LECTURE 7 (F)

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OCT 6, 2014

GRA 6035 MATHEMATICS

Plan:

1. Constrained optimization
2. Admissible sets
3. Lagrange problems
4. Kuhn-Tucker problems

Reading:

[ME] 18.1–18.7, (12.3–12.5), 21.1

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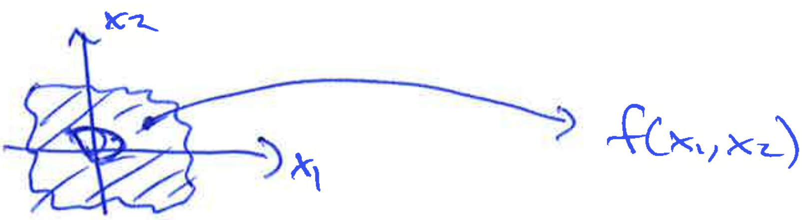
(1) Constrained optimization

max/min $f(x_1, x_2, \dots, x_n)$ when $\underline{x}$ satisfies certain constraints

$\quad\quad\quad\quad\quad\quad \shortparallel$

$\quad\quad\quad\quad\quad f(\underline{x})$



D = set of admissible pts

(pts that satisfy all constraints)

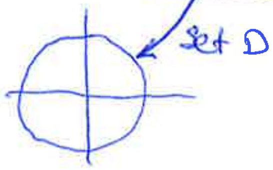
D: subset of $\mathbb{R}^n$

Both the function f and the set D will affect the constrained optimization problem

Ex:

max/min $x+3y$ $\quad$ ($= f(x,y)$)

when $x^2+y^2=10$



Ex:

max/min x^2+3y^2 $\quad$ ($= f(x,y)$)

when $x \geq 0, y \geq 0, 2x+3y \leq 30$

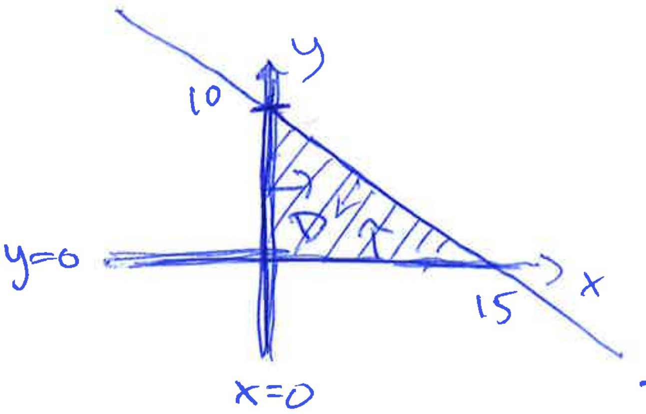
a) Lagrange problems : Equality constraints $(=)$

b) Kuhn-Tucker problems : Closed inequality constraints $(\leq\ \geq)$

2 The set of admissible points

Ex: ~~x ≥ 0 ...~~

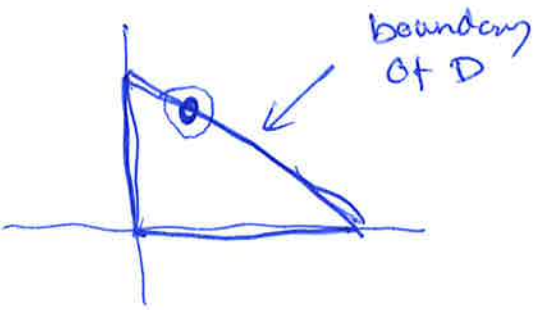
$$\begin{cases} x \geq 0 \\ y \geq 0 \\ 2x+3y \leq 30 \end{cases} \rightsquigarrow \boxed{\begin{aligned} x&=0 \\ y&=0 \\ 2x+3y&=30 \end{aligned}}$$



$$2x+3y=30$$
$$3y = 30-2x$$
$$y = 10 - \tfrac{2}{3}x$$

$$2x+3y < 30$$
$$3y < 30-2x$$
$$y < 10-2/3x$$

Boundary of D:



Defn: The boundary of a set D are the set of points such that any neighbourhood around it contains pts in D and pts outside D.

Defn:

A set D is <u>closed</u> if all boundary pts of D are in D.

A set D is <u>open</u> if all boundary pts of D are outside D (no boundary pts are in D).

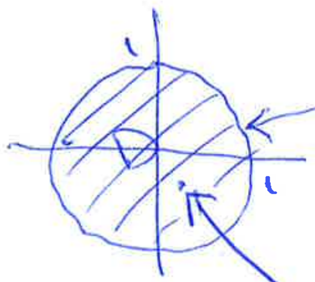
Sets defined by $=$ $\leq$ $\geq$ are closed.

— " — $<$ $>$ are open.

Fact:

The set D of admissible pts in a Lagrange or Kuhn-Tucker problem is closed.

Ex: $D = \{(x,y) : x^2+y^2 \leq 1\}$ ← all pts (x,y) such that $x^2+y^2\leq 1$



$x^2+y^2=1$ = circle with radius 1 = boundary of D

$x^2+y^2<1$

D is closed

$(x-x_0)^2+(y-y_0)^2 = r^2$ is a circle with center in (x_0,y_0) and radius $r>0$.

Extreme value theorem (Weierstrass' thm)

If f is a continuous function, and D is a closed and <u>bounded</u> set, then

$$\max/\min f(\underline{x}) \quad \text{wrt } x \in D$$

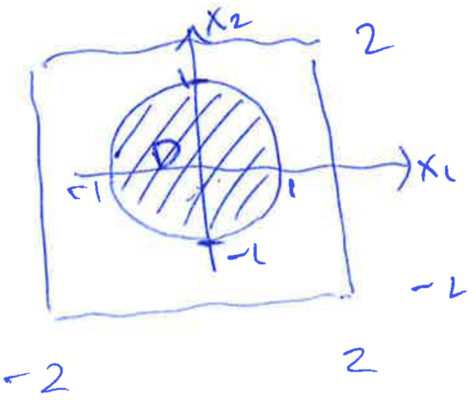
has a solution.

The set D is <u>bounded</u> if there is a finite "box" that contains all pts in D, i.e. that there are (finite) numbers

$$a_1 < b_1$$
$$a_2 < b_2$$
$$\vdots$$
$$a_n < b_n$$

Such that $a_1 \leq x_1 \leq b_1$, $a_2 \leq x_2 \leq b_2, \ldots,$ $a_n \leq x_n \leq b_n$ for all pts $(x_1, \ldots, x_n)$ in D.

<u>Ex:</u> $D: x^2 + y^2 \leq 1$

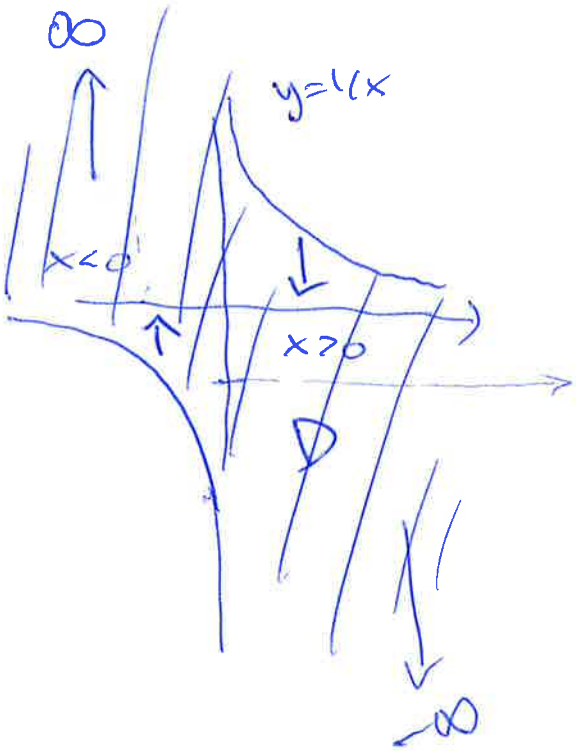


D is <u>bounded</u>

A set is <u>compact</u> if it is closed and bounded

Ex: D: $xy \le 1$ $\longrightarrow$ Boundary: $xy = 1$

$y = 1/x$



$xy < 1$:

$x > 0$: $y < 1/x$

$x < 0$: $y > 1/x$

D is not bounded

To check if a set D is bounded

* Sketch the set D
* consider the equalities / inequalities (constraints)

} Is there a biggest and smallest value of each variable for pts in D.

$xy \le 1$: $x = 1000$, $y = \frac{1}{1000}$

$x = a$, $y = 1/a$ in D as $a \to \infty$

Ex: a) $x^2 + y^2 + z^2 \le 16$ bounded

$-4 \le x \le 4$

$-4 \le y \le 4$

$-4 \le z \le 4$

b) $x^2 + y^2 - z^2 \le 16$ not bounded

$x = a$, $y = 0$, $z = a$ is in D for all values of a

c) $x^3 + y^3 + z^3 \le 8$ not bounded

$x = a$, $y = z = 0$ is in D for $a \le 2$

③ Lagrange problems

$$\boxed{\max/\min\ f(x_1, \dots, x_n) \quad \text{when} \begin{cases} g_1(x_1, \dots, x_n) = a_1 \\ g_2(x_1, \dots, x_n) = a_2 \\ \vdots \\ g_m(x_1, \dots, x_n) = a_m \end{cases}}$$

objective function

D: set of admissible points

Ex: max/min $\underbrace{x+3y}_{f(x,y)}$ when $\underbrace{x^2+y^2}_{g(x,y)} = \underbrace{10}_{a}$

Alternative:
$x^2+y^2-10=0$
$g(x,y) = x^2+y^2-10$
$a=0$

Method of Lagrange multipliers

Lagrangian: $\mathcal{L} = f(x_1 \dots x_n) - \lambda_1 \cdot g(x_1 \dots x_n) - \dots - \lambda_m g_m(x_1 \dots x_n)$

$$\mathcal{L}(x,y;\lambda) = x+3y-\lambda\cdot(x^2+y^2)$$

First order conditions:
(FOC)

$$\boxed{\begin{array}{l} \mathcal{L}'_{x_1} = 0 \\ \mathcal{L}'_{x_2} = 0 \\ \vdots \\ \mathcal{L}'_{x_n} = 0 \end{array}}$$

$\mathcal{L}'_x = 1 - \lambda \cdot 2x = 0$
$\mathcal{L}'_y = 3 - \lambda \cdot 2y = 0$

Consider the Lagrange conditions: FOC+C

$$\boxed{\begin{aligned} \mathcal{L}'_{x_1} &= 0 \\ \mathcal{L}'_{x_2} &= 0 \\ &\vdots \\ \mathcal{L}'_{x_n} &= 0 \end{aligned}}$$

FOC

$$\boxed{\begin{aligned} g_1(x_1,\dots,x_n) &= a_1 \\ g_2(x_1,\dots,x_n) &= a_2 \\ &\vdots \\ g_m(x_1,\dots,x_n) &= a_m \end{aligned}}$$

C

$n+m$ equations in $n+m$ variables

Solutions of FOC+C are candidates for max/min

Ex: max/min $x+3y$ when $x^2+y^2=10$

FOC: $\mathcal{L}'_x = 1-\lambda\cdot 2x = 0$

$\mathcal{L}'_y = 3-\lambda\cdot 2y = 0$

C: $x^2+y^2=10$

no sol'n with $\lambda=0$

$x = 1/2\lambda$, $\lambda \neq 0$

$y = 3/2\lambda$

$$\left(\frac{1}{2\lambda}\right)^2+\left(\frac{3}{2\lambda}\right)^2=10$$

$$\frac{1}{(2\lambda)^2}+\frac{9}{(2\lambda)^2}=10$$

$$\frac{10}{(2\lambda)^2}=10$$

$$(2\lambda)^2=1$$

$$2\lambda=\pm 1$$

$$\lambda=\pm 1/2$$

$\lambda=1/2$: $x=1$, $y=3$; $\lambda=1/2$

$\lambda=-1/2$: $x=-1$, $y=-3$; $\lambda=-1/2$

Candidates for max/min

Thm:

If $\underline{x}^*$ is a solution to a Lagrange problem, then either

i) $(\underline{x}^*; \underline{\lambda}^*)$ solves the Lagrange conditions (FOC+C) for a unique $\underline{\lambda}^*$.

or

ii) $\underline{x}^*$ does not satisfy the NDCQ (non-degenerate constraint qualification) condition:

$$\boxed{\text{NDCQ:} \quad rk\begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \vdots & & & \\ \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_n} \end{pmatrix} = m}$$

$$\boxed{\underline{x}^* \text{ is max/min} \Longrightarrow \begin{array}{c} (\underline{x}^*, \underline{\lambda}^*) \text{ solves FOC+C} \\ \text{or} \\ \underline{x}^* \text{ fails NDCQ} \end{array}}$$

In other words, candidates for max/min in Lagrange problems are

i) solutions to FOC+C

ii) points where NDCQ fails

Ex: max/min $\underset{f(x,y)}{x+3y}$ when $\underset{g(x,y)}{x^2+y^2=10}$

Candidates for max/min:

FOC+C: $(1,3;\,1/2) \leftarrow f=10$

$(-1,-3;\,-1/2) \leftarrow f=-10$

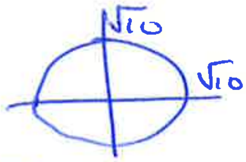
NDCQ fails: no candidates

If there is a max, it must be $(1,3)$

If there is a min, it must be $(-1,-3)$

Since D is bounded, the extreme value thm. says that there is a max/min.

$x^2+y^2=10$ is bounded



$(x,y)=(1,3)$ is max, $f=10$

$(x,y)=(-1,-3)$ is min, $f=-10$

NDCQ: $\operatorname{rk}\begin{pmatrix}2x & 2y\end{pmatrix}=1$

NDCQ fails: $\operatorname{rk}\begin{pmatrix}2x & 2y\end{pmatrix}=0 \Leftrightarrow 2x=0,\ 2y=0$

$x=0,\ y=0$

not possible

(not admissible)

All admissible pts satisfy NDCQ

Ex: max $\overset{f(x,y)}{\overset{\shortparallel}{y}}$ when $x^2+y^3=0$

$$\mathcal{L} = y - \lambda\cdot(x^2+y^3)$$

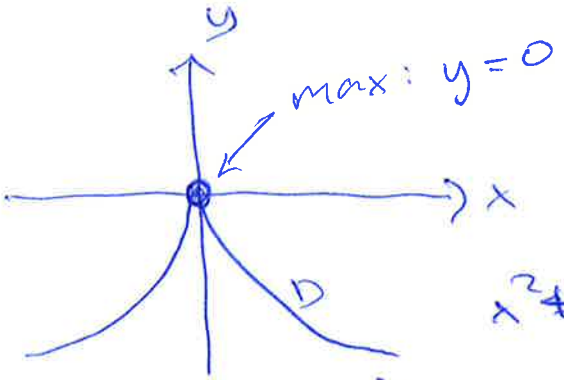
FOC: $\mathcal{L}'_x = -\lambda\cdot 2x = 0$ ~~$\lambda=0$~~ or $x=0$

$\mathcal{L}'_y = 1-\lambda\cdot 3y^2 = 0$

C: $x^2+y^3=0$

$1=0$ not possible | $y^3=0$, $y=0$, $1=0$ not possible

no solutions



$x^2+y^3=0$

$y^3=-x^2$

$y=\sqrt[3]{-x^2}=-x^{2/3}$

NDCQ: $\operatorname{rk}\begin{pmatrix}2x & 3y^2\end{pmatrix}=1$

NDCQ fails: $\operatorname{rk}\begin{pmatrix}2x & 3y^2\end{pmatrix}=0 \iff 2x=0,\ 3y^2=0$

$x=0,\ y=0$

$(0,0)$ is admissible

Candidates for max:

Sol'ns of FOC+C: no candidates

NDCQ fails: $(0,0)$ $f=0$ candidate for max

Lagrange method:

1. Write down and solve Lagrange conditions for (FOC + C). These points are candidates for max/min.

2. Find all pts where NDCQ fails (if any). These points are also candidates for max/min.

3. Compute $f(\underline{x})$ for each candidate point from 1 and 2 $\longrightarrow$ find the best candidate for max and min.

(4) Kuhn-Tucker problems

$$\max f(x_1,\dots,x_n) \quad \text{when} \quad \begin{cases} g_1(x_1,\dots,x_n) \le a_1 \\ g_2(x_1,\dots,x_n) \le a_2 \\ \vdots \\ g_m(x_1,\dots,x_n) \le a_m \end{cases}$$

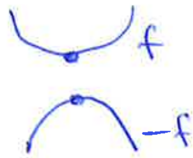
std. form

Ex: $\min x+3y$ when $x^2+y^2 \le 10$

$\shortparallel$

$\max -(x+3y)$ when $x^2+y^2 \le 10$

min f = max −f



Ex: $\max x^2+y^2$ when $\begin{cases} x \ge 0 & -x \le 0 \\ y \ge 0 & -y \le 0 \\ 2x+3y \le 30 & 2x+3y \le 30 \end{cases}$

We transform all Kuhn-Tucker problems into std. form before we form the Lagrangian.

Method for solving Kuhn-Tucker problems:

(assuming std. form)

We form the Lagrangian

$$\mathcal{L}(x_1,\dots,x_n;\lambda_1,\dots,\lambda_m) = f(\underline{x}) - \lambda_1 \cdot g_1(\underline{x}) - \dots - \lambda_m g_m(\underline{x})$$

and consider the Kuhn-Tucker conditions:

FOC + C + CSC

(complementary slackness conditions)

FOC:

$$\begin{array}{|l|}\hline \mathcal{L}'_{x_1} = 0 \\ \mathcal{L}'_{x_2} = 0 \\ \vdots \\ \mathcal{L}'_{x_n} = 0 \\ \hline \end{array}$$

C:

$$\begin{array}{|l|}\hline g_1(\underline{x}) \leq a_1 \\ g_2(\underline{x}) \leq a_2 \\ \vdots \\ g_m(\underline{x}) \leq a_m \\ \hline \end{array}$$

CSC:

$$\begin{array}{|llll|}\hline \lambda_1 \geq 0 & \text{and} & \lambda_1 \cdot (g_1(\underline{x}) - a_1) = 0 \\ \lambda_2 \geq 0 & \text{"} & \lambda_2 \, (g_2(\underline{x}) - a_2) = 0 \\ \vdots & & \vdots \\ \lambda_m \geq 0 & & \lambda_m \cdot (g_m(\underline{x}) - a_m) = 0 \\ \hline \end{array}$$

For each constraint $g_i(x_1, \ldots, x_n) \leq a_i$, the constraint is <u>binding</u> if $g_i(x_1, \ldots, x_n) = a_i$ and non-binding if $g_i(x_1, \ldots, x_n) < a_i$.

$\lambda_i \geq 0$ and $\lambda_i \cdot (g_i(\underline{x}) - a_i) = 0$

means that

$$\boxed{\lambda_i \geq 0 \text{ and if the constraint is non-binding, then } \lambda_i = 0}$$

Thm

If $\underline{x}^*$ is max in a Kuhn-Tucker problem in Std form, then either

i) $(x^*; \lambda^*)$ solves FOC + C + CSC for a unique $\underline{\lambda}^*$

or

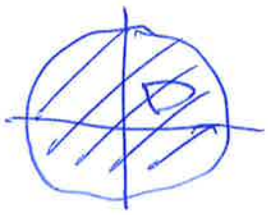
ii) $\underline{x}^*$ fails the NDCQ condition

NDCQ in the Kuhn-Tucker case:

$$\text{rk}\left(\partial g_i / \partial x_j \right) = \# \text{ binding constraints}$$

include only rows where the constraint is binding

Ex: max $x+3y$ wrt $x^2+y^2 \leq 10$



	binding	non-binding
FOC:	$1-\lambda \cdot 2x=0$ $3-\lambda \cdot 2y=0$	← same ← same
C:	$x^2+y^2=10$	$x^2+y^2<10$
CSC:	$\lambda \geq 0$	$\lambda=0$
	$(1,3;1/2)$ $\cancel{(-1,-3;-1/2)}$	no sol's
NDCQ:	rk $(2x\ 2y)=1$ no cond.	no condition