PLENARY SESSION

GRA 6035 MATHEMATICS

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Problems:

5.5
6.8, 6.20, 6.26
7.7, 7.8, 7.9, 7.10
8.6, 8.7, 8.8, 8.9, 8.13

not time for 7.10 / 8.9 today

5.5

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \qquad D_1 = a \qquad \Delta_1 = a, c$$
$$D_2 = ac - b^2$$

$ac - b^2 < 0$: indefinite

$ac - b^2 > 0$: $a > 0, c > 0$ pos. definite
$\qquad a < 0, c < 0$ neg. definite

$ac - b^2 = 0$: $a = 0,\ c \geq 0$ pos. semidef.
$\qquad c \leq 0$ neg. semidef.

$\qquad a > 0,\ c = 0$ pos. semidef.
$\qquad a < 0,\ c = 0$ neg. semidef.

6.8 $f(x,y) = -6x^2 + (2a+4)xy - y^2 + 4ay$
(a is a parameter)

$f'_x = -12x + (2a+4)y$

$f'_y = (2a+4)x - 2y + 4a$

$$H(f) = \begin{pmatrix} -12 & 2a+4 \\ 2a+4 & -2 \end{pmatrix}$$

$D_1 = -12 < 0 \qquad \Delta_1 = -2$

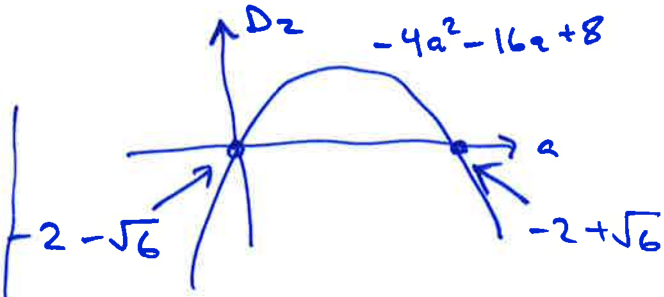
$D_2 = 24 - (2a+4)^2$

$= 24 - (4a^2 + 16a + 16)$

$= \underline{-4a^2 - 16a + 8}$

f is never convex

f is concave
when $\underline{-2-\sqrt{6} \leq a \leq -2+\sqrt{6}}$



$-4a^2 - 16a + 8 = 0$

$$a = \frac{16 \pm \sqrt{16^2 + 16 \cdot 8}}{2 \cdot (-4)}$$

$$= -2 \pm \frac{\sqrt{16}}{4} \cdot \frac{\sqrt{24}}{2}$$

$$= -2 \pm \sqrt{6}$$

6.20 $\quad f(x,y,z) = x^3 + 3xy + 3xz + y^3 + 3yz + z^3$

$$f'_x = 3x^2 + 3y + 3z = 0$$
$$f'_y = 3x + 3y^2 + 3z = 0$$
$$f'_z = 3x + 3y + 3z^2 = 0$$

$$\boxed{\begin{matrix} x^2 + y + z = 0 \\ x + y^2 + z = 0 \\ x + y + z^2 = 0 \end{matrix}}$$

$z = -x^2 - y$: $\quad x + y^2 + (-x^2 - y) = 0$

$\quad x + y + (-x^2 - y)^2 = 0$

$$x - y = x^2 - y^2 = (x-y)(x+y).$$

$\underline{x - y = 0}$ or $\underline{x + y = 1}$

$x + x + (-x^2 - x)^2 = 0$

$2x + x^4 + 2x^3 + x^2 = 0$

$x^4 + 2x^3 + x^2 + 2x = 0$

$x(x^3 + 2x^2 + x + 2) = 0$

$\underline{x = 0}$ or $x^3 + 2x^2 + x + 2 = 0$

$x^2(x+2) + (x+2) = 0$

$(x^2 + 1)(x+2) = 0$

$\underline{x = -2}$

$1 + (-x^2 - y)^2 = 0$

no solutions.

Solutions = stationary pts:

$x = 0$: $(0,0,0)$

$x = -2$: $(-2,-2,-2)$

$$H(f) = \begin{pmatrix} 6x & 3 & 3 \\ 3 & 6y & 3 \\ 3 & 3 & 6z \end{pmatrix}$$

At $(0,0,0)$: saddle pt.

$$H(f)(0,0,0) = \begin{pmatrix} 0 & 3 & 3 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{pmatrix} \quad \begin{matrix} D_1 = 0 \\ D_2 = -9 \\ \text{indef.} \end{matrix}$$

At $(-2,-2,-2)$: local max

$$H(f)(-2,-2,-2) = \begin{pmatrix} -12 & 3 & 3 \\ 3 & -12 & 3 \\ 3 & 3 & -12 \end{pmatrix} \quad \begin{matrix} D_1 = -12 \\ D_2 = 135 \\ D_3 < 0 \end{matrix}$$

neg. defin.

$D_3 = -12 \cdot 135 - 3 \cdot (-45) + 3 \cdot 45$

$= 270 - 12 \cdot 135 < 0$

6.26: $f(x,y,z,w) = x^5 + xy^2 - zw$

a) $f'_x = 5x^4 + y^2 = 0$ $\quad y=0 \quad\quad x=0$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad \Uparrow \quad\quad\quad \Uparrow$

$f'_y = 2xy = 0$ $\quad x=0$ or $y=0$

$f'_z = -w = 0$ $\quad w=0$

$f'_w = -z = 0$ $\quad z=0$

Only one stationary pt: $\underline{(0,0,0,0)}$

b)

$$H(f) = \begin{pmatrix} 20x^3 & 2y & 0 & 0 \\ 2y & 2x & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$H(f)(0,0,0,0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$D_1 = 0$

$D_2 = 0$

$D_3 = 0$

$D_4 = 0$

$\Delta_2 = -1 \Rightarrow$ indefinite

$\Downarrow$

$(0,0,0,0)$ saddle pt.

7.7 / 8.6: max $x+4y+z$ when $\begin{cases} x^2+y^2+z^2=216 \\ x+2y+3z=0 \end{cases}$

$$\mathcal{L} = x+4y+z - \lambda_1\cdot(x^2+y^2+z^2) - \lambda_2\cdot(x+2y+3z)$$

FOC: $\mathcal{L}'_x = 1-\lambda_1\cdot 2x - \lambda_2 = 0$ C: $x^2+y^2+z^2=216$

$\mathcal{L}'_y = 4-\lambda_1\cdot 2y - 2\lambda_2 = 0$ $x+2y+3z=0$

$\mathcal{L}'_z = 1-\lambda_1\cdot 2z - 3\lambda_2 = 0$

$$x = \frac{(1-\lambda_2)\cdot 7}{2\lambda_1\cdot 7} \qquad y = \frac{(4-2\lambda_2)\cdot 7}{(2\lambda_1)\cdot 7} \qquad z = \frac{(1-3\lambda_2)\cdot 7}{2\lambda_1\cdot 7}$$

$\lambda_1=0$: $\lambda_2=1$, $4=2\lambda_2$, $\lambda_2=2$ impossible $\Rightarrow$ no solutions with $\lambda_1=0$

$$\frac{1-\lambda_2}{2\lambda_1} + 2\cdot\frac{4-2\lambda_2}{2\lambda_1} + 3\cdot\frac{1-3\lambda_2}{2\lambda_1} = 0 \qquad \cdot 2\lambda_1$$

$$1-\lambda_2+8-4\lambda_2+3-9\lambda_2=0$$

$$12-14\lambda_2=0 \qquad \lambda_2 = \frac{12}{14} = \underline{\frac{6}{7}} \Rightarrow x = \underline{\frac{1}{14\lambda_1}} \quad y=\underline{\frac{16}{14\lambda_1}} \quad z=\underline{\frac{-11}{14\lambda_1}}$$

$$\left(\frac{1}{14\lambda_1}\right)^2 + \left(\frac{16}{14\lambda_1}\right)^2 + \left(\frac{-11}{14\lambda_1}\right)^2 = 216$$

$$\frac{1+256+121}{14^2\cdot\lambda_1^2} = 216 \qquad \frac{378}{(14\lambda_1)^2} = 216$$

$$(14\lambda_1)^2 = \frac{378}{216} \qquad \lambda_1 = \pm\frac{1}{14}\cdot\sqrt{\frac{378}{216}} = \pm\frac{1}{14}\cdot\sqrt{\frac{7}{4}}$$

$$= \underline{\pm\frac{1}{28}\sqrt{7}}$$

$\underline{\lambda_1 = \frac{1}{28}\sqrt{7}}$: $\quad x = \frac{1}{14 \cdot \frac{1}{28}\sqrt{7}} = \frac{2}{\sqrt{7}} \quad y = \frac{32}{\sqrt{7}} \quad z = \frac{-22}{\sqrt{7}}$

$\lambda_2 = \frac{6}{7} \qquad f = \frac{2 + 4\cdot 32 - 22}{\sqrt{7}} = \underline{\frac{108}{\sqrt{7}}}$

$\underline{\lambda_1 = -\frac{1}{28}\sqrt{7}}$: $\quad x = -\frac{2}{\sqrt{7}} \quad y = -\frac{32}{\sqrt{7}} \quad z = \frac{22}{\sqrt{7}} \quad \lambda_2 = \frac{6}{7} \quad f = \underline{-\frac{108}{\sqrt{7}}}$

<u>Candidate for max:</u> $\quad (x,y,z) = \left(\frac{2}{\sqrt{7}}, \frac{32}{\sqrt{7}}, \frac{-22}{\sqrt{7}}\right)$

$$\mathcal{L}\left(x,y,z; \frac{\sqrt{7}}{28}, \frac{6}{7}\right) = \cancel{x+4y+z} - \underbrace{\lambda_1}_{=\frac{\sqrt{7}}{28}}(x^2+y^2+z^2) - \underbrace{\cancel{\lambda_2(x+2y+3z)}}_{\lambda_2 = \frac{6}{7}}$$

$$\mathcal{L}'' = \begin{pmatrix} -2\cdot\frac{\sqrt{7}}{28} & 0 & 0 \\ 0 & -2\cdot\frac{\sqrt{7}}{28} & 0 \\ 0 & 0 & -2\cdot\frac{\sqrt{7}}{28} \end{pmatrix}$$

neg. definite
$\Downarrow$
concave
$\Downarrow$
<u>the pt. is max</u>

Also possible to conclude using extreme value thm.

7.8 / 8.7 :

$$\underbrace{\max \overset{f(x,y)}{\overset{\shortparallel}{xy}} \quad \text{when} \quad x^2+y^2 \le 1}_{\text{std. form}}$$

$$\mathcal{L} = xy - \lambda \cdot (x^2+y^2)$$

FOC: $\mathcal{L}'_x = y - \lambda \cdot 2x = 0$ C: $x^2+y^2 \le 1$

$\mathcal{L}'_y = x - \lambda \cdot 2y = 0$

CSC: $\lambda \ge 0$ and $\lambda \cdot (x^2+y^2-1) = 0$

a) $x^2+y^2=1$: $\lambda \ge 0$

$y = 2\lambda \cdot x$

$x - 2\lambda \cdot (2\lambda \cdot x) = 0$

$x - (2\lambda)^2 \cdot x = 0$

$x \cdot (1 - (2\lambda)^2) = 0$

$x=0 \quad (2\lambda)^2 = 1$

$x=0$: $y=0$

not adm.

⇓

no solution

$2\lambda = \pm 1 \quad \lambda = \pm 1/2$

$\lambda = 1/2$: $y = x$

$x^2 + x^2 = 1$

$x^2 = 1/2$

$x = y = \pm\sqrt{1/2}$

$\lambda = -1/2$: not possible

Candidates: $(\sqrt{1/2}, \sqrt{1/2}; 1/2) \quad f = 1/2$

$(-\sqrt{1/2}, -\sqrt{1/2}; 1/2) \quad f = 1/2$

b) $x^2+y^2<1$: $\lambda = 0, x=0, y=0$

Candidate: $(0,0;0) \quad f=0$

Solve the max problem:

Cand: $(\sqrt{1/2}, \sqrt{1/2}\,;\ 1/2)$, $(-\sqrt{1/2}, -\sqrt{1/2}\,;\ 1/2)$

$$\mathcal{L} = xy - \frac{1}{2}(x^2+y^2)$$

$$H(\mathcal{L}) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

$D_1 = -1$ $\quad \Delta_1 = -1, -1$

$D_2 = 0$ $\quad \Delta_2 = 0$

neg. semidef. $\Rightarrow$ $\mathcal{L}$ concave

$\Downarrow$

the pts are max pts

7.9 / 8.8: max xyz when $\begin{cases} x+y+z \le 1 \\ x \ge 0 \\ y \ge 0 \\ z \ge 0 \end{cases}$

$= \max xyz$ when $\begin{cases} x+y+z \le 1 \\ -x \le 0 \\ -y \le 0 \\ -z \le 0 \end{cases}$

$$\mathcal{L} = xyz - \lambda\cdot(x+y+z) - \mu_1(-x) - \mu_2(-y) - \mu_3(-z)$$
$$= xyz - \lambda(x+y+z) + \mu_1 x + \mu_2 y + \mu_3 z$$

FOC: $\mathcal{L}'_x = yz - \lambda + \mu_1 = 0$
$\mathcal{L}'_y = xz - \lambda + \mu_2 = 0$
$\mathcal{L}'_z = xy - \lambda + \mu_3 = 0$

C: $x+y+z \le 1$
$x \ge 0$
$y \ge 0$
$z \ge 0$

CSC: $\lambda \ge 0$ and $\lambda\cdot(x+y+z-1) = 0$
$\mu_1 \ge 0$ $\quad \mu_1 x = 0$
$\mu_2 \ge 0$ $\quad \mu_2 y = 0$
$\mu_3 \ge 0$ $\quad \mu_3 z = 0$

a) $\lambda = 0$
b) $\lambda > 0$

a) $\lambda = 0$: $yz + \mu_1 = 0 \Rightarrow yz = 0$ and $\mu_1 = 0$
$xz + \mu_2 = 0$ $\quad xz = 0$ $\quad \mu_2 = 0$
$xy + \mu_3 = 0$ $\quad xy = 0$ $\quad \mu_3 = 0$
$\Downarrow$
$\lambda = 0,\ \mu_1 = \mu_2 = \mu_3 = 0,\ xy = yz = xz = 0$
$\Downarrow$

Candidates:
$x=0, y=0, z \in [0,1]$ $\quad f=0$
$x=0, z=0, y \in [0,1]$ $\quad f=0$
$y=0, z=0, x \in [0,1]$ $\quad f=0$

b) $\lambda > 0$: $x+y+z=1$ (because of csc)

Assume $x>0$: CSC: $\mu_1 \cdot x = 0 \Rightarrow \mu_1 = 0$

FOC: $yz - \lambda + \overset{0}{\mu_1} = 0$ $\qquad \lambda = yz > 0 \Downarrow y>0,\ z>0 \Downarrow \mu_2 = 0,\ \mu_3 = 0$ (csc)

$xz - \lambda + \mu_2 = 0$

$xy - \lambda + \mu_3 = 0$

$\mu_1 = \mu_2 = \mu_3 = 0$, $\lambda > 0$

$\lambda = yz = xz = xy > 0 \Rightarrow x>0,\ y\geq 0,\ z\geq 0$

$\Downarrow$ $\qquad$ $\Downarrow$

$y = x \quad z = y \Rightarrow x = y = z$

Candidate point:

$x = y = z = 1/3$, $\overset{1/9}{\lambda} > 0$, $\mu_1 = \mu_2 = \mu_3 = 0$

$f = xyz = \underline{1/27}$

no more candidate pts.

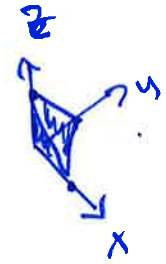
the pt is the max

$x\geq 0, y\geq 0, z\geq 0$
$x+y+z \leq 1$ } **Bounded**

$0 \leq x \leq 1$

$0 \leq y \leq 1$

$0 \leq z \leq 1$



NDCQ:

$$\begin{pmatrix} +1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$\text{rk}\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} = 3$ ✓ when (1), (2), (3) B

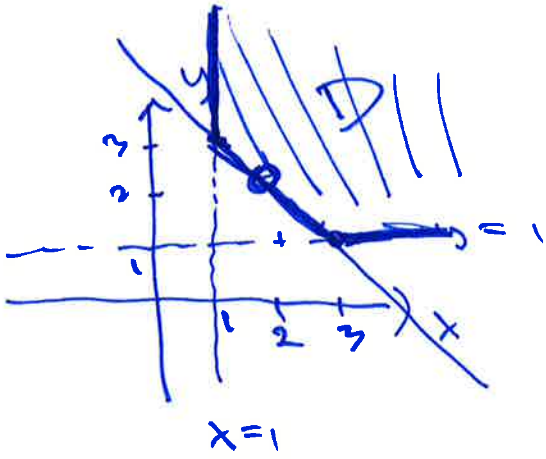
$\text{rk}\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = 3$ ✓ when (1), (2), (4) B

⋮

NDCQ holds at all adm. points.

8.13 $\max \ln(x^2y) - x - y$ when $\begin{cases} x+y \geq 4 \\ x \geq 1 \\ y \geq 1 \end{cases}$

$$= \max \overbrace{2\ln(x)+\ln(y)}^{\ln(x^2y)} - x - y \quad \text{when} \begin{cases} -x-y \leq -4 \\ -x \leq -1 \\ -y \leq -1 \end{cases}$$



$$\mathcal{L} = 2\ln(x)+\ln(y) - x - y - \lambda(-(x+y)) - \mu_1(-x+1) - \mu_2(-y+1)$$
$$= 2\ln x + \ln y - x - y + \lambda(x+y) + \mu_1(x-1) + \mu_2(y-1)$$

$\underline{\text{FOC}}$: $\mathcal{L}'_x = \frac{2}{x} - 1 + \lambda + \mu_1 = 0$

$\mathcal{L}'_y = \frac{1}{y} - 1 + \lambda + \mu_2 = 0$

C: $x+y \geq 4$, $x \geq 1$, $y \geq 1$

$\underline{\text{CSC}}$: $\lambda \geq 0$, $\lambda \cdot (x+y-4) = 0$

$\mu_1 \geq 0$ $\quad \mu_1(x-1) = 0$

$\mu_2 \geq 0$ $\quad \mu_2(y-1) = 0$

a) $x=1$: $1 + \lambda + \mu_1 = 0$ not possible $\Rightarrow$ $x>1$ $\mu_1 = 0$

b) $y=1$: $\lambda + \mu_2 = 0 \Rightarrow \lambda = 0, \mu_2 = 0$, $\frac{2}{x} = 1 \Rightarrow x = 2$

$x+y = 2+1 = 3 \not\geq 4$

not possible $\Rightarrow$ $y>1$ $\mu_2 = 0$

c) $\underline{x+y=4}$: $\left.\begin{array}{l} \frac{2}{x} - 1 + \lambda = 0 \\ \frac{1}{y} - 1 + \lambda = 0 \end{array}\right\}$ $\lambda = 1 - \frac{2}{x} = 1 - \frac{1}{y}$

$\Rightarrow \frac{2}{x} = \frac{1}{y} \Rightarrow x = 2y$

$x+y=4$

$2y+y=4 \quad 3y=4 \quad y=4/3$

c) $\underline{x+y=4}$: $\quad x = 2y \implies y = 4/3, \; x = 8/3$

candidate $\to$

$$\boxed{\begin{array}{l} x = \underline{8/3}, \; y = \underline{4/3} \quad \lambda = 1 - \frac{1}{y} = 1 - \frac{3}{4} = \underline{1/4} \quad \mu_1 = \underline{0}, \; \mu_2 = \underline{0} \\ f = 2\ln x + \ln y - x - y \\ \quad = 2\ln(8/3) + \ln(4/3) - 4 \end{array}}$$

$$L(x,y,\, 1/4, 0, 0) = \underbrace{2\ln x + \ln y} - x - y + \frac{1}{4}(x+y)$$

$$H(L) = \begin{pmatrix} -2/x^2 & 0 \\ 0 & -1/y^2 \end{pmatrix} \qquad D_1 = -\frac{2}{x^2} < 0$$

$$D_2 = \frac{2}{x^2 y^2} > 0$$

L is concave (in it's domain of defn.) $\Longleftarrow$ neg. def. for all (x,y) (where $x \neq 0, y \neq 0$)

$\Downarrow$

$\underline{(8/3, 4/3) \text{ is the max}}$