PLENARY SESSION

GRA 6035

MATHEMATICS

Eivind Eriksen

SEP 22, 2014

<u>Problems</u>

1.4, 1.15

2.18, 2.20c

3.4, 3.5, 3.14, 3.15

4.6, 4.7, 4.8, 4.9, 4.11

1.4

$$x+y+z=1$$
$$x-y+z=4$$
$$x+2y+z=h$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 4 \\ 1 & 2 & 1 & h \end{array}\right) \begin{array}{c} -1 \\ \\ -1 \end{array} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 3 \\ 0 & 1 & 0 & h-1 \end{array}\right) \begin{array}{c} \\ 1/2 \\ \\ \end{array}$$

$$\downarrow$$

$h=-1/2$: one free var. (consistent)

$h \neq -1/2$: inconsistent (no sol's)

$$\Leftarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & h+1/2 \end{array}\right)$$

<u>Answer:</u> $\underline{\underline{h=-1/2}}$

1.15

$$\begin{aligned} x_1 + x_2 + x_3 &= 2q \\ 2x_1 - 3x_2 + 2x_3 &= 4q \\ 3x_1 - 2x_2 + px_3 &= q \end{aligned}$$

parameters: p, q

variables: ~~x, y, z~~ x_1, x_2, x_3

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & 2 \\ 3 & -2 & p \end{pmatrix} \qquad \hat{A} = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2q \\ 2 & -3 & 2 & 4q \\ 3 & -2 & p & q \end{array}\right)$$

rk A:

rk $\hat{A}$:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2q \\ 2 & -3 & 2 & 4q \\ 3 & -2 & p & q \end{array}\right) \begin{array}{l} \\ \cdot(-2) \\ \cdot(-3) \end{array} \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2q \\ 0 & -5 & 0 & 0 \\ 0 & -5 & p-3 & -5q \end{array}\right) \begin{array}{l} \\ \\ \cdot(-1) \end{array}$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2q \\ 0 & -5 & 0 & 0 \\ 0 & 0 & p-3 & -5q \end{array}\right)$$

$$\text{rk } A = \begin{cases} 2, & p = 3 \\ 3 & p \neq 3 \end{cases}$$

$$\text{rk } \hat{A} = \begin{cases} 2, & p = 3, q = 0 \\ 3, & \text{otherwise} \end{cases}$$

no solutions: $p = 3, \ q \neq 0$

one solution: $p \neq 3$

infinitely many solutions: $p = 3, \ q = 0$

2.18

a) $A = \begin{pmatrix} x & 0 & x^2-2 \\ 0 & 1 & 1 \\ -1 & x & x-1 \end{pmatrix}$

$|A| = 1 \cdot \begin{vmatrix} x & x^2-2 \\ -1 & x-1 \end{vmatrix} - x \cdot \begin{vmatrix} x & x^2-2 \\ 0 & 1 \end{vmatrix}$

$= x(x-1) - (x^2-2)\cdot(-1) - x(x-0)$

$= x^2 - x + \cancel{x^2} - 2 \cancel{- x^2}$

$= \underline{x^2 - x - 2}$

$|A| = 0 \quad x^2 - x - 2 = 0 \quad x = \frac{1 \pm \sqrt{1^2 - 4\cdot(-2)}}{2}$

$= \frac{1 \pm 3}{2} = 2, -1$

$$\boxed{\operatorname{rk} A = \begin{cases} 3, & x \neq 2, -1 \\ 2 & x = 2, -1 \end{cases}}$$

$x = -1$: $A = \begin{vmatrix} -1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & -1 & -2 \end{vmatrix}$ $\quad \underline{x = 2}$: $A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{pmatrix}$

$\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} \neq 0 \qquad \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \neq 0$

b) In the same way (see solutions)

2.20 c)

$$\begin{aligned} x_1 - x_2 + 2x_3 + x_4 &= 1 \\ 2x_1 + x_2 - x_3 + 3x_4 &= 3 \\ x_1 + 5x_2 - 8x_3 + x_4 &= 1 \\ 4x_1 + 5x_2 - 7x_3 + 7x_4 &= 7 \end{aligned}$$

$$A = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 3 \\ 1 & 5 & -8 & 1 \\ \cancel{4} & \cancel{5} & \cancel{-7} & \cancel{7} \end{pmatrix} \qquad \hat{A} = \left(A \,\middle|\, \begin{matrix} 1 \\ 3 \\ 1 \\ 7 \end{matrix} \right)$$

$$\begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & 1 \\ 0 & 0 & 0 & -2 \\ 0 & & & \end{pmatrix}$$

$$|A| = \begin{vmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 3 \\ 1 & 5 & -8 & 1 \\ 4 & 5 & -7 & 7 \end{vmatrix} = \quad -2 \quad -1 \quad -4$$

$$-3 \quad -2 \quad = \begin{vmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & 1 \\ 0 & 6 & -10 & 0 \\ 0 & 9 & -15 & 3 \end{vmatrix} = 1 \cdot \begin{vmatrix} 3 & -5 & 1 \\ 6 & -10 & 0 \\ 9 & -15 & 3 \end{vmatrix} = 1 \cdot (0) + 3 \cdot 0$$

$$= 0$$

3-minors: row 1,2,3 col 1,2,4

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \\ 1 & 5 & 1 \end{vmatrix} \end{array} = 1 \cdot (1 - 15) - 2 \cdot (-1 - 5) + 1 \cdot (-3 - 1)$$

$$= -14 + 12 - 4 = -6 \neq 0$$

$\mathrm{rk}\, A = 3$

What is $\mathrm{rk}\, \hat{A}$?

$$\hat{A} = \left(\begin{matrix} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 3 \\ 1 & 5 & -8 & 1 \\ \cancel{4} & \cancel{5} & \cancel{-7} & \cancel{7} \end{matrix} \,\middle|\, \begin{matrix} 1 \\ 3 \\ 1 \\ \cancel{7} \end{matrix} \right)$$

$\mathrm{rk}\, \hat{A} = 3$

Conclusion: infinitely many solutions
one free variable

x_3 free, x_1, x_2, x_4 can be found from eqn. 1, 2, 3

$$x_1 - x_2 + x_4 = 1 - 2x_3$$
$$2x_1 + x_2 + 3x_4 = 3 + x_3$$
$$x_1 + 5x_2 + x_4 = 1 + 8x_3$$

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \\ 1 & 5 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 - 2x_3 \\ 3 + x_3 \\ 1 + 8x_3 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \\ 1 & 5 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 - 2x_3 \\ 3 + x_3 \\ 1 + 8x_3 \end{pmatrix}$$

(or use Gaussian elimination)

3.4

i) $\underline{a}, \underline{b}, \underline{c}$ linearly independent $\Longrightarrow$ $\underline{a}+\underline{b}$, $\underline{b}+\underline{c}$, $\underline{a}+\underline{c}$ are linearly independent

Consider $x_1 \cdot (\underline{a}+\underline{b}) + x_2 \cdot (\underline{b}+\underline{c}) + x_3 \cdot (\underline{a}+\underline{c}) = \underline{0}$

$$(x_1+x_3)\underline{a} + (x_1+x_2)\underline{b} + (x_2+x_3)\underline{c} = \underline{0}$$

since $\underline{a}, \underline{b}, \underline{c}$ are lin. independent $\Downarrow$

$x_1+x_3=0$ $x_1+x_2=0$ $x_2+x_3=0$

$x_1 = -x_3 = \underline{0}$ $\cancel{x_2 = x_1}$ $x_2 = -x_3 = \underline{0}$

$-x_3 - x_3 = 0$

$-2x_3 = 0$

$\underline{x_3 = 0}$

Hence the vectors are linearly independent.

ii) See solutions.

3.5

$\underline{v}_1, \dots, \underline{v}_n$: m-vectors

$\left.\begin{array}{l}\text{at least one of the vectors} \\ \text{is a lin. comb. of} \\ \text{the others}\end{array}\right\} \Longleftrightarrow$ $x_1\underline{v}_1 + \dots + x_n\underline{v}_n = \underline{0}$ has non-trivial solutions

$\Longrightarrow$ Assume $\underline{v}_1 = x_2\underline{v}_2 + x_3\underline{v}_3 + \dots + x_n\underline{v}_n$

Then $\underline{0} = -\underline{v}_1 + x_2\underline{v}_2 + \dots + x_n\underline{v}_n \Rightarrow (-1, x_2, \dots, x_n)$ is a solution

nontrivial solution

$\Longleftarrow$ Assume $x_1\underline{v}_1 + \dots + x_n\underline{v}_n = 0$ with $x_1 \neq 0$

Then $x_1\underline{v}_1 = -x_2\underline{v}_2 - \dots - x_n\underline{v}_n$

$$\underline{v}_1 = -\frac{x_2}{x_1}\underline{v}_2 - \dots - \frac{x_n}{x_1}\underline{v}_n$$

3.14

$$\underline{w} = \begin{pmatrix} -4 \\ 3 \\ h \end{pmatrix} \qquad \underline{v_1} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \quad \underline{v_2} = \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix} \quad \underline{v_3} = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$$

$$\underline{w} = x_1 \underline{v_1} + x_2 \underline{v_2} + x_3 \underline{v_3}$$

$$\left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{array}\right) \begin{matrix}]1 \\ ]+2 \\ \end{matrix} \rightarrow \left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{array}\right) \begin{matrix} \\ ]-3 \\ \end{matrix}$$

$$\downarrow$$

Consistent when $\underline{h=5}$

$$\left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{array}\right)$$

3.15

$$\underline{v_1} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \quad \underline{v_2} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \underline{v_3} = \begin{pmatrix} h+1 \\ h \\ h-2 \end{pmatrix}$$

When are $\underline{v_1}, \underline{v_2}, \underline{v_3}$ lin. independent?

$$\begin{vmatrix} 2 & 1 & h+1 \\ 3 & 2 & h \\ -1 & 1 & h-2 \end{vmatrix} = 2\cdot\big(2(h-2)-h\big) - 3\big(h-2-(h+1)\big) + (-1)\cdot\big(h-2(h+1)\big)$$

$$= 2h-8+9+h+2 = \underline{3h+3}$$

$h=-1$: $|A| = 0 \Leftrightarrow$ lin. dependent

$\underline{h \neq -1}$: $|A| \neq 0 \Leftrightarrow$ lin. independent

4.6 $A = \begin{pmatrix} 2 & -7 \\ 3 & -8 \end{pmatrix}$

$\begin{vmatrix} 2-\lambda & -7 \\ 3 & -8-\lambda \end{vmatrix} = 0$ $\quad \lambda^2 + 6\lambda + 5 = 0$

$\lambda_1 = -5 \quad \lambda_2 = -1$

$\lambda_1 = -5$:

$\begin{pmatrix} 7 & -7 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad \underline{v} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \underline{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -1$:

$\begin{pmatrix} 3 & -7 \\ 3 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad \underline{v} = t\begin{pmatrix} 7 \\ 3 \end{pmatrix} \quad \underline{v_2} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$

$D = \begin{pmatrix} -5 & 0 \\ 0 & -1 \end{pmatrix} \quad P = \begin{pmatrix} 1 & 7 \\ 1 & 3 \end{pmatrix} \quad P^{-1}AP = D$

4.7 $A = \begin{pmatrix} 3 & 5 \\ 0 & 3 \end{pmatrix}$ $\quad$ not diagonalizable

$\lambda_1 = \lambda_2 = 3$

$\lambda_1 = 3$: $\begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$D = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ $\quad \begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \underline{v_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$P = \begin{pmatrix} 1 & ? \\ 0 & ? \end{pmatrix}$

Only one lin indep. eigenvector.

4.8 $\quad \underline{s} = \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix} \qquad T = \begin{pmatrix} 0.85 & 0.45 \\ 0.15 & 0.55 \end{pmatrix}$

i) $\quad T\underline{s} = \begin{pmatrix} 0.53 \\ 0.47 \end{pmatrix}$

ii) $\quad \begin{vmatrix} 0.85-\lambda & 0.45 \\ 0.15 & 0.55-\lambda \end{vmatrix} = 0$

$$\lambda^2 - 1.40\lambda + 0.40 = 0$$

$$\underline{\lambda_1 = 1} \quad \underline{\lambda_2 = 0.40}$$

$\underline{\lambda = 1:} \quad \begin{pmatrix} -0.15 & 0.45 \\ 0.15 & -0.45 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \underline{t \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix}}$

$\underline{\lambda = 0.40:} \quad \begin{pmatrix} 0.45 & 0.45 \\ 0.15 & 0.15 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \underline{t \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$

iii) $\quad P = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 0.40 \end{pmatrix} \qquad \begin{matrix} P^{-1}TP = D \\ T = PDP^{-1} \end{matrix}$

iv) $\quad D^n = \begin{pmatrix} 1^n & 0 \\ 0 & 0.40^n \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ as $n \to \infty$

$$T^n\underline{s} = (PDP^{-1})^n \cdot \underline{s} = PD^nP^{-1} \cdot \underline{s} \to P \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} P^{-1} \cdot \underline{s}$$

$$= \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \cdot \frac{1}{4} \cdot \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \cdot \frac{1}{4}\begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 3/4 & 3/4 \\ 1/4 & 1/4 \end{pmatrix} \cdot \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix}}} = \underline{\underline{\begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}}}$$

4.11

$$A = \begin{pmatrix} 1 & 1 & -4 \\ 0 & t+2 & t-8 \\ 0 & -5 & 5 \end{pmatrix} \quad (t \text{ parameter})$$

a) $|A| = 1 \cdot \left((t+2)\cdot 5 - (t-8)\cdot(-5)\right)$

$= 5t+10+5t-40 = \underline{10t-30}$

$$\operatorname{rk} A = \begin{cases} 3, & t \neq 3 \\ 2, & t = 3 \end{cases} \qquad \begin{vmatrix} 1 & 1 \\ 0 & 5 \end{vmatrix} = 5 \neq 0$$

b) Eigenvalues:

$$\begin{vmatrix} 1-\lambda & 1 & -4 \\ 0 & t+2-\lambda & t-8 \\ 0 & -5 & 5-\lambda \end{vmatrix} = 0 \quad \left(\begin{array}{l} \lambda \text{ variable} \\ t \text{ parameter} \end{array}\right)$$

$$(1-\lambda)\cdot\left((t+2-\lambda)(5-\lambda)+5(t-8)\right) = 0$$

$\underline{\lambda_1 = 1}$ $\qquad \lambda^2 - (t+7)\cdot\lambda + (10t-30) = 0$

$$\lambda = \frac{(t+7) \pm \sqrt{(t+7)^2 - 4\cdot 1\cdot(10t-30)}}{2}$$

$$= \frac{t+7}{2} \pm \frac{\sqrt{t^2+14t+49-40t+120}}{2}$$

$$= \frac{t+7}{2} \pm \frac{\sqrt{t^2-26t+169}}{2} \quad (t-13)^2$$

$$= \frac{t+7}{2} \pm \frac{t-13}{2} =$$

$\lambda_2 = \underline{t-3} \qquad \lambda_3 = \underline{10}$

c) When is A diagonalizable?

$\lambda_1 = 1 \qquad \lambda_2 = t-3 \qquad \lambda_3 = 10$

If the eigenvalues are distinct, A is diag.

$t \neq 4, 13 \Rightarrow$ A diagonalizable

$t=4$: $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 10$

$$\left.\begin{matrix} t=4 \\ \lambda=1 \end{matrix}\right\} \begin{pmatrix} 1-\lambda & 1 & -4 \\ 0 & t+2-\lambda & t-8 \\ 0 & -5 & 5-\lambda \end{pmatrix} = \begin{pmatrix} 0 & 1 & -4 \\ 0 & 5 & -4 \\ 0 & -5 & 4 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & -4 \\ 0 & 0 & 16 \\ 0 & 0 & 0 \end{pmatrix}$$

one free var

$\Rightarrow$ not diag. for $t=4$

$t=13$: similar, not diagonalizable

Concl: A diagonalizable for $t \neq 4, 13$

4.10 $A = \begin{pmatrix} 1 & 7 & -2 \\ 0 & s & 0 \\ 1 & 1 & 4 \end{pmatrix}$ $\underline{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

a) $\det(A) =$ (see solutions)

$rk(A) =$

b) Eigenvalues:

Eigenvector: $\underline{v}$?

$$A\underline{v} = \lambda \cdot \underline{v}$$

$$A\underline{v} = \begin{pmatrix} 1 & 7 & -2 \\ 0 & s & 0 \\ 1 & 1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ s \\ 6 \end{pmatrix}$$

$$\lambda \underline{v} = \lambda \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda \\ \lambda \end{pmatrix}$$

$$\begin{pmatrix} 6 \\ s \\ 6 \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda \\ \lambda \end{pmatrix} \qquad \begin{array}{l} \lambda = 6 \\ s = 6 \end{array}$$

Concl: If $s = 6$, then $\underline{v}$ is an eigenvector ($\lambda = 6$)

If $s \neq 6$, then $\underline{v}$ is not an eigenvector.

4.9 $A = \begin{pmatrix} 4 & 1 & 2 \\ 0 & 3 & 0 \\ 1 & 1 & 5 \end{pmatrix}$

E-values: $\begin{vmatrix} 4-\lambda & 1 & 2 \\ 0 & 3-\lambda & 0 \\ 1 & 1 & 5-\lambda \end{vmatrix} = 0$

$$(3-\lambda) \cdot \begin{vmatrix} 4-\lambda & 2 \\ 1 & 5-\lambda \end{vmatrix} = 0$$

$$(3-\lambda) \cdot (\lambda^2 - 9\lambda + 18) = 0$$

$\lambda = 3$ or $\lambda^2 - 9\lambda + 18$

$$\lambda = \frac{9 \pm \sqrt{81 - 4 \cdot 18}}{2} = \frac{9 \pm 3}{2}$$

$\lambda_1 = 3$ $\lambda_2 = 3$ $\lambda_3 = 6$ $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

E-vectors:

$\lambda = 3$: $\begin{pmatrix} 1 & 1 & 2 \\ \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{1} & \cancel{1} & \cancel{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $x + y + 2z = 0$

y free

z free

$x = -y - 2z$

$y = y$

$z = z$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y-2z \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} -2z \\ 0 \\ z \end{pmatrix}$$

$$= y \cdot \underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}_{= v_1} + z \cdot \underbrace{\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}}_{= v_2}$$

$$P = \begin{pmatrix} -1 & -2 & \\ 1 & 0 & \\ 0 & 1 & \end{pmatrix}$$

$\underline{\lambda=6}$:
$$\begin{pmatrix} -2 & 1 & 2 \\ 0 & -3 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} -2x+y+2z &= 0 \qquad & x &= z \\ -3y &= 0 & y &= 0 \\ x+y-z &= 0 & z &\text{ free} \end{aligned}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ z \end{pmatrix} = z \cdot \underline{\underline{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}} \qquad \underline{\underline{P = \begin{pmatrix} -1 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}}}$$

Yes, $P = \begin{pmatrix} -1 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ diagonalizes A and

$$P^{-1}AP = D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$A = PDP^{-1}$$

$$A^{17} = (PDP^{-1})^{17} = (PDP^{-1})(PDP^{-1})\ldots(PDP^{-1})$$
$$= P \cdot D^{17} \cdot P^{-1} \qquad P^{-1} = \frac{1}{|P|} \cdot (C_{ij})^T$$

$$= \begin{pmatrix} -1 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3^{17} & 0 & 0 \\ 0 & 3^{17} & 0 \\ 0 & 0 & 6^{17} \end{pmatrix} \cdot \frac{1}{3} \cdot \begin{pmatrix} 0 & 3 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$\underline{\underline{= 3^{16} \cdot \begin{pmatrix} 2 & -1 & -2 \\ 0 & 3 & 0 \\ -1 & -1 & 1 \end{pmatrix} + 6^{16} \cdot \begin{pmatrix} 2 & 2 & 4 \\ 0 & 0 & 0 \\ 2 & 2 & 4 \end{pmatrix}}}$$