0 \end{array}\right\}$ inside the circle

or

a) $x^2+y^2=10$	b) $x^2+y^2<10$
$\left.\begin{array}{l} 1-\lambda\cdot 2x=0 \\ 3-\lambda\cdot 2y=0 \\ x^2+y^2=10 \end{array}\right\}$ $\lambda \geq 0$	$1-\lambda\cdot 2x=0$ $3-\lambda\cdot 2y=0$ $x^2+y^2<10$ $\lambda=0$
$(x,y;\lambda) = (1,3;\ 1/2)$ ~~$(-1,-3,-1/2)$~~	$1=0$ no soln.

One candidate: $\left.\begin{array}{l} x=1 \\ y=3 \\ \lambda=1/2 \end{array}\right\} f=10$



On the circle:

$\lambda \geq 0$: max

$\lambda \leq 0$: min

Inside: