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Digital Workbook for GRA 6035 Mathematics

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Contents

Part I
Lectures in GRA6035 Mathematics

Lecture 1
Linear Systems and Gaussian Elimination

1.1 Main concepts

An $m \times n$ *linear system* in the variables $x_1, x_2, \ldots, x_n$ is a system of m linear equations in these variables, and can in general be written as

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = & b_m \end{array}$$

A more compact notation is the matrix form $A\mathbf{x} = \mathbf{b}$ using the coefficient matrix A, and an even more compact notation is the *augmented matrix* $\hat{A}$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \quad \text{and} \quad \hat{A} = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right)$$

A solution of this linear system is an n-couple of numbers $(s_1, s_2, \ldots, s_n)$ such that $x_1 = s_1,\ x_2 = s_2, \ldots, x_n = s_n$ solves all m equations simultaneously.

Lemma 1.1. *Any linear system has either no solutions (inconsistent), one unique solution (consistent) or infinitely many solutions (consistent).*

Gaussian elimination is an efficient algorithm that can be used to solve any linear system. We use *elementary row operations* on the augmented matrix of the system until we reach an *echelon form* of the matrix, transform it back to a linear system and solve it by *back substitution*. The following operations are called elementary row operations:

1. To add a multiple of one row to another row
2. To interchange two rows
3. To multiply a row with a nonzero constant

In any non-zero row, the *leading coefficient* is the first (or leftmost) non-zero entry. A matrix is in echelon form if the following conditions hold:

- All rows of zeros appear below non-zero rows.
- Each leading coefficient appears further to the right than the leading coefficients in the rows above.

A *pivot* is a leading coefficient in an echelon form, and the *pivot positions* of a matrix are the positions where there are pivots in the echelon form of the matrix. Back substitution is the process of solving the equations for the variables in the pivot positions (called *basic variables*), starting from the last non-zero equation and continuing the process in reverse order. The non-basic variables are called *free variables*.

Gauss-Jordan elimination is variation, where we continue with elementary row operations until we reach a *reduced echelon form*. A matrix is in reduced echelon form if it is in echelon form, and the following additional condtions are satisfied:

- Each leading coefficient is 1
- All other entries in columns with leading coefficients are 0

Gauss-Jordan elimination can be convenient for linear systems with infinitely many solutions. Gaussian elimination is more efficient for large linear systems.

Lemma 1.2. *Any matrix can be transformed into an echelon form, and also into a reduced echelon form, using elementary row operations. The reduced echelon form is unique. In general, an echelon form is not unique, but its pivot positions are.*

The *rank* of a matrix A, written $\mathrm{rk}A$, is defined as the number of pivot positions in A. It can be computed by finding an echelon form of A and counting the pivots.

Lemma 1.3. *An $m \times n$ linear system is consistent if $\mathrm{rk}A = \mathrm{rk}\hat{A}$, and inconsistent otherwise. If the linear system is consistent, the number of free variables is given by $n - \mathrm{rk}A$.*

1.2 Problems

1.1. Write down the coefficient matrix and the augmented matrix of the following linear systems:

$$a) \quad \begin{aligned} 2x + 5y &= 6 \\ 3x - 7y &= 4 \end{aligned} \qquad\qquad b) \quad \begin{aligned} x + y - z &= 0 \\ x - y + z &= 2 \\ x - 2y + 4z &= 3 \end{aligned}$$

1.2. Write down the linear system in the variables x, y, z with augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 2 & -3 & 1 & 0 \\ 7 & 4 & 1 & 3 \end{array}\right)$$

1.3. Use substitution to solve the linear system

$$\begin{aligned} x + y + z &= 1 \\ x - y + z &= 4 \\ x + 2y + 4z &= 7 \end{aligned}$$

1.4. For what values of h does the following linear system have solutions?

$$\begin{aligned} x + y + z &= 1 \\ x - y + z &= 4 \\ x + 2y + z &= h \end{aligned}$$

1.5. Solve the following linear systems by Gaussian elimination:

$$a)\quad \begin{aligned} x + y + z &= 1 \\ x - y + z &= 4 \\ x + 2y + 4z &= 7 \end{aligned} \qquad\qquad b)\quad \begin{aligned} 2x + 2y - z &= 2 \\ x + y + z &= -2 \\ 2x + 4y - 3z &= 0 \end{aligned}$$

1.6. Solve the following linear system by Gauss-Jordan elimination:

$$\begin{aligned} x + y + z &= 1 \\ x - y + z &= 4 \\ x + 2y + 4z &= 7 \end{aligned}$$

1.7. Solve the following linear systems by Gaussian elimination:

$$a)\quad \begin{aligned} -4x + 6y + 4z &= 4 \\ 2x - y + z &= 1 \end{aligned} \qquad\qquad b)\quad \begin{aligned} 6x + y &= 7 \\ 3x + y &= 4 \\ -6x - 2y &= 1 \end{aligned}$$

1.8. Discuss the number of solutions of the linear system

$$\begin{aligned} x + 2y + 3z &= 1 \\ -x + ay - 21z &= 2 \\ 3x + 7y + az &= b \end{aligned}$$

for all values of the parameters a and b.

1.9. Find the pivot positions of the following matrix:

$$\begin{pmatrix} 1 & 3 & 4 & 1 & 7 \\ 3 & 2 & 1 & 0 & 7 \\ -1 & 3 & 2 & 4 & 9 \end{pmatrix}$$

1.10. Show that the following linear system has infinitely many solutions, and determine the number of degrees of freedom:

$$\begin{aligned} x + 6y - 7z + 3w &= 1 \\ x + 9y - 6z + 4w &= 2 \\ x + 3y - 8z + 4w &= 5 \end{aligned}$$

Find free variables and express the basic variables in terms of the free ones.

1.11. Solve the following linear systems by substitution and by Gaussian elimination:

$$a) \quad \begin{aligned} x - 3y + 6z &= -1 \\ 2x - 5y + 10z &= 0 \\ 3x - 8y + 17z &= 1 \end{aligned} \qquad b) \quad \begin{aligned} x + y + z &= 0 \\ 12x + 2y - 3z &= 5 \\ 3x + 4y + z &= -4 \end{aligned}$$

1.12. Find the rank of the following matrices:

$$a) \quad \begin{pmatrix} 1 & 2 \\ 8 & 16 \end{pmatrix} \qquad b) \quad \begin{pmatrix} 1 & 3 & 4 \\ 2 & 0 & 1 \end{pmatrix} \qquad c) \quad \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -4 & 7 \\ -1 & -2 & -1 & -2 \end{pmatrix}$$

1.13. Find the rank of the following matrices:

$$(a) \quad \begin{pmatrix} 1 & 3 & 0 & 0 \\ 2 & 4 & 0 & -1 \\ 1 & -1 & 2 & 2 \end{pmatrix} \qquad b) \quad \begin{pmatrix} 2 & 1 & 3 & 7 \\ -1 & 4 & 3 & 1 \\ 3 & 2 & 5 & 11 \end{pmatrix} \qquad c) \quad \begin{pmatrix} 1 & -2 & -1 & 1 \\ 2 & 1 & 1 & 2 \\ -1 & 1 & -1 & -3 \\ -2 & -5 & -2 & 0 \end{pmatrix}$$

1.14. Prove that any 4×6 homogeneous linear system has non-trivial solutions.

1.15. Discuss the ranks of the coefficient matrix A and the augmented matrix $\hat{A}$ of the linear system

$$\begin{aligned} x_1 + x_2 + x_3 &= 2q \\ 2x_1 - 3x_2 + 2x_3 &= 4q \\ 3x_1 - 2x_2 + px_3 &= q \end{aligned}$$

for all values of p and q. Use this to determine the number of solutions of the linear system for all values of p and q.

1.16. Midterm Exam in GRA6035 on 24/09/2010, Problem 3
Compute the rank of the matrix

$$A = \begin{pmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & -2 & 4 \end{pmatrix}$$

1.17. Mock Midterm Exam in GRA6035 on 09/2010, Problem 3
Compute the rank of the matrix

$$A = \begin{pmatrix} 1 & 2 & -5 & 0 & -1 \\ 2 & 5 & -8 & 4 & 3 \\ -3 & -9 & 9 & -7 & -2 \\ 3 & 10 & -7 & 11 & 7 \end{pmatrix}$$

1.18. Midterm Exam in GRA6035 on 24/05/2011, Problem 3
Compute the rank of the matrix

$$A = \begin{pmatrix} 2 & 10 & 6 & 8 \\ 1 & 5 & 4 & 11 \\ 3 & 15 & 7 & -2 \end{pmatrix}$$

1.3 Solutions

General remark: In some of the problems, we compute an echelon form. Since the echelon form is not unique, it is possible to get to another echelon form than the one indicated in the solutions below. However, the pivot positions should be the same.

1.1 The coefficient matrix and the augmented matrix of the system is given by

$$a) \quad \begin{pmatrix} 2 & 5 \\ 3 & -7 \end{pmatrix}, \quad \left(\begin{array}{cc|c} 2 & 5 & 6 \\ 3 & -7 & 4 \end{array}\right) \qquad b) \quad \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -2 & 4 \end{pmatrix}, \quad \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 2 \\ 1 & -2 & 4 & 3 \end{array}\right)$$

1.2 The linear system is given by

$$\begin{aligned} x + 2y \phantom{{}+z} &= 4 \\ 2x - 3y + z &= 0 \\ 7x + 4y + z &= 3 \end{aligned}$$

1.3 We solve the linear system

$$\begin{aligned} x + y + z &= 1 \\ x - y + z &= 4 \\ x + 2y + 4z &= 7 \end{aligned}$$

by substitution. First, we solve the first equation for z and get $z = 1 - x - y$. Then we substitute this expression for z in the last two equations. We get

$$\begin{aligned} -2y &= 3 \\ -3x - 2y &= 3 \end{aligned}$$

We solve the first equation for y, and get $y = -1.5$. Then we substitute this value for y in the second equation, and get $x = 0$. Finally, we substitute both these values in $z = 1 - x - y$ and get $z = 2.5$. The solution is therefore $x = 0,\ y = -1.5,\ z = 2.5$.

1.4 We solve the linear system

$$\begin{aligned} x + y + z &= 1 \\ x - y + z &= 4 \\ x + 2y + z &= h \end{aligned}$$

by substitution. First, we solve the first equation for z and get $z = 1 - x - y$. Then we substitute this expression for z in the last two equations. We get

$$\begin{aligned} -2y &= 3 \\ y &= h - 1 \end{aligned}$$

We solve the first equation for y, and get $y = -1.5$. Then we substitute this value for y in the second equation, and get $-1.5 = h - 1$. If $h = -0.5$, this holds and the system have solutions (x is a free variable, $y = -1.5$ and $z = 1 - x - y = 2.5 - x$). If $h \neq -0.5$, then this leads to a contradiction and the system have no solutions. Therefore, the linear system have solutions if and only if $h = -0.5$.

1.5 The linear systems have the following augmented matrices:

$$a) \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 4 \\ 1 & 2 & 4 & 7 \end{array}\right) \qquad b) \quad \left(\begin{array}{ccc|c} 2 & 2 & -1 & 2 \\ 1 & 1 & 1 & -2 \\ 2 & 4 & -3 & 0 \end{array}\right)$$

a) To solve the system, we reduce the system to an echelon form using elementary row operations. The row operations are indicated.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 4 \\ 1 & 2 & 4 & 7 \end{array}\right) \qquad \begin{array}{l} R_2 \leftarrow R_2 + (-1)R_1 \\ R_3 \leftarrow R_3 + (-1)R_1 \end{array}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 3 \\ 0 & 1 & 3 & 6 \end{array}\right) \qquad R_3 \leftarrow R_3 + (0.5)R_2$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 3 \\ 0 & 0 & 3 & 7.5 \end{array}\right)$$

From the last equation we get $z = 2.5$, substitution in the second equation gives $y = -1.5$, and substitution in the first equation gives $x = 0$. Therefore, the solution of a) is $x = 0,\ y = -1.5,\ z = 2.5$.
b) To solve the system, we reduce the system to an echelon form using elementary row operations. The row operations are indicated.

$$
\left(\begin{array}{ccc|c} 2 & 2 & -1 & 2 \\ 1 & 1 & 1 & -2 \\ 2 & 4 & -3 & 0 \end{array}\right) \qquad \begin{array}{r} R_2 \leftarrow R_2 + (-0.5)R_1 \\ R_3 \leftarrow R_3 + (-1)R_1 \end{array}
$$

$$
\Rightarrow \left(\begin{array}{ccc|c} 2 & 2 & -1 & 2 \\ 0 & 0 & 1.5 & -3 \\ 0 & 2 & -2 & -2 \end{array}\right) \qquad \begin{array}{r} R_2 \leftarrow R_3 \\ R_3 \leftarrow R_2 \end{array}
$$

$$
\Rightarrow \left(\begin{array}{ccc|c} 2 & 2 & -1 & 2 \\ 0 & 2 & -2 & -2 \\ 0 & 0 & 1.5 & -3 \end{array}\right)
$$

From the last equation we get $z = -2$, substitution in the second equation gives $y = -3$, and substitution in the first equation gives $x = 3$. Therefore, the solution of b) is $x = 3,\ y = -3,\ z = -2$.

1.6 We reduce the system to the reduced echelon form using elementary row operations:

$$
\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 4 \\ 1 & 2 & 4 & 7 \end{array}\right) \qquad \begin{array}{r} R_2 \leftarrow R_2 + (-1)R_1 \\ R_3 \leftarrow R_3 + (-1)R_1 \end{array}
$$

$$
\Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 3 \\ 0 & 1 & 3 & 6 \end{array}\right) \qquad R_3 \leftarrow R_3 + (0.5)R_2
$$

$$
\Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 3 \\ 0 & 0 & 3 & 7.5 \end{array}\right) \qquad \begin{array}{r} R_2 \leftarrow (-1/2) \cdot R_2 \\ R_3 \leftarrow (1/3) \cdot R_3 \\ R_1 \leftarrow R_1 + (-1)R_2 + (-1)R_3 \end{array}
$$

$$
\Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1.5 \\ 0 & 0 & 1 & 2.5 \end{array}\right)
$$

$$
\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1.5 \\ 0 & 0 & 1 & 2.5 \end{array}\right)
$$

We read off the solution of the system: $x = 0,\ y = -1.5,\ z = 2.5$.

1.7 a) We reduce the linear system to an echelon form:

$$
\left(\begin{array}{ccc|c} -4 & 6 & 4 & 4 \\ 2 & -1 & 1 & 1 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{ccc|c} -4 & 6 & 4 & 4 \\ 0 & 2 & 3 & 3 \end{array}\right)
$$

We see that the system has infinitely many solutions (z is a free variable and x, y are basic variables). We reduce the system to a reduced echelon form:

$$
\left(\begin{array}{ccc|c} -4 & 6 & 4 & 4 \\ 0 & 2 & 3 & 3 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 1 & -1.5 & -1 & -1 \\ 0 & 1 & 1.5 & 1.5 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 1 & 0 & 1.25 & 1.25 \\ 0 & 1 & 1.5 & 1.5 \end{array}\right)
$$

We see that $x+1.25z=1.25$, $y+1.5z=1.5$. Therefore the solution is given by $x=1.25-1.25z$, $y=1.5-1.5z$ (z is a free variable).
b) We reduce the linear system to an echelon form:

$$\left(\begin{array}{cc|c} 6 & 1 & 7 \\ 3 & 1 & 4 \\ -6 & -2 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 6 & 1 & 7 \\ 0 & 0.5 & 0.5 \\ 0 & -1 & 8 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 6 & 1 & 7 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 9 \end{array}\right)$$

We see that the system has no solutions.

1.8 We find the augmented matrix of the linear system and reduce it to an echelon form:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ -1 & a & -21 & 2 \\ 3 & 7 & a & b \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & a+2 & -18 & 3 \\ 0 & 1 & a-9 & b-3 \end{array}\right)$$

We interchange the last two rows to avoid division with $a+2$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & a-9 & b-3 \\ 0 & a+2 & -18 & 3 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & a-9 & b-3 \\ 0 & 0 & -18-(a-9)(a+2) & 3-(b-3)(a+2) \end{array}\right)$$

We compute $-18-(a-9)(a+2)=7a-a^2$. So when $a\neq 0$ and $a\neq 7$, the system has a unique solution. When $a=0$, we compute $3-(b-3)(a+2)=9-2b$. So when $a=0$ and $b\neq 9/2$, the system is inconsistent, and when $a=0$, $b=9/2$, the system has infinitely many solutions (one degree of freedom). When $a=7$, we compute $3-(b-3)(a+2)=30-9b$. So when $a=7$ and $b\neq 30/9=10/3$, the system is inconsistent, and when $a=7$, $b=10/3$, the system has infinitely many solutions (one degree of freedom).

1.9 We redude the matrix to an echelon form using row elementary row operations:

$$\begin{pmatrix} 1 & 3 & 4 & 1 & 7 \\ 3 & 2 & 1 & 0 & 7 \\ -1 & 3 & 2 & 4 & 9 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 & 4 & 1 & 7 \\ 0 & -7 & -11 & -3 & -14 \\ 0 & 6 & 6 & 5 & 16 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 & 4 & 1 & 7 \\ 0 & -7 & -11 & -3 & -14 \\ 0 & 0 & -24/7 & * & * \end{pmatrix}$$

We have not computed the entries marked $*$ since they are not needed to find the pivot positions. The pivot positions in the matrix are marked with a box:

$$\begin{pmatrix} \boxed{1} & 3 & 4 & 1 & 7 \\ 3 & \boxed{2} & 1 & 0 & 7 \\ -1 & 3 & \boxed{2} & 4 & 9 \end{pmatrix}$$

1.10 We find the augmented matrix and reduce it to an echelon form using elementary row operations:

$$\left(\begin{array}{cccc|c}1&6&-7&3&1\\1&9&-6&4&2\\1&3&-8&4&5\end{array}\right) \Rightarrow \left(\begin{array}{cccc|c}1&6&-7&3&1\\0&3&1&1&1\\0&-3&-1&1&4\end{array}\right) \Rightarrow \left(\begin{array}{cccc|c}1&6&-7&3&1\\0&3&1&1&1\\0&0&0&2&5\end{array}\right)$$

We see that the system has infinitely many solutions and one degree of freedom (z is a free variable and x, y, w are basic variables). To express x, y, w in terms of z, we find the reduced echelon form:

$$\left(\begin{array}{cccc|c}1&6&-7&3&1\\0&3&1&1&1\\0&0&0&2&5\end{array}\right) \Rightarrow \left(\begin{array}{cccc|c}1&6&-7&3&1\\0&1&1/3&1/3&1/3\\0&0&0&1&5/2\end{array}\right) \Rightarrow \left(\begin{array}{cccc|c}1&0&-9&0&-7/2\\0&1&1/3&0&-1/2\\0&0&0&1&5/2\end{array}\right)$$

We see that $x-9z=-7/2$, $y+z/3=-1/2$ and $w=5/2$. This means that the solution is given by $x=9z-7/2$, $y=-z/3-1/2$, $w=5/2$ (z is a free variable).

1.11 a) We find the augmented matrix of the linear system and reduce it to an echelon form:

$$\left(\begin{array}{ccc|c}1&-3&6&-1\\2&-5&10&0\\3&-8&17&1\end{array}\right) \Rightarrow \left(\begin{array}{ccc|c}1&-3&6&-1\\0&1&-2&2\\0&0&1&2\end{array}\right)$$

Back substitution gives the solution $x=5,\ y=6,\ z=2$.
b) We find the augmented matrix of the linear system and reduce it to an echelon form:

$$\left(\begin{array}{ccc|c}1&1&1&0\\12&2&-3&5\\3&4&1&-4\end{array}\right) \Rightarrow \left(\begin{array}{ccc|c}1&1&1&0\\0&1&-2&-4\\0&0&-35&-35\end{array}\right)$$

Back substitution gives the solution $x=1,\ y=-2,\ z=1$.

1.12 a) We find an echelon form of the matrix:

$$\begin{pmatrix}1&2\\8&16\end{pmatrix} \Rightarrow \begin{pmatrix}1&2\\0&0\end{pmatrix}$$

We see that the rank of A is 1 since there is one pivot position.
b) We find an echelon form of the matrix:

$$\begin{pmatrix}1&3&4\\2&0&1\end{pmatrix} \Rightarrow \begin{pmatrix}1&3&4\\0&-6&-7\end{pmatrix}$$

We see that the rank of A is 2 since there are two pivot positions.
c) We find an echelon form of the matrix:

$$\begin{pmatrix}1&2&-1&3\\2&4&-4&7\\-1&-2&-1&-2\end{pmatrix} \Rightarrow \begin{pmatrix}1&2&-1&3\\0&0&-2&1\\0&0&-2&1\end{pmatrix} \Rightarrow \begin{pmatrix}1&2&-1&3\\0&0&-2&1\\0&0&0&0\end{pmatrix}$$

We see that the rank of A is 2 since there are two pivot positions.

1.13 a) We find an echelon form of the matrix:

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 2 & 4 & 0 & -1 \\ 1 & -1 & 2 & 2 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & -2 & 0 & -1 \\ 0 & -4 & 2 & 2 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 2 & 4 \end{pmatrix}$$

We see that the rank of A is 3 by counting pivot positions.
b) We find an echelon form of the matrix:

$$\begin{pmatrix} 2 & 1 & 3 & 7 \\ -1 & 4 & 3 & 1 \\ 3 & 2 & 5 & 11 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 2 & 1 & 3 & 7 \\ 0 & 4.5 & 4.5 & 4.5 \\ 0 & 0.5 & 0.5 & 0.5 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 2 & 1 & 3 & 7 \\ 0 & 4.5 & 4.5 & 4.5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We see that the rank of A is 2 by counting pivot positions.
c) We find an echelon form of the matrix:

$$\begin{pmatrix} 1 & -2 & -1 & 1 \\ 2 & 1 & 1 & 2 \\ -1 & 1 & -1 & -3 \\ -2 & -5 & -2 & 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & -2 & -1 & 1 \\ 0 & 5 & 3 & 0 \\ 0 & -1 & -2 & -2 \\ 0 & -9 & -4 & 2 \end{pmatrix}$$

We interchange the two middle rows to get easier computations:

$$\begin{pmatrix} 1 & -2 & -1 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & 5 & 3 & 0 \\ 0 & -9 & -4 & 2 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & -2 & -1 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & 0 & -7 & -10 \\ 0 & 0 & 14 & 20 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & -2 & -1 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & 0 & -7 & -10 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We see that the rank of A is 3 by counting pivot positions. T

1.14 Let A be the 4×6 coefficient matrix of the homogeneous linear system. Then $n = 6$ (there are 6 variables) while $\mathrm{rk} A \leq 4$ (there cannot be more than one pivot position in each row). So there are at least two degrees of freedom, and the system has non-trivial solutions.

1.15 We find the coefficient matrix A and the augmented matrix $\hat{A}$ of the system:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & 2 \\ 3 & -2 & p \end{pmatrix}, \quad \hat{A} = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2q \\ 2 & -3 & 2 & 4q \\ 3 & -2 & p & q \end{array}\right)$$

Then we compute an echelon form of $\hat{A}$ (which contains an echelon form of A as the first three columns):

$$\hat{A} = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2q \\ 2 & -3 & 2 & 4q \\ 3 & -2 & p & q \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2q \\ 0 & -5 & 0 & 0 \\ 0 & 0 & p-3 & -5q \end{array}\right)$$

By counting pivot positions, we see that the ranks are given by

$$\operatorname{rk} A = \begin{cases} 3 & p \neq 3 \\ 2 & p = 3 \end{cases} \qquad \operatorname{rk} \hat{A} = \begin{cases} 3 & p \neq 3 \text{ or } q \neq 0 \\ 2 & p = 3 \text{ and } q = 0 \end{cases}$$

The linear system has one solution if $p \neq 3$, no solutions if $p = 3$ and $q \neq 0$, and infinitely many solutions (one degree of freedom) if $p = 3$ and $q = 0$.

1.16 Midterm Exam in GRA6035 on 24/09/2010, Problem 3

We compute an echelon form of A using elementary row operations, and get

$$A = \begin{pmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & -2 & 4 \end{pmatrix} \dashrightarrow \begin{pmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 0 & -6 \end{pmatrix}$$

Hence A has rank 3.

1.17 Mock Midterm Exam in GRA6035 on 09/2010, Problem 3

We compute an echelon form of A using elementary row operations, and get

$$A = \begin{pmatrix} 1 & 2 & -5 & 0 & -1 \\ 2 & 5 & -8 & 4 & 3 \\ -3 & -9 & 9 & -7 & -2 \\ 3 & 10 & -7 & 11 & 7 \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 2 & -5 & 0 & -1 \\ 0 & 1 & 2 & 4 & 5 \\ 0 & 0 & 0 & 5 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence A has rank 3.

1.18 Midterm Exam in GRA6035 on 24/05/2011, Problem 3

We compute an echelon form of A using elementary row operations, and get

$$A = \begin{pmatrix} 2 & 10 & 6 & 8 \\ 1 & 5 & 4 & 11 \\ 3 & 15 & 7 & -2 \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 5 & 4 & 11 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence A has rank 2.

Lecture 2
Matrices and Matrix Algebra

2.1 Main concepts

An $m \times n$ *matrix* is a rectangular array of numbers (with m rows and n columns). The entry of the matrix A in row i and column j is denoted a_{ij}. The usual notation is

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

The matrix A is called square if $m = n$. A main diagonal of a square matrix consists of the entries $a_{11}, a_{22}, \dots, a_{nn}$, and the matrix is called a *diagonal* matrix if all other entries are zero. It is called *upper triangular* if all entries below the main diagonal are zero.

We may add, subtract and mulitiply matrices of compatible dimensions, and write $A + B$, $A - B$ and $A \cdot B$. We may also multiply a number with a matrix and *transpose* a matrix, and we write $c \cdot A$ for scalar multiplication and A^T for the transpose of A. A square matrix is called *symmetric* if $A^T = A$.

The *zero matrix* 0 is the matrix where all entries are zero, and the identity matrix I is the diagonal matrix

$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

The identity matrix has the property that $I \cdot A = A \cdot I = A$ for any matrix A. Matrix multiplication is in general not commutative, so $A \cdot B \neq B \cdot A$. Otherwise, computations with matrices follow similar rules as multiplication with numbers.

An inverse of a square matrix A is a matrix A^{-1} such that $A \cdot A^{-1} = A^{-1} \cdot A = I$. If it exists, A is called an *invertible* matrix, and in this case A^{-1} is unique. We cannot define division for matrices since, in general, $A^{-1} \cdot B \neq B \cdot A^{-1}$.

We may compute the *determinant* $\det(A) = |A|$ of any square matrix A, and the result is a number. When A is a 2×2 matrix, then

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

In general, the determinant can be computed by cofactor expansion along any row or column. Along the first row we get the cofactor expension

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{nn}C_{nn}$$

where $C_{ij} = (-1)^{i+j} \cdot M_{ij}$ is the cofactor in position (i, j) and the *minor* M_{ij} is the determinant of the matrix obtained by deleting row i and column j from A. Elementary row operations can be used to simplify the computation of determinants.

1. A is invertible if and only if $\det(A) \neq 0$
2. $\det(AB) = \det(A)\det(B)$
3. $\det(A^T) = \det(A)$
4. $A\mathbf{x} = \mathbf{b}$ has solution $x = A^{-1}\mathbf{b}$ if $\det(A) \neq 0$

Let A be an $m \times n$ matrix. A *minor* of order r in A is the determinant of an $r \times r$ submatrix of A obtained by deleting $m - r$ rows and $n - r$ columns. The rank of A is the maximal order of a non-zero minor in A. In particular, an $n \times n$ matrix A has $\text{rk}(A) = n$ if and only if $\det(A) \neq 0$.

2.2 Problems

2.1. Compute $4A + 2B$, AB, BA, BI and IA when

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 6 \\ 7 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2.2. One of the laws of matrix algebra states that $(AB)^T = B^T A^T$. Prove this when A and B are 2×2-matrices.

2.3. Simplify the following matrix expressions:

$$\begin{aligned} a) &\quad AB(BC - CB) + (CA - AB)BC + CA(A - B)C \\ b) &\quad (A - B)(C - A) + (C - B)(A - C) + (C - A)^2 \end{aligned}$$

2.4. A general $m \times n$-matrix is often written $A = (a_{ij})_{m \times n}$, where a_{ij} is the entry of A in row i and column j. Prove that if $m = n$ and $a_{ij} = a_{ji}$ for all i and j, then

$A = A^T$. Give a concrete example of a matrix with this property, and explain why it is reasonable to call a matrix A symmetric when $A = A^T$.

2.5. Compute D^2, D^3 and D^n when

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

2.6. Write down the 3×3 linear system corresponding to the matrix equation $A\mathbf{x} = \mathbf{b}$ when

$$A = \begin{pmatrix} 3 & 1 & 5 \\ 5 & -3 & 2 \\ 4 & -3 & -1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix}$$

2.7. Initially, three firms A, B and C (numbered 1, 2 and 3) share the market for a certain commodity. Firm A has 20% of the marked, B has 60% and C has 20%. In course of the next year, the following changes occur:

A keeps 85% of its customers, while losing 5% to B and 10% to C
B keeps 55% of its customers, while losing 10% to A and 35% to C
C keeps 85% of its customers, while losing 10% to A and 5% to B

We can represent market shares of the three firms by means of *a market share vector*, defined as a column vector $\mathbf{s}$ whose components are all non-negative and sum to 1. Define the matrix $\mathbf{T}$ and the initial share vector $\mathbf{s}$ by

$$T = \begin{pmatrix} 0.85 & 0.10 & 0.10 \\ 0.05 & 0.55 & 0.05 \\ 0.10 & 0.35 & 0.85 \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} 0.2 \\ 0.6 \\ 0.2 \end{pmatrix}$$

The matrix T is called the *transition matrix*. Compute the vector $T\mathbf{s}$, show that it is also a market share vector, and give an interpretation. What is the interpretation of $T^2\mathbf{s}$ and $T^3\mathbf{s}$? Finally, compute $T\mathbf{q}$ when

$$\mathbf{q} = \begin{pmatrix} 0.4 \\ 0.1 \\ 0.5 \end{pmatrix}$$

and give an interpretation.

2.8. Compute the following matrix product using partitioning. Check the result by ordinary matrix multiplication:

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 0 & -1 \end{array}\right) \cdot \left(\begin{array}{cc} 2 & -1 \\ 0 & 1 \\ \hline 1 & 1 \end{array}\right)$$

2.9. If $A = (a_{ij})_{n\times n}$ is an $n\times n$-matrix, then its determinant may be computed by

$$|A| = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

where C_{ij} is the cofactor in position (i, j). This is called cofactor expansion along the first row. Similarly one may compute $|A|$ by cofactor expansion along any row or column. Compute $|A|$ using cofactor expansion along the first column, and then along the third row, when

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 1 & 0 & 8 \end{pmatrix}$$

Check that you get the same answer. Is A invertible?

2.10. Let A and B be 3×3-matrices with $|A| = 2$ and $|B| = -5$. Find $|AB|$, $|-3A|$ and $|-2A^T|$. Compute $|C|$ when C is the matrix obtained from B by interchanging two rows.

2.11. Compute the determinant using elementary row operations:

$$\begin{vmatrix} 3 & 1 & 5 \\ 9 & 3 & 15 \\ -3 & -1 & -5 \end{vmatrix}$$

2.12. Without computing the determinants, show that

$$\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & a^2+c^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix} = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2$$

2.13. Find the inverse matrix A^{-1}, if it exists, when A is the matrix given by

$$a)\quad A = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \qquad b)\quad A = \begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix} \qquad c)\quad A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

2.14. Compute the cofactor matrix, the adjoint matrix and the inverse matrix of these matrices:

$$a)\quad A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 1 & 0 & 8 \end{pmatrix} \qquad b)\quad B = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Verify that $AA^{-1} = I$ and that $BB^{-1} = I$.

2.15. Write the linear system of equations

$$\begin{aligned} 5x_1 + x_2 &= 3 \\ 2x_1 - x_2 &= 4 \end{aligned}$$

on matrix form $A\mathbf{x} = \mathbf{b}$ and solve it using A^{-1}.

2.16. There is an efficient way of finding the inverse of a square matrix using row operations. Suppose we want to find the inverse of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 5 & 7 \end{pmatrix}$$

To do this we form the partitioned matrix

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 3 & 3 & 0 & 1 & 0 \\ 2 & 5 & 7 & 0 & 0 & 1 \end{array}\right)$$

and then reduced it to reduced echelon form using elementary row operations: First, we add (-1) times the first row to the second row

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 3 & 3 & 0 & 1 & 0 \\ 2 & 5 & 7 & 0 & 0 & 1 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 2 & 5 & 7 & 0 & 0 & 1 \end{array}\right)$$

Then we add (-2) times the first row to the last row

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 2 & 5 & 7 & 0 & 0 & 1 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \end{array}\right)$$

Then we add (-1) times the second row to the third

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

Next, we add (-3) times the last row to the first

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & 3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

Then we add (-2) times the second row to the first

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & 3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & 1 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

We now have the partitioned matrix $(I|A^{-1})$ and thus

$$A^{-1} = \begin{pmatrix} 6 & 1 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

Use the same technique to find the inverse of the following matrices:

$$a)\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad b)\ \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad c)\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad d)\ \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

2.17. Describe all minors of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$$

It is not necessary to compute all the minors.

2.18. Determine the ranks of these matrices for all values of the parameters:

$$a)\ \begin{pmatrix} x & 0 & x^2-2 \\ 0 & 1 & 1 \\ -1 & x & x-1 \end{pmatrix} \qquad b)\ \begin{pmatrix} t+3 & 5 & 6 \\ -1 & t-3 & -6 \\ 1 & 1 & t+4 \end{pmatrix}$$

2.19. Give an example where $\mathrm{rk}(AB) \neq \mathrm{rk}(BA)$. Hint: Try some 2×2 matrices.

2.20. Use minors to determine if the systems have solutions. If they do, determine the number of degrees of freedom. Find all solutions and check the results.

$$a)\ \begin{aligned} -2x_1 - 3x_2 + x_3 &= 3 \\ 4x_1 + 6x_2 - 2x_3 &= 1 \end{aligned} \qquad b)\ \begin{aligned} x_1 + x_2 - x_3 + x_4 &= 2 \\ 2x_1 - x_2 + x_3 - 3x_4 &= 1 \end{aligned}$$

$$c)\ \begin{aligned} x_1 - x_2 + 2x_3 + x_4 &= 1 \\ 2x_1 + x_2 - x_3 + 3x_4 &= 3 \\ x_1 + 5x_2 - 8x_3 + x_4 &= 1 \\ 4x_1 + 5x_2 - 7x_3 + 7x_4 &= 7 \end{aligned} \qquad d)\ \begin{aligned} x_1 + x_2 + 2x_3 + x_4 &= 5 \\ 2x_1 + 3x_2 - x_3 - 2x_4 &= 2 \\ 4x_1 + 5x_2 + 3x_3 \phantom{{}+ x_4} &= 7 \end{aligned}$$

2.21. Let $A\mathbf{x} = \mathbf{b}$ be a linear system of equations in matrix form. Prove that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are both solutions of the system, then so is $\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$ for every number λ. Use this fact to prove that a linear system of equations that is consistent has either one solution or infinitely many solutions.

2.22. Find the rank of A for all values of the parameter t, and solve $A\mathbf{x} = \mathbf{b}$ when $t = -3$:

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & t \\ 4 & 7-t & -6 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 11 \\ 3 \\ 6 \end{pmatrix}$$

2.23. Midterm Exam in GRA6035 on 24/09/2010, Problem 1
Consider the linear system

$$\begin{pmatrix} 1 & -3 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 4 \\ 0 \end{pmatrix}$$

Which statement is true?

1. The linear system is inconsistent.
2. The linear system has a unique solution.
3. The linear system has one degree of freedom
4. The linear system has two degrees of freedom
5. I prefer not to answer.

2.24. Mock Midterm Exam in GRA6035 on 09/2010, Problem 1
Consider the linear system

$$\begin{pmatrix} 3 & -9 & 12 & -9 & 0 \\ 0 & 2 & -4 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -9 \\ -14 \\ 4 \\ 7 \end{pmatrix}$$

Which statement is true?

1. The linear system has a unique solution.
2. The linear system has one degree of freedom
3. The linear system has two degrees of freedom
4. The linear system is inconsistent.
5. I prefer not to answer.

2.25. Midterm Exam in GRA6035 on 24/05/2011, Problem 3
Consider the linear system

$$\begin{pmatrix} 1 & 2 & -3 & -1 & 0 \\ 0 & 1 & 7 & 3 & -4 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

Which statement is true?

1. The linear system is inconsistent
2. The linear system has a unique solution
3. The linear system has one degree of freedom
4. The linear system has two degrees of freedom
5. I prefer not to answer.

2.3 Solutions

2.1 We have

$$4A+2B=\begin{pmatrix}12 & 24\\ 30 & 4\end{pmatrix},\quad AB=\begin{pmatrix}25 & 12\\ 15 & 24\end{pmatrix},\quad BA=\begin{pmatrix}28 & 12\\ 14 & 21\end{pmatrix},\quad BI=B,\quad IA=A$$

2.2 Let

$$A=\begin{pmatrix}a & b\\ c & d\end{pmatrix},\quad B=\begin{pmatrix}x & y\\ z & w\end{pmatrix}$$

Then we have

$$AB=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}x & y\\ z & w\end{pmatrix}=\begin{pmatrix}ax+bz & bw+ay\\ cx+dz & dw+cy\end{pmatrix}\implies (AB)^T=\begin{pmatrix}ax+bz & cx+dz\\ bw+ay & dw+cy\end{pmatrix}$$

and

$$A^T=\begin{pmatrix}a & c\\ b & d\end{pmatrix},\quad B^T=\begin{pmatrix}x & z\\ y & w\end{pmatrix}\implies B^TA^T=\begin{pmatrix}ax+bz & cx+dz\\ bw+ay & dw+cy\end{pmatrix}$$

Comparing the expressions, we see that $(AB)^T=B^TA^T$.

2.3 We have

$$\begin{aligned}
(a)\quad & AB(BC-CB)+(CA-AB)BC+CA(A-B)C=ABBC-ABCB+CABC\\
& -ABBC+CAAC-CABC=-ABCB+CAAC=-ABCB+CA^2C\\
(b)\quad & (A-B)(C-A)+(C-B)(A-C)+(C-A)^2=AC-A^2-BC+BA+CA\\
& -C^2-BA+BC+C^2-CA-AC+A^2=0
\end{aligned}$$

2.4 The entry in position (j,i) in A^T equals the entry in position (i,j) in A. Therefore, a square matrix A satisfies $A^T=A$ if $a_{ij}=a_{ji}$. The matrix

$$A=\begin{pmatrix}13 & 3 & 2\\ 3 & -2 & 4\\ 2 & 4 & 3\end{pmatrix}$$

has this property. The condition that $a_{ij}=a_{ji}$ is a symmetry along the diagonal of A, so it is reasonable to call a matrix with $A^T=A$ symmetric.

2.5 We compute

$$D^2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D^3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix}^3 = \begin{pmatrix} 8 & 0 & 0 \\ 0 & -27 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$D^n = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix}^n = \begin{pmatrix} 2^n & 0 & 0 \\ 0 & (-3)^n & 0 \\ 0 & 0 & (-1)^n \end{pmatrix}$$

2.6 We compute

$$A\mathbf{x} = \begin{pmatrix} 3 & 1 & 5 \\ 5 & -3 & 2 \\ 4 & -3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 + x_2 + 5x_3 \\ 5x_1 - 3x_2 + 2x_3 \\ 4x_1 - 3x_2 - x_3 \end{pmatrix}$$

Thus we see that $A\mathbf{x} = \mathbf{b}$ if and only if

$$\begin{aligned} 3x_1 + x_2 + 5x_3 &= 4 \\ 5x_1 - 3x_2 + 2x_3 &= -2 \\ 4x_1 - 3x_2 - x_3 &= -1 \end{aligned}$$

2.7 We compute

$$T\mathbf{s} = \begin{pmatrix} 0.85 & 0.10 & 0.10 \\ 0.05 & 0.55 & 0.05 \\ 0.10 & 0.35 & 0.85 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.6 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.35 \\ 0.4 \end{pmatrix}$$

This vector is a market share vector since $0.25 + 0.35 + 0.4 = 1$, and it represents the market shares after one year. We have $T^2\mathbf{s} = T(T\mathbf{s})$ and $T^3\mathbf{s} = T(T^2\mathbf{s})$, so these vectors are the marked share vectors after two and three years. Finally, we compute

$$T\mathbf{q} = \begin{pmatrix} 0.85 & 0.10 & 0.10 \\ 0.05 & 0.55 & 0.05 \\ 0.10 & 0.35 & 0.85 \end{pmatrix} \begin{pmatrix} 0.4 \\ 0.1 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.1 \\ 0.5 \end{pmatrix}$$

We see that $T\mathbf{q} = \mathbf{q}$; if the market share vector is $\mathbf{q}$, then it does not change. Hence $\mathbf{q}$ is an *equilibrium*.

2.8 We write the matrix product as

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 0 & -1 \end{array}\right) \cdot \left(\begin{array}{cc} 2 & -1 \\ 0 & 1 \\ \hline 1 & 1 \end{array}\right) = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = AC + BD$$

We compute

$$AC = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -2 & 1 \end{pmatrix}, \quad BD = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

Hence, we get

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 0 & -1 \end{array}\right) \cdot \left(\begin{array}{cc} 2 & -1 \\ 0 & 1 \\ \hline 1 & 1 \end{array}\right) = \begin{pmatrix} 2 & 0 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -3 & 0 \end{pmatrix}$$

Ordinary matrix multiplication gives the same result.

2.9 We first calculate $|A|$ using cofactor expansion along the first column:

$$\begin{aligned} |A| &= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} \\ &= (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 5 & 6 \\ 0 & 8 \end{vmatrix} + (-1)^{2+1} \cdot 0 \cdot \begin{vmatrix} 2 & 3 \\ 0 & 8 \end{vmatrix} + (-1)^{3+1} \cdot 1 \cdot \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} \\ &= (5 \cdot 8 - 0 \cdot 6) + 0 + (2 \cdot 6 - 5 \cdot 3) \\ &= 40 + 12 - 15 = 37 \end{aligned}$$

We then calculate $|A|$ using cofactor expansion along the third row:

$$\begin{aligned} |A| &= a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} \\ &= (-1)^{3+1} \cdot 1 \cdot \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} + (-1)^{3+2} \cdot 0 \cdot \begin{vmatrix} 1 & 3 \\ 0 & 6 \end{vmatrix} + (-1)^{3+3} \cdot 8 \cdot \begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix} \\ &= (2 \cdot 6 - 5 \cdot 3) + 0 + 8 \cdot (1 \cdot 5 - 0 \cdot 2) \\ &= 12 - 15 + 8 \cdot 5 = 37 \end{aligned}$$

We see that $\det(A) = 37 \neq 0$ using both methods, hence A is invertible.

2.10 We compute

$$\begin{aligned} |AB| &= |A||B| = 2 \cdot (-5) = -10 \\ |-3A| &= (-3)^3|A| = (-27) \cdot 2 = -54 \\ |-2A^T| &= (-2)^3|A^T| = (-8) \cdot |A| = (-8) \cdot 2 = -16 \\ |C| &= -|B| = -(-5) = 5 \end{aligned}$$

2.11 If we add the first row to the last row to simplify the determinant, we get

$$\begin{vmatrix} 3 & 1 & 5 \\ 9 & 3 & 15 \\ -3 & -1 & -5 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 5 \\ 9 & 3 & 15 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

2.12 We have that

$$A = \begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix} \quad \Rightarrow \quad A^2 = \begin{pmatrix} b^2+c^2 & ab & ac \\ ab & a^2+c^2 & bc \\ ac & bc & a^2+b^2 \end{pmatrix}$$

This implies that

$$\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = |A|^2 = |A||A| = |AA| = |A^2| = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & a^2+c^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$$

2.13 To determine which matrices are invertible, we calculate the determinants:

$$a) \quad \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0, \quad b) \quad \begin{vmatrix} 1 & 3 \\ -1 & 3 \end{vmatrix} = 6 \neq 0, \quad c) \quad \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

Hence the matrices in b) and c) are invertible, and we have

$$b) \quad \begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix}, \quad c) \quad \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

2.14 In order to find the cofactor matrix, we must find all the cofactors of A:

$$C_{11} = (-1)^{1+1} \cdot \begin{vmatrix} 5 & 6 \\ 0 & 8 \end{vmatrix} = 40, \; C_{12} = (-1)^{1+2} \cdot \begin{vmatrix} 0 & 6 \\ 1 & 8 \end{vmatrix} = 6, \; C_{13} = (-1)^{3+1} \cdot \begin{vmatrix} 0 & 5 \\ 1 & 0 \end{vmatrix} = -5$$

$$C_{21} = (-1)^{2+1} \cdot \begin{vmatrix} 2 & 3 \\ 0 & 8 \end{vmatrix} = -16, \; C_{22} = (-1)^{2+2} \cdot \begin{vmatrix} 1 & 3 \\ 1 & 8 \end{vmatrix} = 5, \; C_{23} = (-1)^{2+3} \cdot \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2$$

$$C_{31} = (-1)^{3+1} \cdot \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = -3, \; C_{32} = (-1)^{3+2} \cdot \begin{vmatrix} 1 & 3 \\ 0 & 6 \end{vmatrix} = -6, \; C_{33} = (-1)^{3+3} \cdot \begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix} = 5$$

From this we find the cofactor matrix and the adjoint matrix of A:

$$\begin{pmatrix} 40 & 6 & -5 \\ -16 & 5 & 2 \\ -3 & -6 & 5 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 40 & 6 & -5 \\ -16 & 5 & 2 \\ -3 & -6 & 5 \end{pmatrix}^T = \begin{pmatrix} 40 & -16 & -3 \\ 6 & 5 & -6 \\ -5 & 2 & 5 \end{pmatrix}$$

The determinant $|A|$ of A is 37 from the problem above. The inverse matrix is then

$$A^{-1} = \frac{1}{37} \begin{pmatrix} 40 & -16 & -3 \\ 6 & 5 & -6 \\ -5 & 2 & 5 \end{pmatrix} = \begin{pmatrix} \frac{40}{37} & -\frac{16}{37} & -\frac{3}{37} \\ \frac{6}{37} & \frac{5}{37} & -\frac{6}{37} \\ -\frac{5}{37} & \frac{2}{37} & \frac{5}{37} \end{pmatrix}$$

Similarly, we find the cofactor matrix and the adjoint matrix of B to be

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & -b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We compute that $|B| = 1$, and it follows that B^{-1} is given by

$$B^{-1} = \begin{pmatrix} 1 & 0 & -b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We verify that $AA^{-1} = BB^{-1} = I$.

2.15 We note that

$$\begin{pmatrix} 5x_1 + x_2 \\ 2x_1 - x_2 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

This means that

$$\begin{aligned} 5x_1 + x_2 &= 3 \\ 2x_1 - x_2 &= 4 \end{aligned}$$

is equivalent to

$$\begin{pmatrix} 5 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

We thus have

$$A = \begin{pmatrix} 5 & 1 \\ 2 & -1 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

Since $|A| = 5(-1) - 2 \cdot 1 = -7 \neq 0$, A is invertible. By the formula for the inverse of an 2×2-matrix, we get

$$A^{-1} = \begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{2}{7} & -\frac{5}{7} \end{pmatrix}.$$

If we multiply the matrix equation $A\mathbf{x} = \mathbf{b}$ on the left by A^{-1}, we obtain

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}.$$

Now, the important point is that $A^{-1}A = I$ and $I\mathbf{x} = \mathbf{x}$. Thus we get that $\mathbf{x} = A^{-1}\mathbf{b}$. From this we find the solution:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{2}{7} & -\frac{5}{7} \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

In other words $x_1 = 1$ and $x_2 = -2$.

2.16 (a) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$

2.17 Removing a column gives a 3-minor. Thus there are 4 minors of order 3. To get a 2-minor, we must remove a row and two columns. There are $3 \cdot 4 \cdot 3/2 = 18$ ways to do this, so there are 18 minors of order 2. The 1-minors are the entries of the matrix, so there are $3 \cdot 4 = 12$ minors of order 1.

2.18 (a) We compute the determinant

$$\begin{vmatrix} x & 0 & x^2-2 \\ 0 & 1 & 1 \\ -1 & x & x-1 \end{vmatrix} = x^2 - x - 2.$$

We have that $x^2 - x - 2 = 0$ if and only if $x = -1$ or $x = 2$, so if $x \neq -1$ and $x \neq 2$, then $r(A) = 3$. If $x = -1$, then

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & -1 & -2 \end{pmatrix}.$$

Since for instance $\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -1 \neq 0$, it follows that $r(A) = 2$. If $x = 2$, then

$$A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{pmatrix}.$$

Since for instance $\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2 \neq 0$, we see that $r(A) = 2$.
(b) We compute the determinant

$$\begin{vmatrix} t+3 & 5 & 6 \\ -1 & t-3 & -6 \\ 1 & 1 & t+4 \end{vmatrix} = (t+4)(t+2)(t-2)$$

Hence the rank is 3 if $t \neq -4$, $t \neq -2$, and $t \neq 2$. The rank is 2 if $t = -4$,$t = -2$, or $t = 2$, since there is a non-zero minor of order 2 in each case.

2.19 The examples that are easiest to find, are 2×2-matrices A, B such that $AB \neq 0$ but $BA = 0$. Then $\text{rk}(AB) \geq 1$ and $\text{rk}(BA) = 0$. For instance we can choose

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Then $AB = A$ has rank one, and $BA = 0$ has rank zero.

2.20 We compute the minors in the coefficient matrix A and augmented matrix $\hat{A}$ in each case, and use this to determine the ranks and the number of solutions.
(a)

$$A = \begin{pmatrix} -2 & -3 & 1 \\ 4 & 6 & -2 \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} -2 & -3 & 1 & 3 \\ 4 & 6 & -2 & 1 \end{pmatrix}$$

We see that all three 2-minors in A are zero, so $\text{rk}(A) = 1$ (since $A \neq 0$, the rank of A must be at least one). Moreover, $\text{rk}(\hat{A}) = 2$ since the 2-minor obtained by keeping column 1 and 4 is non-zero:

$$\begin{vmatrix} -2 & 3 \\ 4 & 1 \end{vmatrix} = -2 - 12 = -14 \neq 0$$

Since $\text{rk} A < \text{rk} \hat{A}$, there are no solutions.
(b)

$$A = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 2 & -1 & 1 & -3 \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} 1 & 1 & -1 & 1 & 2 \\ 2 & -1 & 1 & -3 & 1 \end{pmatrix}$$

We see that the first 2-minor in A is non-zero, since

$$\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -1 - 2 = -3 \neq 0$$

This means that $\text{rk}(A) = \text{rk}(\hat{A}) = 2$, and that x_3 and x_4 are free variables. To find the solution of the system, we solve for the basic variables:

$$\begin{aligned} x_1 + x_2 &= x_3 - x_4 + 2 \\ 2x_1 - x_2 &= -x_3 + 3x_4 + 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= \tfrac{2}{3}x_4 + 1 \\ x_2 &= x_3 - \tfrac{5}{3}x_4 + 1 \end{aligned}$$

(c)

$$A = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 3 \\ 1 & 5 & -8 & 1 \\ 4 & 5 & -7 & 7 \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} 1 & -1 & 2 & 1 & 1 \\ 2 & 1 & -1 & 3 & 3 \\ 1 & 5 & -8 & 1 & 1 \\ 4 & 5 & -7 & 7 & 7 \end{pmatrix}$$

We compute the determinant of A and see that $|A| = 0$. Next, we look for a 3-minor in A that is non-zero, and (after a while) try the one obtained by keeping the first three rows and column 1, 2 and 4:

$$\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \\ 1 & 5 & 1 \end{vmatrix} = 1(1-15) - 2(-1-5) + 1(-3-1) = -14 + 12 - 4 = -6 \neq 0$$

Since the 3-minor is non-zero, $\text{rk}(A) = 3$. Since column 4 and 5 in $\hat{A}$ are equal, all 4-minors in $\hat{A}$ are zero, and $\text{rk}(\hat{A}) = \text{rk}(A) = 3$. This means that x_3 is a free variable. To find the solution of the system, we use the first three equations to solve for the basic variables:

$$\begin{array}{rcl} x_1 - x_2 + x_4 &=& -2x_3 + 1 \\ 2x_1 + x_2 + 3x_4 &=& x_3 + 3 \\ x_1 + 5x_2 + x_4 &=& 8x_3 + 1 \end{array} \quad \Rightarrow \quad \begin{array}{rcl} x_1 &=& -\frac{1}{3}x_3 \\ x_2 &=& \frac{5}{3}x_3 \\ x_4 &=& \phantom{-\frac{1}{3}x_3}\; 1 \end{array}$$

(d)

$$A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & -1 & -2 \\ 4 & 5 & 3 & 0 \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} 1 & 1 & 2 & 1 & 5 \\ 2 & 3 & -1 & -2 & 2 \\ 4 & 5 & 3 & 0 & 7 \end{pmatrix}$$

We compute all four 3-minors in A, and see that they are all zero. The 2-minor obtained by keeping the first two rows and columns is non-zero:

$$\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3 - 2 = 1 \neq 0$$

Therefore, $\text{rk}(A) = 2$. We compute the 3-minors of $\hat{A}$, and try the one obtained by keeping column 1, 2 and 5:

$$\begin{vmatrix} 1 & 1 & 5 \\ 2 & 3 & 2 \\ 4 & 5 & 7 \end{vmatrix} = 1(21-10) - 1(14-8) + 5(10-12) = 11 - 6 - 10 = -5 \neq 0$$

Since $\text{rk}A < \text{rk}\hat{A} = 3$, there are no solutions.

2.21 $A(\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) = \lambda A\mathbf{x}_1 + (1-\lambda)A\mathbf{x}_2 = \lambda\mathbf{b} + (1-\lambda)\mathbf{b} = \mathbf{b}$. This shows that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are different solutions, then so are all points on the straight line through $\mathbf{x}_1$ and $\mathbf{x}_2$.

2.22 We compute the determinant of A, and by cofactor exppansion along the first row we get

$$\det(A) = \begin{vmatrix} 1 & 3 & 2 \\ 2 & 5 & t \\ 4 & 7-t & -6 \end{vmatrix} = 1(-30 - t(7-t)) - 3(-12-4t) + 2(2(7-t) - 20)$$

This means that $\det(A) = -6 + t + t^2$, and that $\det(A) = 0$ if and only if $t^2 + t - 6 = 0$, or when $t = 2, -3$. Hence $\text{rk}(A) = 3$ for $t \neq 2, -3$, and $\text{rk}(A) = 2$ when $t = 2, -3$ since the 2-minor

$$\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = 5 - 6 = -1 \neq 0$$

is non-zero. When $t = -3$, the rank of A is two, and the augmented matrix $\hat{A}$ is given by

$$\hat{A} = \begin{pmatrix} 1 & 3 & 2 & 11 \\ 2 & 5 & -3 & 3 \\ 4 & 10 & -6 & 6 \end{pmatrix}$$

We compute all four 3-minors of $\hat{A}$, and find that they are all zero. Therefore, $\text{rk}(\hat{A}) = 2$ and the linear system has one degree of freedom. We can choose x_3 free, and we can solve the first two equations for the basic variables x_1 and x_2:

$$\begin{aligned} x_1 + 3x_2 &= -2x_3 + 11 \\ 2x_1 + 5x_2 &= 3x_3 + 3 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= 19x_3 - 46 \\ x_2 &= -7x_3 + 19 \end{aligned}$$

2.23 Midterm Exam in GRA6035 24/09/2010, Problem 1

Since the augmented matrix of the system is in echelon form, we see that the system is consistent and has two free variables, x_3 and x_5. Hence the correct answer is alternative **4**.

2.24 Mock Midterm Exam in GRA6035 09/2010, Problem 1

Since the augmented matrix of the system is in echelon form, we see that the system is inconsistent. Hence the correct answer is alternative **4**.

2.25 Midterm Exam in GRA6035 24/05/2011, Problem 3

Since the augmented matrix of the system is in echelon form, we see that the system is inconsistent. Hence the correct answer is alternative **1**.

Lecture 3
Vectors and Linear Independence

3.1 Main concepts

An m-vector (or a column vector) $\mathbf{v}$ is an $m \times 1$ matrix (or a matrix with a single column). We may add and subtract vectors, and multiply vectors by scalars. A vector $\mathbf{w}$ is a *linear combination* of the the collection $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ of m-vectors if

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

for some numbers $c_1, \dots, c_n$. This vector equation can be rewritten as a matrix equation $A \cdot \mathbf{c} = \mathbf{w}$, where A is the matrix with the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ as columns. This is an $m \times n$ linear system.

We say that the collection $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ of m-vectors is *linearly independent* if the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

only has the trivial solution $c_1 = c_2 = \cdots = c_n = 0$, and *linearly dependent* if there are also non-trivial solutions. The vector equation can be reformulated as a linear system $A\mathbf{c} = \mathbf{0}$, and linearly independence (resp. dependence) corresponds to one unique solution (resp. infinitely many solutions). The vectors are linearly dependent if and only if one of the vectors can be written as a linear combination of the others.

Lemma 3.1. *Let A be the $m \times n$ matrix with the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ as columns.*

1. *If $m = n$, then the vectors are linearly independent if and only if* $\det(A) \neq 0$.
2. *In general, the rank of A is the maximal number of linearly independent vectors among $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$. In particular, all vectors are linearly independent if and only if* $\operatorname{rk} A = n$.

In fact, if we compute the rank of A by Gaussian elimination, the vectors corresponding to pivot columns are linearly independent.

3.2 Problems

3.1. Express the vector $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ when

$$\mathbf{w} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

Draw the three vectors in a two-dimensional coordinate system.

3.2. Determine if the following pairs of vectors are linearly independent:

$$\text{(a)} \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ -6 \end{pmatrix} \quad \text{(b)} \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad \text{(c)} \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Draw the vectors in the plane in each case and explain geometrically.

3.3. Show that the following vectors are linearly dependent:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

3.4. Assume that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent m-vectors.

1. Show that $\mathbf{a}+\mathbf{b}, \mathbf{b}+\mathbf{c}$ and $\mathbf{a}+\mathbf{c}$ are linearly independent.
2. Is the same true of $\mathbf{a}-\mathbf{b}, \mathbf{b}+\mathbf{c}$ and $\mathbf{a}+\mathbf{c}$?

3.5. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be m-vectors. Show that at least one of the vectors can be written as a linear combinations of the others if and only if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$$

has non-trivial solutions.

3.6. Prove that the following vectors are linearly independent:

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

3.7. Using the definition of rank of a matrix, prove that any set of n vectors in $\mathbb{R}^m$ must be linearly dependent if $n > m$.

3.8. Show that the vectors

$$\begin{pmatrix} 3 \\ 4 \\ -1 \\ 2 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

are linearly independent by computing a minor of order two.

3.9. We consider the column vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ of the matrix

$$\begin{pmatrix} 1 & 2 & 4 \\ 3 & 7 & 0 \\ 5 & 11 & 8 \end{pmatrix}$$

1. Show that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent.
2. Find a non-trivial solution of $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$, and use this solution to express one of the three vectors as a linear combination of the two others.

3.10. We consider the column vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ of the matrix

$$\begin{pmatrix} 1 & 2 & 4 & -1 \\ 3 & 7 & 0 & 4 \\ 5 & 11 & 8 & 2 \\ -2 & -5 & 4 & -5 \end{pmatrix}$$

1. Show that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent.
2. Are there three linearly independent vectors among these four vectors? If not, what about two linearly independent vectors among them?

3.11. Let A be any matrix. The *Gram matrix* of A is the square matrix A^TA. A theorem states that for any matrix A, we have that

$$\text{rk}(A) = \text{rk}(A^TA)$$

Use this theorem to find the rank of the following matrices:

$$a) \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \\ 4 & 8 \end{pmatrix} \qquad b) \quad A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 6 \end{pmatrix}$$

3.12. Let A be any $m \times n$-matrix. We define the *null space* of A to be the collection of all n-vectors $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$. In other words, the null space of A is the solution set of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$.

1. Show that the null space of A is equal to the null space of the Gram matrix A^TA.
2. Prove that $\text{rk}(A) = \text{rk}(A^TA)$.

3.13. Midterm Exam in GRA6035 on 24/09/2010, Problem 2
Consider the set of vectors $\mathscr{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ h \end{pmatrix}$$

and h is a parameter. **Which statement is true?**

1. $\mathscr{B}$ is a linearly independent set of vectors for all h
2. $\mathscr{B}$ is a linearly independent set of vectors exactly when $h = 3$
3. $\mathscr{B}$ is a linearly independent set of vectors exactly when $h \neq 1/7$
4. $\mathscr{B}$ is a linearly independent set of vectors exactly when $h \neq 3$
5. I prefer not to answer.

3.14. Mock Midterm Exam in GRA6035 on 09/2010, Problem 1
Consider the vector $\mathbf{w}$ and the set of vectors $\mathscr{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{w} = \begin{pmatrix} -4 \\ 3 \\ h \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$$

Which statement is true?

1. $\mathbf{w}$ is a linear combination of the vectors in $\mathscr{B}$ for all values of h
2. $\mathbf{w}$ is a linear combination of the vectors in $\mathscr{B}$ exactly when $h \neq 5$
3. $\mathbf{w}$ is a linear combination of the vectors in $\mathscr{B}$ exactly when $h = 5$
4. $\mathbf{w}$ is not a linear combination of the vectors in $\mathscr{B}$ for any value of h
5. I prefer not to answer.

3.15. Midterm Exam in GRA6035 on 24/05/2011, Problem 3
Consider the set of vectors $\mathscr{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} h+1 \\ h \\ h-2 \end{pmatrix}$$

and h is a parameter. **Which statement is true?**

1. $\mathscr{B}$ is a linearly independent set of vectors for all h
2. $\mathscr{B}$ is a linearly independent set of vectors exactly when $h = 0$
3. $\mathscr{B}$ is a linearly independent set of vectors exactly when $h \neq 5$
4. $\mathscr{B}$ is a linearly independent set of vectors exactly when $h \neq -1$
5. I prefer not to answer.

3.3 Solutions

3.1 We must find numbers c_1 and c_2 so that

$$\begin{pmatrix} 8 \\ 9 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$$

Multiplying with the inverse from the left, we get that

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 8 \\ 9 \end{pmatrix} = \frac{1}{11} \begin{pmatrix} 3 & 1 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

Hence $c_1 = 3$ and $c_2 = -2$.

3.2 We form the 2×2-matrix with the pair of vectors as columns in each case. We know that the vectors are linearly independent if and only if the determinant of the matrix is non-zero; therefore we compute the determinant in each case:

$$a) \quad \begin{vmatrix} -1 & 3 \\ 2 & -6 \end{vmatrix} = 0 \qquad b) \quad \begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix} = 11 \neq 0 \qquad c) \quad \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 0$$

It follows that the vectors in b) are linearly independent, while the vectors in a) and the vectors in c) are linearly dependent.

An alternative solution is to consider the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$ for each pair of vectors $\mathbf{v}_1, \mathbf{v}_2$ and solve the equation. The equation has non-trivial solutions (one degree of freedom) in a) and c), but only the trivial solution $c_1 = c_2 = 0$ in b). The conclusion is therefore the same as above.

3.3 We consider the vector equation

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} + c_4 \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 0 & 4 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We use elementary row operations to simplify the coefficient matrix of this linear system:

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 0 & 4 & -2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 1 \\ 0 & -2 & 1 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 5 & -5 \end{pmatrix}$$

We see that the linear system has one degree of freedom, and therefore there are non-trivial solutions. Hence the vectors are linearly dependent.

An alternative solution is to argue that the coefficient matrix can maximally have rank 3, and therefore must have at least one degree of freedom.

3.4 For the vectors in a), we consider the vector equation

$$c_1(\mathbf{a}+\mathbf{b}) + c_2(\mathbf{b}+\mathbf{c}) + c_3(\mathbf{a}+\mathbf{c}) = \mathbf{0} \quad \Leftrightarrow \quad (c_1+c_3)\mathbf{a} + (c_1+c_2)\mathbf{b} + (c_2+c_3)\mathbf{c} = \mathbf{0}$$

Since $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are linearly independent, this gives the equations

$$\begin{aligned} c_1 \phantom{{}+c_2} + c_3 &= 0 \\ c_1 + c_2 \phantom{{}+c_3} &= 0 \\ c_2 + c_3 &= 0 \end{aligned} \quad \Leftrightarrow \quad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

But the determinant of the coefficient matrix is

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1(1) - 1(-1) = 2 \neq 0$$

so the only solution is the trivial solutions. Hence the vectors are linearly independent. For the vectors in b), we consider the vector equation

$$c_1(\mathbf{a}-\mathbf{b})+c_2(\mathbf{b}+\mathbf{c})+c_3(\mathbf{a}+\mathbf{c})=\mathbf{0} \quad \Leftrightarrow \quad (c_1+c_3)\mathbf{a}+(-c_1+c_2)\mathbf{b}+(c_2+c_3)\mathbf{c}=\mathbf{0}$$

Since $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are linearly independent, this gives the equations

$$\begin{array}{rcrcrcl} c_1 & & & + & c_3 & = & 0 \\ -c_1 & + & c_2 & & & = & 0 \\ & & c_2 & + & c_3 & = & 0 \end{array} \quad \Leftrightarrow \quad \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

But the determinant of the coefficient matrix is

$$\begin{vmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1(1) - (-1)(-1) = 0$$

so there is at least one degree of freedom and non-trivial solutions. Hence the vectors are linearly dependent.

3.5 Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be m-vectors. We consider the following statements:

1. At least one of the vectors can be written as a linear combinations of the others.
2. The vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$ has non-trivial solutions.

We must show that $(a) \Leftrightarrow (b)$. If statement (a) holds, then one of the vectors, say $\mathbf{v}_n$, can be written as a linear combination of the other vectors,

$$\mathbf{v}_n = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{n-1}\mathbf{v}_{n-1}$$

This implies that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{n-1}\mathbf{v}_{n-1} + (-1)\mathbf{v}_n = \mathbf{0}$$

which is a non-trivial solution (since $c_n = -1 \neq 0$). So (a) implies (b). Conversely, if (b) holds, then there is a non-trivial solution to the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{n-1}\mathbf{v}_{n-1} + c_n\mathbf{v}_n = \mathbf{0}$$

So at least one of the variables are non-zero; say $c_n \neq 0$. Then we can rewrite this vector equation as

$$\mathbf{v}_n = -\frac{1}{c_n}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{n-1}\mathbf{v}_{n-1}$$

Hence one of the vectors can be written as a linear combination of the others, and (b) implies (a).

3.6 We compute the determinant

$$\begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 1(1) + 1(2) = 3$$

Since the determinant is non-zero, the vectors are linearly independent.

3.7 We consider n vectors in $\mathbb{R}^m$, and form the $m \times n$-matrix with these vectors as columns. Since $n > m$, the maximal rank that the matrix can have is m, and therefore the corresponding homogeneous linear system has at least $n - m > 0$ degrees of freedom. Therefore, the vectors must be linearly dependent.

3.8 We form the matrix with the two vectors as columns, and compute the 2-minor obtained by deleting the last two rows. We get

$$\begin{vmatrix} 3 & 0 \\ 4 & 1 \end{vmatrix} = 3 \neq 0$$

Hence the matrix has rank two, and there are no degrees of freedom. Therefore, the vectors are linearly independent.

3.9 We reduce the matrix to an echelon form using elementary row operations:

$$\begin{pmatrix} 1 & 2 & 4 \\ 3 & 7 & 0 \\ 5 & 11 & 8 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & -12 \\ 0 & 1 & -12 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & -12 \\ 0 & 0 & 0 \end{pmatrix}$$

We see that the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has one degree of freedom, and it follows that the vectors are linearly dependent. Moreover, we see that x_3 is a free variable and that the solutions are given by

$$x_2 = 12x_3, \quad x_1 = -2(12x_3) - 4x_3 = -28x_3$$

In particular, one solution is $x_1 = -28,\ x_2 = 12,\ x_3 = 1$. This implies that

$$-28\mathbf{v}_1 + 12\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0} \quad \Rightarrow \quad \mathbf{v}_3 = 28\mathbf{v}_1 - 12\mathbf{v}_2$$

3.10 We use elementary row operations to reduce the matrix to an echelon form:

$$\begin{pmatrix} 1 & 2 & 4 & -1 \\ 3 & 7 & 0 & 4 \\ 5 & 11 & 8 & 2 \\ -2 & -5 & 4 & -5 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & 2 & 4 & -1 \\ 0 & 1 & -12 & 7 \\ 0 & 1 & -12 & 7 \\ 0 & -1 & 12 & -7 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & 2 & 4 & -1 \\ 0 & 1 & -12 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We see that there are are two degrees of freedom, since the matrix has rank 2. Therefore, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent, and there are not three linearly independent vectors among them either. However, there are two linearly independent column vectors since the rank is two. In fact, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent since there are pivot positions in column 1 and 2.

3.11 We compute the Gram matrix $A^T A$ and its determinant in each case: For the matrix in a) we have

$$A^T A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \\ 4 & 8 \end{pmatrix}^T \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \\ 4 & 8 \end{pmatrix} = \begin{pmatrix} 30 & 60 \\ 60 & 120 \end{pmatrix} \quad \Rightarrow \quad \det(A^T A) = \begin{vmatrix} 30 & 60 \\ 60 & 120 \end{vmatrix} = 0$$

This means that $\mathrm{rk}(A) < 2$, and we see that $\mathrm{rk}(A) = 1$ since $A \neq 0$. For the matrix in b) we have

$$A^T A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 6 \end{pmatrix}^T \begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 14 & 18 \\ 18 & 41 \end{pmatrix} \quad \Rightarrow \quad \det(A^T A) = \begin{vmatrix} 14 & 18 \\ 18 & 41 \end{vmatrix} = 250$$

This means that $\mathrm{rk}(A) = 2$.

3.12 To show that the null space of A equals the null space of $A^T A$, we have to show that $A\mathbf{x} = \mathbf{0}$ and $A^T A\mathbf{x} = \mathbf{0}$ have the same solutions. It is clear that multiplication with A^T from the left gives

$$A\mathbf{x} = \mathbf{0} \quad \Rightarrow \quad A^T A\mathbf{x} = \mathbf{0}$$

so any solution of $A\mathbf{x} = \mathbf{0}$ is also a solution of $A^T A\mathbf{x} = \mathbf{0}$. Conversely, suppose that $\mathbf{x}$ is a solution of $A^T A\mathbf{x} = \mathbf{0}$, and write

$$\begin{pmatrix} h_1 \\ h_2 \\ \dots \\ h_m \end{pmatrix} = A\mathbf{x}$$

Then we have

$$A^T A\mathbf{x} = \mathbf{0} \quad \Rightarrow \quad \mathbf{x}^T A^T A\mathbf{x} = \mathbf{0}$$

by left multiplication by $\mathbf{x}^T$ on both sides. This means that

$$\mathbf{x}^T A^T A\mathbf{x} = (A\mathbf{x})^T (A\mathbf{x}) = 0 \quad \Rightarrow \quad h_1^2 + h_2^2 + \cdots + h_m^2 = 0$$

The last equation implies that $h_1 = h_2 = \cdots = h_m = 0$; that is, that $A\mathbf{x} = \mathbf{0}$. This means that any solution of $A^T A\mathbf{x} = \mathbf{0}$ is also a solution of $A\mathbf{x} = \mathbf{0}$. Hence the two solution sets are equal, and this proves a). To prove b), consider the two linear systems $A\mathbf{x} = \mathbf{0}$ and $\mathbf{A}^T A\mathbf{x} = \mathbf{0}$ in n variables. Since the two linear systems have the same solutions, they must have the same number d of free variables. This gives

$$\text{rk}(A) = n - d, \quad \text{rk}(A^T A) = n - d$$

so $\text{rk}(A) = \text{rk}(A^T A)$.

3.13 Midterm Exam in GRA6035 24/09/2010, Problem 2
We compute the determinant

$$\begin{vmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ -1 & 1 & h \end{vmatrix} = h - 3$$

Hence the vectors are linearly independent exactly when $h \neq 3$, and the correct answer is alternative **D**. This question can also be answered using Gauss elimination.

3.14 Mock Midterm Exam in GRA6035 09/2010, Problem 1
The vector $\mathbf{w}$ is a linear combination of the vectors in $\mathscr{B}$ if and only if the linear system

$$x_1 \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} + x_2 \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ h \end{pmatrix}$$

is consistent. We write down the augmented matrix of the system and reduce it to echelon form

$$\begin{pmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{pmatrix}$$

The system is consistent if and only if $h = 5$. Hence the correct answer is alternative **C**. This question can also be answered using minors.

3.15 Midterm Exam in GRA6035 24/05/2011, Problem 3
We compute the determinant

$$\begin{vmatrix} 2 & 1 & h+1 \\ 3 & 2 & h \\ -1 & 1 & h-2 \end{vmatrix} = 3h + 3$$

Hence the vectors are linearly independent exactly when $h \neq -1$, and the correct answer is alternative **D**. This question can also be answered using Gauss elimination.

Lecture 4
Eigenvalues and Diagonalization

4.1 Main concepts

Let A be an $n \times n$ matrix. We say that a vector $\mathbf{v}$ is an *eigenvector* for A with *eigenvalue* λ if

$$A\mathbf{v} = \lambda\mathbf{v}$$

and $\mathbf{v} \neq \mathbf{0}$. This equation can also be written $(A - \lambda I)\mathbf{v} = \mathbf{0}$. The *characteristic equation* of A is the n'th order polynomial equation

$$\det(A - \lambda I) = 0$$

in λ, and its solutions are the eigenvalues of A. For every eigenvalue λ, the non-zero solutions of the linear system $(A - \lambda I)\mathbf{v} = \mathbf{0}$ are the eigenvectors of A with eigenvalue λ.

The expression $\det(A - \lambda I)$ is called the *characteristic polynomial* of A. It can be written in the form

$$\det(A - \lambda I) = (\lambda - \lambda_1) \cdot (\lambda - \lambda_2) \cdots (\lambda - \lambda_r) \cdot Q(\lambda)$$

where $Q(\lambda)$ is a polynomial of degree $n - r$ such that $Q(\lambda) = 0$ have no solutions, and $\lambda_1, \lambda_2, \dots, \lambda_r$ are the eigenvalues of A. It may happen that the same eigenvalue is repeated m times, and in this case it is called an eigenvalue of *multiplicity* m. When $r = n$, we say that A has n eigenvalues counted with multiplicity. In this case, we have

$$\det(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n \quad \text{and} \quad \operatorname{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

where the *trace* of A is defined by $\operatorname{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$, the sum of the diagonal entries in A.

Lemma 4.1. *If λ is an eigenvalue of A of multiplicity m, then the linear system $(A - \lambda I)\mathbf{v} = \mathbf{0}$ has at most m degrees of freedom.*

If there is an invertible matrix P such that $D = P^{-1}AP$ is a diagonal matrix, we say that A is *diagonalizable*. In that case, the matrix P can be used to compute A^N for any N, since

$$A^N = (PDP^{-1})^N = (PDP^{-1})(PDP^{-1})\cdots(PDP^{-1}) = PD^NP^{-1}$$

and D^N is easier to compute. To find a matrix P such that $D = P^{-1}AP$ is diagonal, we must find n linearly independent eigenvectors of A and let P be the matrix with these vectors as columns. The matrix D will be the diagonal matrix with the corresponding eigenvalues on the diagonal.

Lemma 4.2. *The $n \times n$ matrix A is diagonalizable if and only if the following conditions are satisfied:*

1. *There are n eigenvalues of A, counted with multiplicity.*
2. *If λ is an eigenvalue of A of multiplicty m, then $\text{rk}(A - \lambda I) = n - m$.*

Theorem 4.1. *If A is a symmetric matrix, then A is diagonalizable.*

4.2 Problems

4.1. Check if the vector $\mathbf{v}$ is an eigenvector of the matrix A when

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

If $\mathbf{v}$ is an eigenvector, what is the corresponding eigenvalue?

4.2. Find the eigenvalues and eigenvectors of the following matrices:

$$a) \quad \begin{pmatrix} 2 & -7 \\ 3 & -8 \end{pmatrix} \qquad b) \quad \begin{pmatrix} 2 & 4 \\ -2 & 6 \end{pmatrix} \qquad c) \quad \begin{pmatrix} 1 & 4 \\ 6 & -1 \end{pmatrix}$$

4.3. Find the eigenvalues and eigenvectors of the following matrices:

$$a) \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \qquad b) \quad \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 2 & 0 & -2 \end{pmatrix}$$

4.4. Let A be a square matrix and let λ be an eigenvalue of A. Suppose that A is an invertible matrix, and prove that $\lambda \neq 0$ and that $1/\lambda$ is an eigenvalue of A^{-1}.

4.5. Consider the square matrix A and the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ given by

$$A = \begin{pmatrix} 1 & 18 & 30 \\ -2 & -11 & -10 \\ 2 & 6 & 5 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

Show that $\mathbf{v}_i$ is an eigenvector for A for $i = 1,2,3$ and find the corresponding eigenvalues. Use this to find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

4.6. Find an invertible matrix P such that $D = P^{-1}AP$ is diagonal when

$$A = \begin{pmatrix} 2 & -7 \\ 3 & -8 \end{pmatrix}$$

4.7. Show that the following matrix is not diagonalizable:

$$A = \begin{pmatrix} 3 & 5 \\ 0 & 3 \end{pmatrix}$$

4.8. Initially, two firms A and B (numbered 1 and 2) share the market for a certain commodity. Firm A has 20% of the marked and B has 80%. In course of the next year, the following changes occur:

A keeps 85% of its customers, while losing 15% to B
B keeps 55% of its customers, while losing 45% to A

We can represent market shares of the two firms by means of a *market share vector*, defined as a column vector $\mathbf{s}$ whose components are all nonnegative and sum to 1. Define the matrix $\mathbf{T}$ and the initial share vector $\mathbf{s}$ by

$$T = \begin{pmatrix} 0.85 & 0.45 \\ 0.15 & 0.55 \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix}$$

The matrix T is called the *transition matrix*.

1. Compute the vector $T\mathbf{s}$, and show that it is also a market share vector.
2. Find the eigenvalues and eigenvectors of T.
3. Find a matrix P such that $D = P^{-1}TP$ is diagonal, and show that $T^n = PD^nP^{-1}$.
4. Compute the limit of D^n as $n \to \infty$ and use this to find the limit of $T^n\mathbf{s}$ as $n \to \infty$. Explain that the we will approach an *equilibrium*, a situation where the market shares of A and B are constant. What are the equilibrium marked shares?

4.9. Determine if the following matrix is diagonalizable:

$$A = \begin{pmatrix} 4 & 1 & 2 \\ 0 & 3 & 0 \\ 1 & 1 & 5 \end{pmatrix}$$

If this is the case, find an invertible matrix P such that $P^{-1}AP$ is diagonal, and use this to compute A^{17}.

4.10. Final Exam in GRA6035 on 10/12/2010, Problem 2
We consider the matrix A and the vector $\mathbf{v}$ given by

$$A = \begin{pmatrix} 1 & 7 & -2 \\ 0 & s & 0 \\ 1 & 1 & 4 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

1. Compute the determinant and the rank of A.
2. Find all eigenvalues of A. Is $\mathbf{v}$ an eigenvector for A?
3. Determine the values of s such that A is diagonalizable.

4.11. Mock Final Exam in GRA6035 on 12/2010, Problem 1
We consider the matrix A given by

$$A = \begin{pmatrix} 1 & 1 & -4 \\ 0 & t+2 & t-8 \\ 0 & -5 & 5 \end{pmatrix}$$

1. Compute the determinant and the rank of A.
2. Find all eigenvalues of A.
3. Determine the values of t such that A is diagonalizable.

4.12. Final Exam in GRA6035 on 30/05/2011, Problem 2
We consider the matrix A and the vector $\mathbf{v}$ given by

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & s & s^2 \\ 1 & -1 & 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

1. Compute the determinant and the rank of A.
2. Find all values of s such that $\mathbf{v}$ is an eigenvector for A.
3. Compute all eigenvalues of A when $s = -1$. Is A diagonalizable when $s = -1$?

4.3 Advanced Matrix Problems

The advanced problems are challenging and optional, and are meant for advanced students. It is recommended that you work through the ordinary problems and exam problems and make sure that you master them before you attempt these problems.

4.13. Solve the equation

$$\begin{vmatrix} x & 2 & 3 \\ 2 & x & 3 \\ 2 & 3 & x \end{vmatrix} = 0$$

4.14. Solve the equation

$$\begin{vmatrix} x+1 & 0 & x & 0 & x-1 & 0 \\ 0 & x & 0 & x-1 & 0 & x+1 \\ x & 0 & x-1 & 0 & x+1 & 0 \\ 0 & x-1 & 0 & x+1 & 0 & x \\ x-1 & 0 & x+1 & 0 & x & 0 \\ 0 & x+1 & 0 & x & 0 & x-1 \end{vmatrix} = 9$$

4.15. Solve the linear system

$$\begin{array}{rcl} x_2 + x_3 + \ldots + x_{n-1} + x_n &=& 2 \\ x_1 \quad\;\; + x_3 + \ldots + x_{n-1} + x_n &=& 4 \\ x_1 + x_2 \quad\;\; + \ldots + x_{n-1} + x_n &=& 6 \\ \vdots \qquad\qquad\qquad & & \vdots \\ x_1 + x_2 + x_3 + \ldots + x_{n-1} \qquad &=& 2n \end{array}$$

4.4 Solutions

4.1 We compute that

$$A\mathbf{v} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\mathbf{v}$$

This means that $\mathbf{v}$ is an eigenvector with eigenvalue $\lambda = 3$.

4.2 a) We solve the characteristic equation to find the eigenvalues:

$$\begin{vmatrix} 2-\lambda & -7 \\ 3 & -8-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 5 = 0 \quad \Rightarrow \lambda = -1, -5$$

For each eigenvalue, we compute the eigenvectors using an echelon form of the coefficient matrix, and express the eigenvectors in terms of the free variables. For $\lambda = -1$, we get eigenvectors

$$\begin{pmatrix} 3 & -7 \\ 3 & -7 \end{pmatrix} \dashrightarrow \begin{pmatrix} 3 & -7 \\ 0 & 0 \end{pmatrix} \Rightarrow 3x - 7y = 0 \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{7}{3}y \\ y \end{pmatrix} = y \begin{pmatrix} 7/3 \\ 1 \end{pmatrix}$$

For $\lambda = -5$, we get eigenvectors

$$\begin{pmatrix} 7 & -7 \\ 3 & -3 \end{pmatrix} \dashrightarrow \begin{pmatrix} 7 & -7 \\ 0 & 0 \end{pmatrix} \Rightarrow 7x - 7y = 0 \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

b) We solve the characteristic equation to find the eigenvalues:

$$\begin{vmatrix} 2-\lambda & 4 \\ -2 & 6-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 20 = 0 \quad \Rightarrow \quad \text{no solutions}$$

Since there are no solutions, there are no eigenvalues and no eigenvectors.
c) We solve the characteristic equation to find the eigenvalues:

$$\begin{vmatrix} 1-\lambda & 4 \\ 6 & -1-\lambda \end{vmatrix} = \lambda^2 - 25 = 0 \quad \Rightarrow \lambda = 5, -5$$

For each eigenvalue, we compute the eigenvectors using an echelon form of the coefficient matrix, and express the eigenvectors in terms of the free variables. For $\lambda = 5$, we get eigenvectors

$$\begin{pmatrix} -4 & 4 \\ 6 & -6 \end{pmatrix} \dashrightarrow \begin{pmatrix} -4 & 4 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad -4x+4y=0 \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda = -5$, we get eigenvectors

$$\begin{pmatrix} 6 & 4 \\ 6 & 4 \end{pmatrix} \dashrightarrow \begin{pmatrix} 6 & 4 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad 6x+4y=0 \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{2}{3}y \\ y \end{pmatrix} = y \begin{pmatrix} -2/3 \\ 1 \end{pmatrix}$$

4.3 a) We solve the characteristic equation to find the eigenvalues:

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda)(4-\lambda) = 0 \quad \Rightarrow \lambda = 2, 3, 4$$

For each eigenvalue, we compute the eigenvectors using an echelon form of the coefficient matrix, and express the eigenvectors in terms of the free variables. For $\lambda = 2$, we get eigenvectors

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \dashrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad y=0,\ 2z=0 \quad \Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda = 3$, we get eigenvectors

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dashrightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad x=z=0 \quad \Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} = y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda = 4$, we get eigenvectors

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad x=y=0 \quad \Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} = z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

b) We solve the characteristic equation to find the eigenvalues. Since the equation (of degree three) is a bit difficult to solve, we first change the determinant by adding the second row to the first row:

$$\begin{vmatrix} 2-\lambda & 1 & -1 \\ 0 & 1-\lambda & 1 \\ 2 & 0 & -2-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 2-\lambda & 0 \\ 0 & 1-\lambda & 1 \\ 2 & 0 & -2-\lambda \end{vmatrix}$$

Then we transpose the matrix, and subtract the first row from the second row:

$$= \begin{vmatrix} 2-\lambda & 0 & 2 \\ 2-\lambda & 1-\lambda & 0 \\ 0 & 1 & -2-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 0 & 2 \\ 0 & 1-\lambda & -2 \\ 0 & 1 & -2-\lambda \end{vmatrix} = (2-\lambda)(\lambda^2+\lambda) = 0$$

We see that the eigenvalues are $\lambda = 2, 0, -1$. For each eigenvalue, we compute the eigenvectors using an echelon form of the coefficient matrix, and express the eigenvectors in terms of the free variables. For this operation, we must use the matrix **before** the transposition, since the operation of transposing the coefficient matrix will not preserve the solutions of the linear system (but it will preserve the determinant). For $\lambda = 2$, we get eigenvectors

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 2 & 0 & -4 \end{pmatrix} \dashrightarrow \begin{pmatrix} 2 & 0 & -4 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad y = z,\ x = 2z \quad \Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda = 0$, we get eigenvectors

$$\begin{pmatrix} 2 & 2 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & -2 \end{pmatrix} \dashrightarrow \begin{pmatrix} 2 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad 2x = 2z,\ y = -z \quad \Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

For $\lambda = -1$, we get eigenvectors

$$\begin{pmatrix} 3 & 3 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & -1 \end{pmatrix} \dashrightarrow \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad 2x = z,\ 2y = -z \quad \Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

4.4 If A is invertible, then $\det(A) \neq 0$. Hence $\lambda = 0$ is not an eigenvalue. If it were, then $\lambda = 0$ would solve the characteristic equation, and we would have $\det(A - 0 \cdot I_n) = \det(A) = 0$; this is a contradiction. For the last part, notice that if $\mathbf{v}$ is an eigenvector for A with eigenvalue λ, then we have

$$A\mathbf{v} = \lambda v \quad \Rightarrow \quad \mathbf{v} = A^{-1}\lambda\mathbf{v} = \lambda A^{-1}\mathbf{v} \quad \Rightarrow \quad \lambda^{-1}\mathbf{v} = A^{-1}\mathbf{v}$$

This means that $\lambda^{-1} = 1/\lambda$ is an eigenvalue of A^{-1} (with eigenvector $\mathbf{v}$).

4.5 We form that matrix P with $\mathbf{v}_i$ as the i'th column, $i = 1, 2, 3$, and compute AP:

$$AP = \begin{pmatrix} 1 & 18 & 30 \\ -2 & -11 & -10 \\ 2 & 6 & 5 \end{pmatrix} \begin{pmatrix} -3 & -5 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 15 & 25 & 15 \\ -5 & 0 & -5 \\ 0 & -5 & 5 \end{pmatrix}$$

We have $AP = (A\mathbf{v}_1|A\mathbf{v}_2|A\mathbf{v}_3)$, and looking at the columns of AP we see that

$$A\mathbf{v}_1 = -5\mathbf{v}_1,\ A\mathbf{v}_2 = -5\mathbf{v}_2,\ A\mathbf{v}_3 = 5\mathbf{v}_3$$

The vectors are therefore eigenvectors, with eigenvalues $\lambda = -5, -5, 5$. The matrix P is invertible since the three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent, and we have $AP = PD \quad \Rightarrow \quad A = PDP^{-1}$ when D is the diagonal matrix

$$D = \begin{pmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

4.6 We use the eigenvalues and eigenvectors we found in Problem a). Since there are two distinct eigenvalues, the matrix A is diagonalizable, and $D = P^{-1}AP$ when we put

$$P = \begin{pmatrix} 7/3 & 1 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 \\ 0 & -5 \end{pmatrix}$$

4.7 We find the eigenvalues of A by solving the characteristic equation:

$$\begin{vmatrix} 3-\lambda & 5 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 = 0 \quad \Rightarrow \quad \lambda = 3$$

Hence there is only one eigenvalue (with multiplicity 2). The corresponding eigenvectors are found by reducing the coefficient matrix to an echelon form. For $\lambda = 3$, we get eigenvectors

$$\begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad 5y = 0 \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

In particular, there is only one free variable and therefore not more than one linearly independent eigenvector. This means that there are too few linearly independent eigenvectors (only one eigenvector while $n = 2$), hence A is not diagonalizable.

4.8

1. We compute the matrix product

$$T\mathbf{s} = \begin{pmatrix} 0.53 \\ 0.47 \end{pmatrix}$$

   and see that the result is a market share vector.
2. We find the eigenvalues of T by solving the characteristic equation

$$\begin{vmatrix} 0.85-\lambda & 0.45 \\ 0.15 & 0.55-\lambda \end{vmatrix} = \lambda^2 - 1.4\lambda + 0.4 = 0$$

   This gives eigenvalues $\lambda = 1, 0.4$. The eigenvectors for $\lambda = 1$ is given by $-0.15x + 0.45y = 0$, or $x = 3y$; and for $\lambda = 0.4$, the eigenvectors are given by

$0.45x + 0.45y = 0$, or $x = -y$. Hence eigenvectors are given in terms of the free variables by

$$\mathbf{v}_1 = y\begin{pmatrix}3\\1\end{pmatrix},\ \mathbf{v}_2 = y\begin{pmatrix}-1\\1\end{pmatrix}$$

3. From the eigenvectors, we see that

$$P = \begin{pmatrix}3 & -1\\1 & 1\end{pmatrix}$$

This gives $T^n = (PDP^{-1})^n = PD^nP^{-1}$.

4. When $n \to \infty$, we get that

$$D^n = \begin{pmatrix}1 & 0\\0 & 0.4\end{pmatrix}^n = \begin{pmatrix}1 & 0\\0 & 0.4^n\end{pmatrix} \to \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$$

Hence the limit of T^n as $n \to \infty$ is given by

$$T^n \to \begin{pmatrix}3 & -1\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}3 & -1\\1 & 1\end{pmatrix}^{-1} = \begin{pmatrix}3 & 0\\1 & 0\end{pmatrix}\frac{1}{4}\begin{pmatrix}1 & 1\\-1 & 3\end{pmatrix} = \begin{pmatrix}3/4 & 3/4\\1/4 & 1/4\end{pmatrix}$$

and

$$T^n\mathbf{s} \to \begin{pmatrix}3/4 & 3/4\\1/4 & 1/4\end{pmatrix}\begin{pmatrix}0.2\\0.8\end{pmatrix} = \begin{pmatrix}0.75\\0.25\end{pmatrix}$$

The equilibrium marked shares are 75% for A and 25% for B.

4.9 We find the eigenvalues of A by solving the characteristic equation:

$$\begin{vmatrix}4-\lambda & 1 & 2\\0 & 3-\lambda & 0\\1 & 1 & 5-\lambda\end{vmatrix} = (3-\lambda)(\lambda^2 - 9\lambda + 18) = 0 \quad \Rightarrow \quad \lambda = 3,3,6$$

Hence there is one eigenvalue $\lambda = 3$ with multiplicity 2, and one eigenvalue $\lambda = 6$ with multiplicity 1. The corresponding eigenvectors are found by reducing the coefficient matrix to an echelon form. For $\lambda = 3$, we get eigenvectors

$$\begin{pmatrix}1 & 1 & 2\\0 & 0 & 0\\1 & 1 & 2\end{pmatrix} \dashrightarrow \begin{pmatrix}1 & 1 & 2\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix} \quad \Rightarrow \quad x = -y - 2z$$

Since there is two degrees of freedom, there are two linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ for $\lambda = 3$, given by

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-y-2z\\y\\z\end{pmatrix} = y\begin{pmatrix}-1\\1\\0\end{pmatrix} + z\begin{pmatrix}-2\\0\\1\end{pmatrix} = y\mathbf{v}_1 + z\mathbf{v}_2$$

Since $\lambda = 6$ is an eigenvalue of multiplicity one, we get one eigenvector $\mathbf{v}_3$ given by

$$\begin{pmatrix} -2 & 1 & 2 \\ 0 & -3 & 0 \\ 1 & 1 & -1 \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \quad x = z,\ y = 0 \quad \Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Hence A is diagonalizable, and we have that $P^{-1}AP = D$ is diagonal with

$$P = \begin{pmatrix} -1 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

We use this to compute A^{17}, since $A = PDP^{-1}$. We do not show the computation of P^{-1}, which is straight-forward:

$$A^{17} = (PDP^{-1})^{17} = PD^{17}P^{-1} = \begin{pmatrix} -1 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3^{17} & 0 & 0 \\ 0 & 3^{17} & 0 \\ 0 & 0 & 6^{17} \end{pmatrix} \frac{1}{3} \begin{pmatrix} 0 & 3 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

This gives

$$A^{17} = 3^{16} \begin{pmatrix} 2 & -1 & -2 \\ 0 & 3 & 0 \\ -1 & -1 & 1 \end{pmatrix} + 6^{16} \begin{pmatrix} 2 & 2 & 4 \\ 0 & 0 & 0 \\ 2 & 2 & 4 \end{pmatrix}$$

4.10 Final Exam in GRA6035 on 10/12/2010, Problem 2

1. The determinant of A is given by

$$\det(A) = \begin{vmatrix} 1 & 7 & -2 \\ 0 & s & 0 \\ 1 & 1 & 4 \end{vmatrix} = s(4+2) = 6s$$

   It follows that the rank of A is 3 if $s \neq 0$ (since $\det(A) \neq 0$). When $s = 0$, A has rank 2 since $\det(A) = 0$ but the minor

$$\begin{vmatrix} 1 & -2 \\ 1 & 4 \end{vmatrix} = 6 \neq 0$$

   Therefore, we get

$$\operatorname{rk}(A) = \begin{cases} 3, & s \neq 0 \\ 2, & s = 0 \end{cases}$$

2. We compute the characteristic equation of A, and find that

$$\begin{vmatrix} 1-\lambda & 7 & -2 \\ 0 & s-\lambda & 0 \\ 1 & 1 & 4-\lambda \end{vmatrix} = (s-\lambda)(\lambda^2-5\lambda+6) = (s-\lambda)(\lambda-2)(\lambda-3)$$

Therefore, the eigenvalues of A are $\lambda = s, 2, 3$. Furthermore, we have that

$$A\mathbf{v} = \begin{pmatrix} 6 \\ s \\ 6 \end{pmatrix}$$

We see that $\mathbf{v}$ is an eigenvector for A if and only if $s = 6$, in which case $A\mathbf{v} = 6\mathbf{v}$.

3. If $s \neq 2, 3$, then A has three distinct eigenvalues, and therefore A is diagonalizable. If $s = 2$, we check the eigenspace corresponding to the double root $\lambda = 2$: The coefficient matrix of the system has echelon form

$$\begin{pmatrix} -1 & 7 & -2 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 8 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of rank two, so there is only one free variable. If $s = 3$, we check the eigenspace corresponding to the double root $\lambda = 3$: The coefficient matrix of the system has echelon form

$$\begin{pmatrix} -2 & 7 & -2 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of rank two, so there is only one free variable. In both cases, there are too few linearly independent eigenvectors, and A is not diagonalizable. Hence A is diagonalizable if $s \neq 2, 3$.

4.11 Mock Final Exam in GRA6035 on 12/2010, Problem 1

1. The determinant of A is given by

$$\det(A) = \begin{vmatrix} 1 & 1 & -4 \\ 0 & t+2 & t-8 \\ 0 & -5 & 5 \end{vmatrix} = 10t - 30 = 10(t-3)$$

It follows that the rank of A is 3 if $t \neq 3$ (since $\det(A) \neq 0$). When $t = 3$, A has rank 2 since $\det(A) = 0$ but the minor

$$\begin{vmatrix} 1 & -4 \\ 0 & 5 \end{vmatrix} = 5 \neq 0$$

Therefore, we get

$$\operatorname{rk}(A) = \begin{cases} 3, & t \neq 3 \\ 2, & t = 3 \end{cases}$$

2. We compute the characteristic equation of A, and find that

$$\begin{vmatrix} 1-\lambda & 1 & -4 \\ 0 & t+2-\lambda & t-8 \\ 0 & -5 & 5-\lambda \end{vmatrix} = (1-\lambda)(\lambda^2-(t+7)\lambda+10(t-3)) = 0$$

Since $\lambda^2-(t+7)\lambda+10(t-3) = 0$ has solutions $\lambda = 10$ and $\lambda = t-3$, the eigenvalues of A are $\lambda = 1, 10, t-3$.

3. When A has three distinct eigenvalues, it is diagonalizable. We see that this happens for all values of t except $t = 4$ and $t = 13$. Hence A is diagonalizable for $t \neq 4, 13$. If $t = 4$, we check the eigenspace corresponding to the double root $\lambda = 1$: The coefficient matrix of the system has echelon form

$$\begin{pmatrix} 0 & 1 & -4 \\ 0 & 5 & -4 \\ 0 & -5 & 4 \end{pmatrix} \dashrightarrow \begin{pmatrix} 0 & 1 & -4 \\ 0 & 0 & 16 \\ 0 & 0 & 0 \end{pmatrix}$$

of rank two, so there is only one free variable. If $t = 13$, we check the eigenspace corresponding to the double root $\lambda = 10$: The coefficient matrix of the system has echelon form

$$\begin{pmatrix} -9 & 1 & -4 \\ 0 & 5 & 5 \\ 0 & -5 & -5 \end{pmatrix} \dashrightarrow \begin{pmatrix} -9 & 1 & -4 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

of rank two, so there is only one free variable. In both cases, there are too few linearly independent eigenvectors, and A is not diagonalizable. Hence A is diagonalizable if $t \neq 4, 13$.

4.12 Final Exam in GRA6035 on 30/05/2011, Problem 2

1. The determinant of A is given by

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & s & s^2 \\ 1 & -1 & 1 \end{vmatrix} = 2s^2 - 2 = 2(s-1)(s+1)$$

It follows that the rank of A is 3 if $s \neq \pm 1$ (since $\det(A) \neq 0$). When $s = \pm 1$, A has rank 2 since $\det(A) = 0$ but there is a non-zero minor of order two in each case. Therefore, we get

$$\text{rk}(A) = \begin{cases} 3, & s \neq \pm 1 \\ 2, & s = \pm 1 \end{cases}$$

2. We compute that

$$A\mathbf{v} = \begin{pmatrix} 1 \\ 1+s-s^2 \\ -1 \end{pmatrix}, \quad \lambda\mathbf{v} = \begin{pmatrix} \lambda \\ \lambda \\ -\lambda \end{pmatrix}$$

and see that $\mathbf{v}$ is an eigenvector for A if and only if $\lambda = 1$ and $1 + s - s^2 = 1$, or $s = s^2$. This gives $s = 0, 1$.

3. We compute the characteristic equation of A when $s = -1$, and find that

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & -1-\lambda & 1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = \lambda(2+\lambda-\lambda^2) = -\lambda(\lambda-2)(\lambda+1)$$

Therefore, the eigenvalues of A are $\lambda = 0, 2, -1$ when $s = -1$. Since A has three distinct eigenvalues when $s = -1$, it follows that A is diagonalizable.

Lecture 5
Quadratic Forms and Definiteness

5.1 Main concepts

A *quadratic form* is a polynomial function $f(x_1, x_2, \ldots, x_n) = f(\mathbf{x})$ where all terms have degree two. It has the form

$$f(\mathbf{x}) = c_{11}x_1^2 + c_{12}x_1x_2 + \cdots + c_{nn}x_n^2$$

where c_{ij} are given numbers. There is a unique symmetric $n \times n$ matrix A, called the symmetric matrix of the quadratic form, such that $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$. This matrix is given by

$$A = (a_{ij}), \quad \text{where } a_{ii} = c_{ii} \text{ and } a_{ij} = a_{ji} = c_{ij}/2 \text{ for } i \neq j$$

The quadratic form f, and the symmetric matrix A, is called

1. *positive definite* if $f(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$
2. *negative definite* if $f(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$
3. *positive semidefinite* if $f(\mathbf{x}) \geq 0$ for all $\mathbf{x}$
4. *negative semidefinite* if $f(\mathbf{x}) \leq 0$ for all $\mathbf{x}$
5. *indefinite* if f takes both positive and negative values

If f is positive definite, then $\mathbf{x} = \mathbf{0}$ is a global minimum for f, and if f is negative definite, then $\mathbf{x} = \mathbf{0}$ is a global maximum for f.

Proposition 5.1. *Let f be a quadratic form with symmetric matrix A, and let $\lambda_1, \ldots \lambda_n$ be the eigenvalues of A.*

1. *A is positive definite if and only if $\lambda_1, \ldots, \lambda_n > 0$*
2. *A is negative definite if and only if $\lambda_1, \ldots, \lambda_n < 0$*
3. *A is positive semidefinite if and only if $\lambda_1, \ldots, \lambda_n \geq 0$*
4. *A is negative semidefinite if and only if $\lambda_1, \ldots, \lambda_n \leq 0$*
5. *A is indefinite if and only if it has both positive and negative eigenvalues*

Let A be a symmetric $n \times n$ matrix. The *leading principal minor* D_i of order i is the minor obtained by keeping the first i rows and columns of A, and deleting the remaining rows and columns. A *principal minor* of order i is a minor of order i obtained by deleting $n-i$ rows and the $n-i$ columns with the same numbering. There are in general many principal minors of order i, and we denote any one of them by Δ_i.

Proposition 5.2. *Let f be a quadratic form with symmetric matrix A.*

1. *A is positive definite if and only if $D_1, D_2, \dots, D_n > 0$*
2. *A is negative definite if and only if $D_i < 0$ when i is odd and $D_i > 0$ when i is even*
3. *A is positive semidefinite if and only if $\Delta_i \geq 0$ for any principal minor Δ_i*
4. *A is negative semidefinite if and only if $\Delta_i \leq 0$ for any principal minor of odd order i and $\Delta_i \geq 0$ for any principal minor of even order i*

5.2 Problems

5.1. Find the symmetric matrix of the following quadratic forms:

1. $Q(x,y) = x^2 + 2xy + y^2$
2. $Q(x,y) = ax^2 + bxy + cy^2$
3. $Q(x,y,z) = 3x^2 - 2xy + 3xz + 2y^2 + 3z^2$

5.2. Find the symmetric matrix and determine the definiteness of the following quadratic forms:

1. $Q(\mathbf{x}) = x_1^2 + 3x_2^2 + 5x_3^2$
2. $Q(\mathbf{x}) = x_1^2 + 2x_1x_2 + 3x_2^2 + 5x_3^2$

5.3. Compute all leading principal minors and all principal minors of the following matrices:

$$a) \quad A = \begin{pmatrix} -3 & 4 \\ 4 & -5 \end{pmatrix} \qquad b) \quad A = \begin{pmatrix} -3 & 4 \\ 4 & -6 \end{pmatrix}$$

In each case, write down the corresponding quadratic form $Q(x,y) = \mathbf{x}^T A\mathbf{x}$, and determine its definiteness. Use this to classify the stationary point $(x,y) = (0,0)$ of the quadratic form.

5.4. Compute all leading principal minors and all principal minors of the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix}$$

Write down the corresponding quadratic form $Q(x,y,z) = \mathbf{x}^T A\mathbf{x}$, and determine its definiteness. Use this to classify the stationary point $(x,y,z) = (0,0,0)$ of the quadratic form.

5.5. For which values of the parameters a, b, c is the symmetric matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

positive (semi)definite and negative (semi)definite?

5.6. Determine the definiteness of the following quadratic forms:

1. $f(x,y) = 2x^2 - 2xy + y^2$
2. $f(x,y) = -3x^2 + 8xy - 5y^2$
3. $f(x,y) = -3x^2 + 8xy - 6y^2$
4. $f(x,y) = 2x^2 + 8xy + 8y^2$
5. $f(x,y,z) = x^2 + 4xy + 4y^2 + 10yz + 6z^2$
6. $f(x,y,z) = -x^2 + 2xy - y^2 - 2z^2$
7. $f(x,y,z,w) = x^2 + 6xz + 2y^2 + 10yw + 4z^2 + 6w^2$

5.7. Give a detailed proof that $A^T A$ is positive definite for any invertible matrix A.

5.8. Midterm in GRA6035 24/09/2010, Problem 6
Consider the quadratic form

$$Q(x_1, x_2) = x_1^2 - 4x_1x_2 + 4x_2^2$$

Which statement is true?

1. Q is positive semidefinite but not positive definite
2. Q is negative semidefinite but not negative definite
3. Q is indefinite
4. Q is positive definite
5. I prefer not to answer.

5.9. Mock Midterm in GRA6035 09/2010, Problem 6
Consider the function

$$f(x_1, x_2, x_3) = x_1^2 + 6x_1x_2 + 3x_2^2 + 2x_3^2$$

Which statement is true?

1. f is not a quadratic form
2. f is a positive definite quadratic form
3. f is an indefinite quadratic form
4. f is a negative definite quadratic form
5. I prefer not to answer.

5.10. Midterm in GRA6035 24/05/2011, Problem 6
Consider the quadratic form

$$Q(x_1, x_2) = -2x_1^2 + 12x_1x_2 + 2x_2^2$$

Which statement is true?

1. Q is positive semidefinite but not positive definite
2. Q is negative semidefinite but not negative definite
3. Q is indefinite
4. Q is positive definite
5. I prefer not to answer.

5.3 Solutions

5.1 The symmetric matrices are given by

$$a)\quad A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad b)\quad A = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \qquad c)\quad A = \begin{pmatrix} 3 & -1 & 3/2 \\ -1 & 2 & 0 \\ 3/2 & 0 & 3 \end{pmatrix}$$

5.2 The symmetric matrices are given by

$$a)\quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \qquad b)\quad A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

In a) the eigenvalues $\lambda = 1, 3, 5$ are all positive, so the quadratic form Q and the matrix A are positive definite. In b), we compute the eigenvalues using the characteristic equation

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (5-\lambda)(\lambda^2 - 4\lambda + 2) = 0$$

and get $\lambda = 5$ and $\lambda = 2 \pm \sqrt{2}$. Since all eigenvalues are positive, the quadratic form Q and the matrix A are positive definite. Another way to determine the definiteness is to compute the leading principal minors $D_1 = 1$, $D_2 = 2$, and $D_3 = 10$.

5.3 In a) the leading principal minors are $D_1 = -3$ and $D_2 = -1$, and the principal minors are $\Delta_1 = -3, -5$ and $\Delta_2 = -1$. From the leading principal minors, we see that A is indefinite, and this means that $(x, y) = (0, 0)$ is a saddle point for the quadratic form. In b) the leading principal minors are $D_1 = -3$ and $D_2 = 2$, and the principal minors are $\Delta_1 = -3, -6$ and $\Delta_2 = 2$. From the leading principal minors, we see that A is negative definite, and this means that $(x, y) = (0, 0)$ is a global maximum for the quadratic form.

5.4 The leading principal minors and principal minors of the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix}$$

are given by

$$D_1 = 1,\ D_2 = 0,\ D_3 = |A| = 1(24-25) - 2(12) = -25$$

and

$$\Delta_1 = 1, 4, 6,\ \Delta_2 = 0, 6, -1,\ \Delta_3 = -25$$

The quadratic form of A is given by $Q(x,y,z) = x^2 + 4xy + 4y^2 + 10yz + 6z^2$. Since there is a principal minor $\Delta_2 = -1$ of order two that is negative, the quadratic form is indefinite, and the stationary point $(x,y,z) = (0,0,0)$ is a saddle point.

5.5 The leading principal minors are given by $D_1 = a,\ D_2 = ac - b^2$ and the principal minors are given by $\Delta_1 = a, c,\ \Delta_2 = ac - b^2$. Hence we have that

$$\text{positive definite} \Leftrightarrow a > 0, ac - b^2 > 0$$
$$\text{negative definite} \Leftrightarrow a < 0, ac - b^2 > 0$$
$$\text{positive semidefinite} \Leftrightarrow a \geq 0, c \geq 0, ac - b^2 \geq 0$$
$$\text{negative semidefinite} \Leftrightarrow a \leq 0, c \leq 0, ac - b^2 \geq 0$$

5.6

1. The symmetric matrix of the quadratic form is given by

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

   The leading principal minors are $D_1 = 2$ and $D_2 = 1$. Since both are positive, the quadratic form is positive definite.
2. The symmetric matrix of the quadratic form is given by

$$A = \begin{pmatrix} -3 & 4 \\ 4 & -5 \end{pmatrix}$$

   The leading principal minors are $D_1 = -3$ and $D_2 = -1$. Since $D_2 < 0$, the quadratic form is indefinite.
3. The symmetric matrix of the quadratic form is given by

$$A = \begin{pmatrix} -3 & 4 \\ 4 & -6 \end{pmatrix}$$

   The leading principal minors are $D_1 = -3$ and $D_2 = 2$. Since $D_1 < 0$ and $D_2 > 0$, the quadratic form is negative definite.
4. The symmetric matrix of the quadratic form is given by

$$A = \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix}$$

The leading principal minors are $D_1 = 2$ and $D_2 = 0$, and the other principal minors are $\Delta_1 = 8$. Since $\Delta_1, \Delta_2 \geq 0$, the quadratic form is positive semidefinite.

5. The symmetric matrix of the quadratic form is given by

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix}$$

The leading principal minors are $D_1 = 1$, $D_2 = 0$ and $D_3 = -25$. Since D_1 and D_3 have opposite signs, the quadratic form is indefinite.

6. The symmetric matrix of the quadratic form is given by

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The leading principal minors are $D_1 = -1$, $D_2 = 0$ and $D_3 = 0$. The remaining principal minors are $\Delta_1 = -1, -2$ and $\Delta_2 = 2, 2$. Since $Delta_1 \leq 0$, $\Delta_2 \geq 0$ and $\Delta_3 \geq 0$ for all principal minors, the quadratic form is negative semidefinite.

7. The symmetric matrix of the quadratic form is given by

$$A = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 5 \\ 3 & 0 & 4 & 0 \\ 0 & 5 & 0 & 6 \end{pmatrix}$$

The first three leading principal minors are $D_1 = 1$, $D_2 = 2$, $D_3 = -10$. Since D_1 and D_3 have opposite signs, the quadratic form is indefinite (and there is no need to compute the last leading principal minor).

5.7 We consider the quadratic form $\mathbf{x}^T(A^TA)\mathbf{x}$, and let $\mathbf{y} = A\mathbf{x}$. Then we have

$$\mathbf{x}^T(A^TA)\mathbf{x} = (A\mathbf{x})^T(A\mathbf{x}) = \mathbf{y}^T\mathbf{y} = y_1^2 + y_2^2 + \cdots + y_n^2 \geq 0$$

This means $B = A^TA$ is positive semidefinite. Any eigenvalue λ of B therefore satisfies $\lambda \geq 0$. If $\lambda = 0$, then $\det(B) = 0$ and this is not possible since $\det(B) = \det(A^T)\det(A) = \det(A)^2$ and A is invertible so $\det(A) \neq 0$. Hence any eigenvalue λ of B satisfies $\lambda > 0$, and B is positive definite.

5.8 Midterm in GRA6035 24/09/2010, Problem 6

The symmetric matrix associated with Q is

$$A = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$$

We compute its eigenvalues to be 0 and 5. Hence the correct answer is alternative **A**. This question can also be answered using the fact that the principal minors are $1, 4$ (of order one) and 0 (of order two), or the fact that $Q(x_1, x_2) = (x_1 - 2x_2)^2$.

5.9 Mock Midterm in GRA6035 09/2010, Problem 6

Since all terms of f have degree two, it is a quadratic form, and its symmetric matrix is

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

The characteristic equation of A is $(\lambda^2 - 4\lambda - 6)(2 - \lambda) = 0$, and the eigenvalues are $\lambda = 2$ and $\lambda = 2 \pm \sqrt{10}$. Hence the correct answer is alternative **C**. This problem can also be solved using the principal leading minors, which are $D_1 = 1$, $D_2 = -6$ and $D_3 = -12$.

5.10 Midterm in GRA6035 24/05/2011, Problem 6

The symmetric matrix associated with Q is

$$A = \begin{pmatrix} -2 & 6 \\ 6 & 2 \end{pmatrix}$$

We compute its eigenvalues to be $\pm\sqrt{40}$. Hence the correct answer is alternative **C**. This problem can also be solved using the principal leading minors, which are $D_1 = -2$ and $D_2 = -40$.

Lecture 6
Unconstrained Optimization

6.1 Main concepts

Let $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$ be a function in n variables. We consider the *unconstrained optimization problem*

$$\max / \min f(\mathbf{x}) \quad \text{when } \mathbf{x} \text{ is any point in } \mathbb{R}^n$$

We say that $\mathbf{x}^*$ is a *stationary point* for f if $f_1'(\mathbf{x}^*) = f_2'(\mathbf{x}^*) = \cdots = f_n'(\mathbf{x}^*) = 0$. The Hessian matrix $H(f)$ of f is the $n \times n$ symmetric matrix of second order partial derivatives, given by

$$H(f) = \begin{pmatrix} f''_{11} & f''_{12} & \cdots & f''_{1n} \\ f''_{21} & f''_{22} & \cdots & f''_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f''_{n1} & f''_{n2} & \cdots & f''_{nn} \end{pmatrix}$$

In general, each coefficient in the Hessian matrix is a function of $\mathbf{x}$.

Lemma 6.1. *If f has a local max/min at $\mathbf{x}^*$, then $\mathbf{x}^*$ is a stationary point.*

Lemma 6.2. *Let $\mathbf{x}^*$ be a stationary point of f. Then we have:*

1. *If $H(f)(\mathbf{x}^*)$ is positive definite, then $\mathbf{x}^*$ is a local minimum for f*
2. *If $H(f)(\mathbf{x}^*)$ is negative definite, then $\mathbf{x}^*$ is a local maximum for f*
3. *If $H(f)(\mathbf{x}^*)$ is indefinite, then $\mathbf{x}^*$ is a saddle point for f*

The function f is *convex* if and only if $H(f)(\mathbf{x})$ is positive semidefinite for all $\mathbf{x}$ in $\mathbb{R}^n$, and *concave* if $H(f)(\mathbf{x})$ is negative semidefinite for all $\mathbf{x}$ in $\mathbb{R}^n$.

Proposition 6.1. *If f is convex, then any stationary point is a global minimum for f. If f is concave, then any stationary point is a global maximum for f.*

6.2 Problems

6.1. Find the stationary points and classify their type:

1. $f(x,y) = x^4 + x^2 - 6xy + 3y^2$
2. $f(x,y) = x^2 - 6xy + 2y^2 + 10x + 2y - 5$
3. $f(x,y) = xy^2 + x^3y - xy$
4. $f(x,y) = 3x^4 + 3x^2y - y^3$

6.2. Find the stationary points and classify their type:

1. $f(x,y,z) = x^2 + 6xy + y^2 - 3yz + 4z^2 - 10x - 5y - 21z$
2. $f(x,y,z) = (x^2 + 2y^2 + 3z^2)e^{-(x^2+y^2+z^2)}$

6.3. Find the stationary points and classify their type:

1. $f(x,y) = e^{x+y}$
2. $f(x,y) = e^{xy}$
3. $f(x,y) = \ln(x^2 + y^2 + 1)$

6.4. Is the function convex? Is it concave?

1. $f(x,y) = e^{x+y}$
2. $f(x,y) = e^{xy}$
3. $f(x,y) = \ln(x^2 + y^2 + 1)$

6.5. Consider the set $\{(x,y) : x \geq 0, y \geq 0, xy \leq 1\}$. Sketch the set in a coordinate system, and determine if it is a convex set.

6.6. Compute the Hessian of the function $f(x,y) = x - y - x^2$, and show that f is a concave function defined on $D_f = \mathbb{R}^2$. Determine if

$$g(x,y) = e^{x-y-x^2}$$

is a convex or a concave function on $\mathbb{R}^2$.

6.7. Let $f(x,y) = ax^2 + bxy + cy^2 + dx + ey + f$ be the general polynomial in two variables of degree two. For which values of the parameters is this function (strictly) convex and (strictly) concave?

6.8. Determine the values of the parameter a for which the function

$$f(x,y) = -6x^2 + (2a+4)xy - y^2 + 4ay$$

is convex and concave on $\mathbb{R}^2$.

6.9. Determine if the function $f(x,y,z) = \ln(xyz)$ defined on the domain of definition $D_f = \{(x,y,z) : x > 0, y > 0, z > 0\}$ is convex or concave.

6.10. Consider the function $f(x,y) = x^4 + 16y^4 + 32xy^3 + 8x^3y + 24x^2y^2$ defined on $\mathbb{R}^2$. Determine if this function is convex or concave.

6.11. Consider the function $f(x,y) = e^{x+y} + e^{x-y}$ defined on $\mathbb{R}^2$. Determine if this function is convex or concave.

6.12. Find all extremal points for the function $f(x,y,z) = x^4 + y^4 + z^4 + x^2 + y^2 + z^2$.

6.13. Show that the function $f(x,y) = x^3 + y^3 - 3x - 2y$ defined on the convex set $S = \{(x,y) : x > 0,\ y > 0\}$ is (strictly) convex, and find its global minimum.

6.14. A company produces two output goods, denoted by A and B. The cost per day is

$$C(x,y) = 0.04x^2 - 0.01xy + 0.01y^2 + 4x + 2y + 500$$

when x units of A and y units of B are produced ($x > 0,\ y > 0$). The firm sells all it produces at prices 13 per unit of A and 8 per unit of B. Find the profit function π and the values of x and y that maximizes profit.

6.15. The function $f(x,y,z) = x^2 + 2xy + y^2 + z^3$ is defined on $S = \{(x,y,z) : z > 0\}$. Show that S is a convex set. Find the stationary points of f and the Hessian matrix. Is f convex or concave? Does f have a global extremal point?

6.16. Show that the function $f(x,y,z) = x^4 + y^4 + z^4 + x^2 - xy + y^2 + yz + z^2$ is convex.

6.17. Find all local extremal points for the function $f(x,y,z) = -2x^4 + 2yz - y^2 + 8x$ and classify their type.

6.18. The function $f(x,y,z) = x^2 + y^2 + 3z^2 - xy + 2xz + yz$ defined on $\mathbb{R}^3$ has only one stationary point. Show that it is a local minimum.

6.19. Find all local extremal points for the function $f(x,y) = x^3 - 3xy + y^3$ and classify their type.

6.20. The function $f(x,y,z) = x^3 + 3xy + 3xz + y^3 + 3yz + z^3$. Find all local extremal points for f and classify their type.

6.21. Midterm in GRA6035 on 24/09/2010, Problem 7
Consider the function

$$f(x_1,x_2,x_3) = 3x_1^2 + 2x_1x_2 + 3x_2^2 + x_3^2 + x_1 - x_2$$

defined on $\mathbb{R}^3$. **Which statement is true?**

1. f is a convex function but not a concave function
2. f is a convex function and a concave function
3. f is not a convex function but a concave function
4. f is neither a convex nor a concave function
5. I prefer not to answer.

6.22. Mock Midterm in GRA6035 on 09/2010, Problem 7
Consider the function
$$f(x_1,x_2) = 3 - a \cdot Q(x_1,x_2)$$
defined on $\mathbb{R}^2$, where $a \in \mathbb{R}$ is a number and Q is a positive definite quadratic form. **Which statement is true?**

1. f is convex for all values of a
2. f is concave for all values of a
3. f is convex if $a \geq 0$ and concave if $a \leq 0$
4. f is convex if $a \leq 0$ and concave if $a \geq 0$
5. I prefer not to answer.

6.23. Midterm in GRA6035 on 24/05/2011, Problem 7
Consider the function
$$f(x_1,x_2,x_3) = -x_1^2 + 2x_1x_2 - 3x_2^2 - x_3^2 - x_1 - x_3$$
defined on $\mathbb{R}^3$. **Which statement is true?**

1. f is a convex function but not a concave function
2. f is a convex function and a concave function
3. f is not a convex function but a concave function
4. f is neither a convex nor a concave function
5. I prefer not to answer.

6.24. Final Exam in GRA6035 on 10/12/2010, Problem 1
We consider the function $f(x,y,z) = x^2e^x + yz - z^3$.

1. Find all stationary points of f.
2. Compute the Hessian matrix of f. Classify the stationary points of f as local maxima, local minima or saddle points.

6.25. Mock Final Exam in GRA6035 on 12/2010, Problem 2

1. Find all stationary points of $f(x,y,z) = e^{xy+yz-xz}$.
2. The function $g(x,y,z) = e^{ax+by+cz}$ is defined on $\mathbb{R}^3$. Determine the values of the parameters a,b,c such that g is convex. Is it concave for any values of a,b,c?

6.26. Final Exam in GRA6035 on 30/05/2011, Problem 1
We consider the function $f(x,y,z,w) = x^5 + xy^2 - zw$.

1. Find all stationary points of f.
2. Compute the Hessian matrix of f. Classify the stationary points of f as local maxima, local minima or saddle points.

6.3 Solutions

6.1 In each case, we compute the first order partial derivatives and solve the first order conditions to find the stationary points. Then we compute the Hessian and determine the definiteness of the Hessian at each stationary point to classify it.

1. The first order conditions (FOC) are given by
$$f'_x = 4x^3 + 2x - 6y = 0, \quad f'_y = -6x + 6y = 0$$
From the second equation $x = y$, and the first then gives $4x^3 - 4x = 4x(x^2 - 1) = 0$, or $x = 0$ or $x = \pm 1$. The stationary points are $(0,0)$, $(1,1)$ and $(-1,-1)$. The Hessian matrix is given by
$$H(f) = \begin{pmatrix} 12x^2 + 2 & -6 \\ -6 & 6 \end{pmatrix}$$
At $(0,0)$ we have $D_1 = 2$ and $D_2 = 12 - 36 = -24$. This point is a saddle point since D_2 is negative. At $(\pm 1, \pm 1)$ we have $D_1 = 14$ and $D_2 = 84 - 36 = 48$. These points are local minima since $D_1, D_2 > 0$.
2. The first order conditions (FOC) are given by
$$f'_x = 2x - 6y + 10 = 0, \quad f'_y = -6x + 4y + 2 = 0$$
We combine the equations to get $-14y + 32 = 0$ or $y = 32/14 = 16/7$, and then $x = 3(16/7) - 5 = 13/7$. The unique stationary point is $(13/7, 16/7)$. The Hessian matrix is given by
$$H(f) = \begin{pmatrix} 2 & -6 \\ -6 & 4 \end{pmatrix}$$
and $D_1 = 2$, $D_2 = 8 - 36 = -28$. The stationary point is a saddle point since D_2 is negative.
3. The first order conditions (FOC) are given by
$$f'_x = y^2 + 3x^2y - y = y(y + 3x^2 - 1) = 0, \quad f'_y = 2xy + x^3 - x = x(2y + x^2 - 1) = 0$$
From the second equation either $x = 0$ or $x^2 = 1 - 2y$. In the first equation, $x = 0$ gives $y^2 - y = 0$, that is $y = 0$ or $y = 1$, and the stationary points in this case are $(0,0)$ and $(0,1)$. The other possibility is $x \neq 0$ but $x^2 = 1 - 2y$. In this case, the first equation gives $y = 0$ or $y + 3x^2 - 1 = 0$. If $y = 0$ then $x = \pm 1$, and if $y = 1 - 3x^2$, then $x^2 = 1 - 2(1 - 3x^2) = 6x^2 - 1$, or $x = \pm 1/\sqrt{5}$. This gives stationary points $(\pm 1, 0)$ and $(\pm 1/\sqrt{5}, 2/5)$. The Hessian matrix is given by
$$H(f) = \begin{pmatrix} 6xy & 2y + 3x^2 - 1 \\ 2y + 3x^2 - 1 & 2x \end{pmatrix}$$

At $(0,0)$ and $(0,1)$, we have $D_1 = 0$ and $D_2 = -1$. These points are saddle points since D_2 is negative. At $(\pm 1,0)$ we have $D_1 = 0$ and $D_2 = -4$. These points are also saddle points since D_2 is negative. At $(\pm 1/\sqrt{5}, 3/5)$, we have $D_1 = 12x/5$ and $D_2 = 12(1/5)(2/5) - (4/5+3/5-1)^2 = 20/25$. This is positive definite when x is positive, and negative definite when x is negative. Therefore $(1/\sqrt{5}, 2/5)$ is a local minimum and $(-1/\sqrt{5}, 2/5)$ is a local maximum

4. The first order conditions (FOC) are given by

$$f'_x = 12x^3 + 6xy = 0, \quad f'_y = 3x^2 - 3y^2 = 0$$

From the second equation $x^2 = y^2$, and therefore $x = y$ or $x = -y$. If $x = y$, the first equation gives $12x^3 + 6x^2 = 6x^2(2x+1) = 0$, that is $x = 0$ or $x = -1/2$. This gives stationary points $(0,0)$ and $(-1/2,-1/2)$. If $x = -y$ then the first equation gives $12x^3 - 6x^2 = 6x^2(2x-1)$, that is $x = 0$ or $x = 1/2$. This gives stationary points $(0,0$ (again) and $(1/2,-1/2)$. The Hessian matrix is given by

$$H(f) = \begin{pmatrix} 36x^2 + 6y & 6x \\ 6x & -6y \end{pmatrix}$$

At $(\pm 1/2, -1/2)$ we have $D_1 = 6$ and $D_2 = 18 - 9 = 9$. These points are local minima since D_1, D_2 are positive. At $(0,0)$ the Hessian matrix is the zero matrix, and the second derivative test is inconclusive. Along the path where $x = 0$ and $y = a$ (the y-axis), we have that $f(0,a) = -a^3$. Since $f(0,a)$ is negative for $a > 0$ and positiv for $a < 0$, the stationary point $(0,0)$ at $a = 0$ is a saddle point.

6.2

1. The first order conditions (FOC) are given by

$$f'_x = 2x + 6y - 10 = 0, \quad f'_y = 6x + 2y - 3z - 5 = 0, \quad f'_z = -3y + 8z - 21 = 0$$

The unique stationary point is $(x,y,z) = (2,1,3)$, and we find it by solving the linear system (for instance using Gaussian elimination). The Hessian is indefinite at this stationary point since the Hessian matrix

$$H(f) = \begin{pmatrix} 2 & 6 & 0 \\ 6 & 2 & -3 \\ 0 & -3 & 8 \end{pmatrix}$$

has leading principal minors $D_1 = 2$ and $D_2 = -32$ (there is no need to compute the last leading principal minor). The stationary point $(2,1,3)$ is a saddle point for f.

2. Let us write $u = x^2 + 2y^2 + 3z^2$ and $v = -(x^2 + y^2 + z^2)$. The first order conditions (FOC) are given by

$$f'_x = 2xe^v + ue^v(-2x) = 2xe^v(1-u)$$
$$f'_y = 4ye^v + ue^v(-2y) = 2ye^v(2-u)$$
$$f'_z = 6ze^v + ue^v(-2z) = 2ze^v(3-u)$$

If $u = 1$, then $y = z = 0$, and $x = \pm 1$. If $u = 2$, then $x = z = 0$ and $y = \pm 1$. If $u = 3$, then $x = y = 0$ and $z = \pm 1$. If $u \neq 1,2,3$ then $x = y = z = 0$. We find the stationary points $(\pm 1,0,0)$, $(0,\pm 1,0)$, $(0,0,\pm 1)$ and $(0,0,0)$. To compute the Hessian matrix, notice that

$$f''_{xx} = (2(1-u)+2x(-2x))e^v + 2x(1-u)e^v(-2x) = 2e^v(1-u-4x^2(2-u))$$

The other second order principal minors can be computed similarly. We find that the Hessian matrix is given by

$$H(f) = 2e^v \begin{pmatrix} (1-u)-4x^2(2-u) & -2xy(3-u) & -2xz(4-u) \\ -2xy(3-u) & (2-u)-2y^2(4-u) & -2yz(5-u) \\ -2xz(4-u) & -2yz(5-u) & (3-u)-2z^2(6-u) \end{pmatrix}$$

At all stationary points, $xy = xz = yz$ so that the Hessian matrix is diagonal. At $(0,0,0)$, the diagonal entries are $(2,4,6)$ and are all positive. The point $(0,0,0)$ is therefore a local minimum point. At $(\pm 1,0,0)$, the diagonal entries are $2e^{-1}(-2,1,1)$, and at $(0,\pm 1,0)$ the diagonal entries are $2e^{-1}(-1,-4,1)$. In both cases, the diagonal contains both positive and negative entries, and these stationary points are therefore saddle points. At $(0,0,\pm 1)$, the diagonal entries are $2e^{-1}(-2,-1,-6)$ and are all negative. The points $(0,0,\pm 1)$ are therefore local maxima.

6.3

1. The function $f(x,y) = e^{x+y}$ has partial derivatives $f'_1 = f'_2 = e^{x+y}$, hence there are no stationary points.
2. The function $f(x,y) = e^{xy}$ has partial derivatives $f'_1 = ye^{xy}$ and $f'_2 = xe^{xy}$. Hence the stationary points are given by $ye^{xy} = xe^{xy} = 0$, and we see that there is a unique stationary point $(x,y) = (0,0)$. The Hessian matrix is

$$H(f)(0,0) = \begin{pmatrix} y^2e^{xy} & (1+xy)e^{xy} \\ (1+xy)e^{xy} & x^2e^{xy} \end{pmatrix}(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This matrix is indefinite since $D_2 = -1$, and $(0,0)$ is a saddle point.
3. The function $f(x,y) = \ln(x^2+y^2+1)$ has partial derivatives $f'_1 = 2x/(x^2+y^2+1)$ and $f'_2 = 2y/(x^2+y^2+1)$. Hence there is a unique stationary point $(x,y) = (0,0)$. The Hessian matrix is

$$H(f)(0,0) = \begin{pmatrix} \frac{-2x^2+2y^2+2}{(x^2+y^2+1)^2} & \frac{-4xy}{(x^2+y^2+1)^2} \\ \frac{-4xy}{(x^2+y^2+1)^2} & \frac{2x^2-2y^2+2}{(x^2+y^2+1)^2} \end{pmatrix}(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Since $D_1 = 2$, $D_2 = 4$, the Hessian matrix is posive definte, and $(0,0)$ is a local minimum point for f.

6.4

1. In the case $f(x,y) = e^{x+y}$, we see from the computations above that the Hessian matrix is
$$H(f) = \begin{pmatrix} e^{x+y} & e^{x+y} \\ e^{x+y} & e^{x+y} \end{pmatrix}$$
Hence $D_1 = e^{x+y} > 0$ and $D_2 = 0$, and the remaining principal minors are given by $\Delta_1 = e^{x+y}$, so the Hessian matrix is positive semidefinite for all (x,y) and f is convex.
2. In the case $f(x,y) = e^{xy}$, the Hessian matrix is
$$H(f) = \begin{pmatrix} y^2 e^{xy} & (1+xy)e^{xy} \\ (1+xy)e^{xy} & x^2 e^{xy} \end{pmatrix}$$
Since it is indefinite for $(x,y) = (0,0)$, f is neither convex nor concave.
3. In the case $f(x,y) = \ln(x^2+y^2+1)$, the Hessian matrix is
$$H(f) = \begin{pmatrix} \frac{-2x^2+2y^2+2}{(x^2+y^2+1)^2} & \frac{-4xy}{(x^2+y^2+1)^2} \\ \frac{-4xy}{(x^2+y^2+1)^2} & \frac{2x^2-2y^2+2}{(x^2+y^2+1)^2} \end{pmatrix}$$
Since $D_1(2,0) = -6/25$ and $D_1(0,2) = 20/25$, it follows that f is neither convex nor concave.

6.5 The points $(1/2,2)$ and $(2,1/2)$ are in the set $\{(x,y) : x \geq 0, y \geq 0, xy \leq 1\}$, while the straight line between these two points goes through the point $(5/4,5/4)$ which is not in the set. Hence the set is not convex. This can also be derived from the fact that $y = 1/x$ is not a concave function.

6.6 The Hessian of the function $f(x,y) = x - y - x^2$ is given by
$$H(f) = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$$
since $f'_x = 1-2x$ and $f'_y = -1$. Since $\Delta_1 = -2, 0 \leq 0$ and $\Delta_2 = 0 \geq 0$, the function f is concave on $D_f = \mathbb{R}^2$. The Hessian of the function $g(x,y) = e^{x-y-x^2}$ is given by
$$H(g) = \begin{pmatrix} e^{x-y-x^2}(1-2x)^2 + e^{x-y-x^2}(-2) & e^{x-y-x^2}(-1)(1-2x) \\ e^{x-y-x^2}(-1)(1-2x) & e^{x-y-x^2}(-1)(-1) \end{pmatrix}$$
by the product rule, since $g'_x = e^{x-y-x^2}(1-2x)$ and $g'_y = e^{x-y-x^2}(-1)$. This gives
$$H(g) = e^{x-y-x^2} \begin{pmatrix} 4x^2-4x-1 & 2x-1 \\ 2x-1 & 1 \end{pmatrix}$$

This gives $D_2 = (e^{x-y-x^2})^2(4x^2-4x-1-(2x-1)^2) = -2(e^{x-y-x^2})^2 < 0$, and this means that g is neither convex nor concave.

6.7 The Hessian matrix of the function $f(x,y) = ax^2+bxy+cy^2+dx+ey+f$ is given by

$$H(f) = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$$

This means that $D_1 = 2a$ and $D_2 = 4ac-b^2$. Therefore, f is strictly convex if and only if $a > 0$ and $4ac-b^2 > 0$, and f is strictly concave if and only if $a < 0$ and $4ac-b^2 > 0$. The remaining principal minors are $\Delta_1 = 2c$, and this means that f is convex if and only if $a \geq 0, c \geq 0, 4ac-b^2 \geq 0$, and that f is concave if and only if $a \leq 0, c \leq 0, 4ac-b^2 \geq 0$.

6.8 The Hessian matrix of the function $f(x,y) = -6x^2+(2a+4)xy-y^2+4ay$ is given by

$$H(f) = \begin{pmatrix} -12 & 2a+4 \\ 2a+4 & -2 \end{pmatrix}$$

Hence the leading principal minors are $D_1 = -12$ and $D_2 = 24-(2a+4)^2 = -4a^2-16a+8$, and the remaining principal minor is $\Delta_1 = -2$. We have that

$$-4a^2-16a+8 = -4(a^2+4a-2) \geq 0 \quad \Leftrightarrow \quad -2-\sqrt{6} \leq a \leq -2+\sqrt{6}$$

This implies that f is concave if and only if $-2-\sqrt{6} \leq a \leq -2+\sqrt{6}$, and that f is never convex.

6.9 Since $f'_x = yz/xyz = 1/x$, we have $f'_y = 1/y$ and $f'_z = 1/z$ in the same way (or by the remark $\ln(xyz) = \ln(x)+\ln(y)+\ln(z)$). The Hessian becomes

$$H(f) = \begin{pmatrix} -1/x^2 & 0 & 0 \\ 0 & -1/y^2 & 0 \\ 0 & 0 & -1/z^2 \end{pmatrix}$$

This matrix has $D_1 = -1/x^2 < 0$, $D_2 = 1/(x^2y^2) > 0$, $D_3 = -1/(x^2y^2z^2) < 0$ on $D_f = \{(x,y,z) : x > 0, y > 0, z > 0\}$, hence f is concave.

6.10 Since $f'_x = 4x^3+32y^3+24x^2y+48xy^2$ and $f'_y = 64y^3+96xy^2+8x^3+48x^2y$, we have

$$H(f) = \begin{pmatrix} 12x^2+48xy+48y^2 & 96y^2+24x^2+96xy \\ 96y^2+24x^2+96xy & 192y^2+192xy+48x^2 \end{pmatrix}$$

Completing the squares, we see that

$$H(f) = \begin{pmatrix} 12(x+2y)^2 & 24(x+2y)^2 \\ 24(x+2y)^2 & 48(x+2y)^2 \end{pmatrix} = 12(x+2y)^2 \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

This gives $\Delta_1 = 12(x+2y)^2, 48(x+2y)^2 \geq 0$ and $\Delta_2 = 144(x+2y)^4(4-4) = 0$. This implies that f is convex.

6.11 We have $f'_x = e^{x+y} + e^{x-y}$ and $f'_y = e^{x+y} - e^{x-y}$, and the Hessian is given by

$$H(f) = \begin{pmatrix} e^{x+y} + e^{x-y} & e^{x+y} - e^{x-y} \\ e^{x+y} - e^{x-y} & e^{x+y} + e^{x-y} \end{pmatrix}$$

This implies that $D_1 = e^{x+y} + e^{x-y} > 0$ and that we have

$$D_2 = (e^{x+y} + e^{x-y})^2 - (e^{x+y} + e^{x-y})^2 = 4e^{x+y}e^{x-y} = 4e^{2x} > 0$$

Hence the function f is convex.

6.12 The partial derivatives of $f(x,y,z) = x^4 + y^4 + z^4 + x^2 + y^2 + z^2$ are

$$f'_x = 4x^3 + 2x, \quad f'_y = 4y^3 + 2y, \quad f'_z = 4z^3 + 2z$$

The stationary points are given by $2x(2x^2+1) = 2y(2y^2+1) = 2z(2z^2+1) = 0$, and this means that the unique stationary point is $(x,y,z) = (0,0,0)$. The Hessian of f is

$$H(f) = \begin{pmatrix} 12x^2+2 & 0 & 0 \\ 0 & 12y^2+2 & 0 \\ 0 & 0 & 12z^2+2 \end{pmatrix}$$

We see that $H(f)$ is positive definite, and therefore f is convex and $(0,0,0)$ is a global minimum point.

6.13 The partial derivatives of $f(x,y) = x^3 + y^3 - 3x - 2y$ are

$$f'_x = 3x^2 - 3, \quad f'_y = 3y^2 - 2$$

The stationary points are given by $3x^2 - 3 = 3y^2 - 2 = 0$, and this means that the unique stationary point in S is $(x,y,z) = (1, \sqrt{2/3})$. The Hessian of f is

$$H(f) = \begin{pmatrix} 6x & 0 \\ 0 & 6y \end{pmatrix}$$

We see that $H(f)$ is positive definite since $D_1 = 6x > 0$ and $D_2 = 36xy > 0$, and therefore f is convex and $(1, \sqrt{2/3})$ is a global minimum point.

6.14 The profit function $\pi(x,y)$ is defined on $\{(x,y) : x > 0,\ y > 0\}$, and is given by

$$\pi(x,y) = 13x + 8y - C(x,y) = -0.04x^2 + 0.01xy - 0.01y^2 + 9x + 6y - 500$$

The Hessian of π is given by

$$H(\pi) = \begin{pmatrix} -0.08 & 0.01 \\ 0.01 & -0.02 \end{pmatrix}$$

and it is negative definite since $D_1 = -0.08 < 0$ and $D_2 = 0.016 - 0.0001 = 0.0159 > 0$, and therefore π is concave. The stationary point of π is given by

$$\pi'_x = -0.08x + 0.01y + 9 = 0, \quad \pi'_y = 0.01x - 0.02y + 6 = 0$$

This gives $(x,y) = (160,380)$, which is the unique maximum point.

6.15 To prove that S is a convex set, pick any points $P = (x,y,z)$ and $Q = (x',y',z')$ in S. By definition, $z > 0$ and $z' > 0$, which implies that all points on the line segment $[P,Q]$ have positive z-coordinate as well. This means that $[P,Q]$ is contained in S, and therefore S is convex. The partial derivatives of f are

$$f'_x = 2x + 2y, \quad f'_y = 2x + 2y, \quad f'_z = 3z^2$$

Since $z > 0$, there are no stationary points in S. The Hessian matrix of f is

$$H(f) = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 6z \end{pmatrix}$$

The principal minors are $\Delta_1 = 2,2,6z > 0$, $\Delta_2 = 0,12z,12z > 0$ and $\Delta_3 = 0$, so $H(f)$ is positive semidefinite and f is convex (but not strictly convex) on S. Since f has no stationary points and S is open (so there are no boundary points), f does not have global extremal points.

6.16 The partial derivatives of $f(x,y,z) = x^4 + y^4 + z^4 + x^2 - xy + y^2 + yz + z^2$ are

$$f'_x = 4x^3 + 2x - y, \quad f'_y = 4y^3 - x + 2y + z, \quad f'_z = 4z^3 + y + 2z$$

and the Hessian matrix is

$$H(f) = \begin{pmatrix} 12x^2 + 2 & -1 & 0 \\ -1 & 12y^2 + 2 & 1 \\ 0 & 1 & 12z^2 + 2 \end{pmatrix}$$

Since $D_1 = 12x^2 + 2 > 0$, $D_2 = (12x^2+2)(12y^2+2) - 1 = 144x^2y^2 + 24x^2 + 24y^2 + 3 > 0$ and $D_3 = -1(12x^2+2) + (12z^2+2)D_2 = 1728x^2y^2z^z + 288(x^2y^2 + x^2z^2 + y^2z^2) + 36x^2 + 48y^2 + 36z^2 + 4 > 0$, we see that f is convex.

6.17 The partial derivatives of the function $f(x,y,z) = -2x^4 + 2yz - y^2 + 8x$ is

$$f'_x = -8x^3 + 8, \quad f'_y = 2z - 2y, \quad f'_z = 2y$$

Hence the stationary points are given by $y = 0, z = 0, x = 1$ or $(x,y,z) = (1,0,0)$. The Hessian matrix of f is

$$H(f) = \begin{pmatrix} -24x^2 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & 0 \end{pmatrix} \quad \Rightarrow \quad H(f)(1,0,0) = \begin{pmatrix} -24 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

Since $D_1 = -24 < 0$, $D_2 = 48 > 0$, but $D_3 = 96 > 0$, we see that the stationary point $(1,0,0)$ is a saddle point.

6.18 The Hessian matrix of the function $f(x,y,z) = x^2 + y^2 + 3z^2 - xy + 2xz + yz$ is

$$H(f) = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 6 \end{pmatrix}$$

Since $D_1 = 2 > 0$, $D_2 = 3 > 0$, $D_3 = 2(-5) - 1(4) + 6D_2 = 4 > 0$, we see that $H(f)$ is positive definite, and that the unique stationary point is a local minimum point.

6.19 The partial derivatives of the function $f(x,y) = x^3 - 3xy + y^3$ are

$$f'_x = 3x^2 - 3y, \quad f'_y = -3x + 3y^2$$

The stationary points are therefore given by $3x^2 - 3y = 0$ or $y = x^2$, and $-3x + 3y^2 = 0$ or $y^2 = x^4 = x$. This gives $x = 0$ or $x^3 = 1$, that is, $x = 1$. The stationary points are $(x,y) = (0,0), (1,1)$. The Hessian matrix of f is

$$H(f) = \begin{pmatrix} 6x & -3 \\ -3 & 9y \end{pmatrix} \quad \Rightarrow \quad H(f)(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}, \; H(f)(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 9 \end{pmatrix}$$

In the first case, $D_1 = 0$; $D_2 = -9 < 0$ so $(0,0)$ is a saddle point. In the second case, $D_1 = 6$, $D_2 = 45 > 0$, so $(1,1)$ is a local minimum point.

6.20 The partial derivatives of the function $f(x,y,z) = x^3 + 3xy + 3xz + y^3 + 3yz + z^3$ are

$$f'_x = 3x^2 + 3y + 3z, \quad f'_y = 3x + 3y^2 + 3z, \; f'_z = 3x + 3y + 3z^2$$

The stationary points are given by $x^2 + y + z = 0$, $x + y^2 + z = 0$ and $x + y + z^2 = 0$. The first equation gives $z = -x^2 - y$, and the second becomes $x + y^2 + (-x^2 - y) = 0$, or $x - y = x^2 - y^2 = (x-y)(x+y)$. This implies that $x - y = 0$ or that $x + y = 1$. We see that $x + y = 1$ implies that $1 + z^2 = 0$ from the third equation, and this is impossible, and we infer that $x - y = 0$, or $x = y$. Then $z = -x^2 - x$ from the computation above, and the last equation gives

$$x + y + z^2 = 2x + (-x^2 - x)^2 = x^4 + 2x^3 + x^2 + 2x = (x+2)(x^3 + x) = 0$$

Hence $x = 0$, $x = -2$ or $x^2 + 1 = 0$. The last equation has not solutions, se we get two stationary points $(x,y,z) = (0,0,0), (-2,-2,-2)$. The Hessian matrix of f at $(0,0,0)$ is

$$H(f) = \begin{pmatrix} 6x & 3 & 3 \\ 3 & 6y & 3 \\ 3 & 3 & 6z \end{pmatrix} \quad \Rightarrow \quad H(f)(0,0,0) = \begin{pmatrix} 0 & 3 & 3 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{pmatrix}$$

In this case, $D_1 = 0$; $D_2 = -9 < 0$, so $(0,0,0)$ is a saddle point. At $(-2,-2,-2)$, the Hessian is

$$H(f) = \begin{pmatrix} 6x & 3 & 3 \\ 3 & 6y & 3 \\ 3 & 3 & 6z \end{pmatrix} \quad \Rightarrow \quad H(f)(-2,-2,-2) = \begin{pmatrix} -12 & 3 & 3 \\ 3 & -12 & 3 \\ 3 & 3 & -12 \end{pmatrix}$$

In this case, $D_1 = -12$, $D_2 = 135 > 0$, $D_3 = -50 < 0$, so $(-2,-2,-2)$ is a local maximum point.

6.21 Midterm in GRA6035 on 24/09/2010, Problem 7

The function f is a sum of a linear function and a quadratic form with symmetric matrix

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since A has eigenvalues $\lambda = 1,2,4$, the quadratic form is positive definite and therefore convex (but not concave). Hence the correct answer is alternative **A**.

6.22 Mock Midterm in GRA6035 on 09/2010, Problem 7

The function f is a sum of a constant function and the quadratic form $-aQ(x_1,x_2)$. Since Q is positive definite, it is convex, and $-Q$ is concave. If $a \geq 0$, then $-aQ(x_1,x_2) = a(-Q(x_1,x_2))$ is concave. If $a \leq 0$, then $-a \geq 0$ and $-aQ(x_1,x_2)$ is convex. The correct answer is alternative **D**.

6.23 Midterm in GRA6035 24/05/2011, Problem 7

The function f is a sum of a linear function and a quadratic form with symmetric matrix

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Since A has eigenvalues $\lambda = -2 \pm \sqrt{2}$ and $\lambda = -1$, the quadratic form is negative definite and therefore concave (but not convex). Hence the correct answer is alternative **C**.

6.24 Final Exam in GRA6035 on 10/12/2010, Problem 1

1. We compute the partial derivatives $f'_x = (x^2+2x)e^x$, $f'_y = z$ and $f'_z = y - 3z^2$. The stationary points are given by the equations

$$(x^2+2x)e^x = 0, \quad z = 0, \quad y - 3z^2 = 0$$

and this gives $x = 0$ or $x = -2$ from the first equation and $y = 0$ and $z = 0$ from the last two. The stationary points are therefore $(x,y,z) = (\mathbf{0},\mathbf{0},\mathbf{0}), (-\mathbf{2},\mathbf{0},\mathbf{0})$.
2. We compute the second order partial derivatives of f and form the Hessian matrix

$$f'' = \begin{pmatrix} (x^2+4x+2)e^x & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -6z \end{pmatrix}$$

We see that the second order principal minor obtained from the last two rows and columns is

$$\begin{vmatrix} 0 & 1 \\ 1 & -6z \end{vmatrix} = -1 < 0$$

hence the Hessian is indefinite in all stationary points. Therefore, both stationary points are **saddle points**.

6.25 Mock Final Exam in GRA6035 on 12/2010, Problem 2

1. We write $f(x,y,z) = e^u$ with $u = xy + yz - xz$, and compute

$$f'_x = e^u(y-z),\ f'_y = e^u(x+z),\ f'_z = e^u(y-x)$$

The stationary points of f are therefore given by

$$y - z = 0,\ x + z = 0,\ y - x = 0$$

which gives $(x,y,z) = (0,0,0)$. This is the unique stationary points of f.

2. We write $f(x,y,z) = e^u$ with $u = ax + by + cz$, and compute that

$$g'_x = e^u \cdot a,\ g'_y = e^u \cdot b,\ g'_z = e^u \cdot c$$

and this gives Hessian matrix

$$H(g) = \begin{pmatrix} a^2e^u & abe^u & ace^u \\ abe^u & b^2e^u & bce^u \\ ace^u & bce^u & c^2e^u \end{pmatrix} = e^u \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix}$$

This gives principal minors $\Delta_1 = e^u a^2, e^u b^2, e^u c^2 \geq 0$, $\Delta_2 = 0,0,0$ and $\Delta_3 = 0$. Hence g is convex for all values of a,b,c, and g is concave if and only if $a = b = c = 0$.

6.26 Final Exam in GRA6035 on 30/05/2011, Problem 1

1. We compute the partial derivatives $f'_x = 5x^4 + y^2$, $f'_y = 2xy$, $f'_z = -w$ and $f'_w = -z$. The stationary points are given by

$$5x^4 + y^2 = 0, \quad 2xy = 0, \quad -w = 0, \quad -z = 0$$

and this gives $z = w = 0$ from the last two equations, and $x = y = 0$ from the first two. The stationary points are therefore $(x,y,z,w) = (\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0})$.

2. We compute the second order partial derivatives of f and form the Hessian matrix

$$f'' = \begin{pmatrix} 20x^3 & 2y & 0 & 0 \\ 2y & 2x & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

We see that the second order principal minor obtained from the last two rows and columns is

$$\begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} = -1 < 0$$

hence the Hessian is indefinite. Therefore, the stationary point is a **saddle point**.

Lecture 7
Constrained Optimization and First Order Conditions

7.1 Main concepts

A *constrained optimization problem* is called a *Lagrange problem* if it has equality constraints, and a *Kuhn-Tucker problem* if it has closed inequality constraints. A Lagrange problem can be written in the standard form

$$\max / \min f(\mathbf{x}) \quad \text{when} \begin{cases} g_1(\mathbf{x}) = a_1 \\ g_2(\mathbf{x}) = a_2 \\ \vdots \\ g_m(\mathbf{x}) = a_m \end{cases}$$

where $f, g_1, \ldots, g_m$ are functions in n variables and $a_1, \ldots, a_m$ are given numbers, and a Kuhn-Tucker problem can be written in standard form as

$$\max f(\mathbf{x}) \quad \text{when} \begin{cases} g_1(\mathbf{x}) \leq a_1 \\ g_2(\mathbf{x}) \leq a_2 \\ \vdots \\ g_m(\mathbf{x}) \leq a_m \end{cases}$$

Given any Lagrange or Kuhn-Tucker problem in standard form, we can construct the *Lagrangian* function

$$\mathscr{L}(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) = f(\mathbf{x}) - \lambda_1 g_1(\mathbf{x}) - \lambda_2 g_2(\mathbf{x}) - \cdots - \lambda_m g_m(\mathbf{x})$$

in the $n+m$ variables $x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m$. The variables $\lambda_1, \ldots, \lambda_m$ are called the *Lagrange multipliers*. The *first order conditions* (FOC) of the Lagrange or Kuhn-Tucker problem are the conditions

$$\mathscr{L}'_{x_1} = \mathscr{L}'_{x_2} = \cdots = \mathscr{L}'_{x_n} = 0$$

Proposition 7.1. *If the Lagrange problem has a solution* $\mathbf{x}^*$ *and this solution satisfy the NDCQ condition*

$$\mathrm{rk}\begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \ddots & \ddots & \vdots & \ddots \\ \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_n} \end{pmatrix} = m$$

at $\mathbf{x}^*$*, then there are unique Lagrange multipliers* $\lambda_1, \dots, \lambda_m$ *such that* $(\mathbf{x}^*, \lambda_1, \dots, \lambda_m)$ *satisfy the FOC's and the constraints.*

In a Kuhn-Tucker problem, there are additional conditions called *complementary slackness conditions* (CSC), and they have the form

$$\lambda_i \geq 0 \quad \text{and} \quad \lambda_i(g_i(\mathbf{x}) - a_i) = 0$$

for $i = 1, 2, \dots m$. We say that the constraint $g_i(\mathbf{x}) \leq a_i$ is *binding* at $\mathbf{x}$ if $g_i(\mathbf{x}) = a_i$, and *non-binding* if $g_i(\mathbf{x}) < a_i$, and the CSC's say that $\lambda_1, \lambda_2, \dots, \lambda_m \geq 0$ and that $\lambda_i = 0$ for each constraint that is non-binding.

Proposition 7.2. *If the Kuhn-Tucker problem has a solution* $\mathbf{x}^*$ *and this solution satisfy the NDCQ condition* $\mathrm{rk} A = k$*, where* A *is the submatrix of*

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \ddots & \ddots & \vdots & \ddots \\ \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_n} \end{pmatrix}$$

consisting of the k *rows corresponding to binding constraints at* $\mathbf{x}^*$*, then there are unique Lagrange multipliers* $\lambda_1, \dots, \lambda_m$ *such that* $(\mathbf{x}^*, \lambda_1, \dots, \lambda_m)$ *satisfy the FOC's, the CSC's and the constraints.*

The candidates for max/min in a Lagrange problem are the solutions of the *Lagrange conditions* (FOC's and constraints) and the points where the NDCQ fails. The candidates for max in a Kuhn-Tucker problem are the solutions of the *Kuhn-Tucker conditions* (FOC's, CSC's and constraints) and the points where NDCQ fails.

A subset D of $\mathbb{R}^n$ is called *closed* if it contains all its boundary points, and *open* if it does not contain any of its boundary points. A boundary point of D is a point such that any neighbourhood around it contains both points in D and points outside D. The set of admissible point in a Lagrange or Kuhn-Tucker problem is closed.

The set D is called *bounded* if there are numbers $a_1 \leq b_1$, $a_2 \leq b_2, \dots, a_n \leq b_n$ (which are assumed to be finite) such that all points $(x_1, x_2, \dots, x_n)$ in D satisfies

$$a_1 \leq x_1 \leq b_1, \quad a_2 \leq x_2 \leq b_2, \quad \dots, \quad a_n \leq x_n \leq b_n$$

In other words, D is bounded if and only if D is contained in a finite n-dimensional box $K = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$. If D is both closed and bounded, it is called a *compact* set.

Theorem 7.1 (Weierstrass' Extreme Value Theorem). *If f is a continuous function defined on a compact set, then f has a maximum and a minimum value.*

7.2 Problems

7.1. Sketch each set and determine if it is open, closed, bounded or convex:

1. $\{(x,y) : x^2 + y^2 < 2\}$
2. $\{(x,y) : x^2 + y^2 > 8\}$
3. $\{(x,y) : xy \leq 1\}$
4. $\{(x,y) : x \geq 0, y \geq 0\}$
5. $\{(x,y) : x \geq 0, y \geq 0, xy \geq 1\}$
6. $\{(x,y) : \sqrt{x} + \sqrt{y} \leq 2\}$

7.2. Use the Lagrange conditions to find candidates for solution of the Lagrange problem

$$\max f(x,y) = xy \text{ subject to } x + 4y = 16$$

7.3. Use the Lagrange conditions to find candidates for solution of the Lagrange problem

$$\max f(x,y) = x^2 y \text{ subject to } 2x^2 + y^2 = 3$$

7.4. Use the Lagrange conditions to find candidates for solution of the Lagrange problem

$$\max f(x,y,z) = xyz \text{ subject to } \begin{cases} x^2 + y^2 = 1 \\ x + z = 1 \end{cases}$$

7.5. Use the Lagrange conditions to find candidates for solution of the Lagrange problem

$$\max / \min f(x,y,z) = x + y + z^2 \text{ subject to } \begin{cases} x^2 + y^2 + z^2 = 1 \\ y = 0 \end{cases}$$

7.6. Use the Lagrange conditions to find candidates for solution of the Lagrange problem

$$\max f(x,y,z) = xz + yz \text{ subject to } \begin{cases} y^2 + z^2 = 1 \\ xz = 3 \end{cases}$$

7.7. Use the Lagrange conditions to find candidates for solution of the Lagrange problem

$$\max f(x,y,z) = x + 4y + z \text{ subject to } \begin{cases} x^2 + y^2 + z^2 = 216 \\ x + 2y + 3z = 0 \end{cases}$$

7.8. Use the Kuhn-Tucker conditions to find candidates for solution of the Kuhn-Tucker problem

$$\max f(x,y) = xy \text{ subject to } x^2 + y^2 \leq 1$$

7.9. Use the Kuhn-Tucker conditions to find candidates for solution of the Kuhn-Tucker problem

$$\max f(x,y,z) = xyz \text{ subject to } \begin{cases} x + y + z \leq 1 \\ x \geq 0 \\ y \geq 0 \\ z \geq 0 \end{cases}$$

7.10. Use the Lagrange conditions to find candidates for solution of the Lagrange problem

$$\max f(x,y,z) = e^x + y + z \text{ subject to } \begin{cases} x^2 + y^2 + z^2 = 1 \\ x + y + z = 1 \end{cases}$$

7.11. Final Exam in GRA6035 on 06/02/2012, Problem 4a
We consider the optimization problem

$$\min\ x^2 + y^2 + z^2 \text{ subject to } 2x^2 + 6y^2 + 3z^2 \geq 36$$

Write down the first order conditions and the complementary slackness conditions for the optimization problem. Find all admissible points that satisfy the first order conditions and the complementary slackness conditions, and write down the value of λ for each of these points.

7.3 Solutions

7.1 We see that the sets have the following types:

1. The set $\{(x,y) : x^2 + y^2 < 2\}$ is convex, open and bounded, but not closed.
2. The set $\{(x,y) : x^2 + y^2 > 8\}$ is open, but not closed, bounded or convex.
3. The set $\{(x,y) : xy \leq 1\}$ is closed, but not open, bounded or convex.
4. The set $\{(x,y) : x \geq 0, y \geq 0\}$ is closed and convex, but not open or bounded.
5. The set $\{(x,y) : x \geq 0, y \geq 0, xy \geq 1\}$ is closed and convex, but not open or bounded.
6. The set $\{(x,y) : \sqrt{x} + \sqrt{y} \leq 2\}$ is closed and bounded, but not open or convex.

7.2 The Lagrangian is given by $L = xy - \lambda(x + 4y)$, and we have FOC's

$$L'_x = y - \lambda = 0 \quad L'_y = x - 4\lambda = 0$$

When we solve the FOC's, we get $y = \lambda$ and $x = 4\lambda$, and when we substitute this into the constraint $x + 4y = 16$, we get $8\lambda = 16$, or $\lambda = 2$. This gives the candidate $(x, y; \lambda) = (8, 2; 2)$ for max with $f(8, 2) = 16$.

7.3 The Lagrangian is given by $L = x^2y - \lambda(2x^2 + y^2)$, and we have FOC's

$$L'_x = 2xy - \lambda \cdot 4x = 0 \quad L'_y = x^2 - \lambda \cdot 2y = 0$$

We solve the FOC's, and get $x(2y - 4\lambda) = 0$ from the first equation, and this gives $x = 0$ or $y = 2\lambda$. When $x = 0$, the second equation give $\lambda = 0$ or $y = 0$, and the constraint $2x^2 + y^2 = 3$ gives $y = \pm\sqrt{3}$, and then $\lambda = 0$. When $y = 2\lambda$, then the second equation gives $x^2 = 4\lambda^2$, or $x = \pm 2\lambda$, and the constraint gives $12\lambda^2 = 3$, or $\lambda = \pm 1/2$. We have the candidates

$$(x, y; \lambda) = (0, \pm\sqrt{3}; 0),\ (\pm 1, 1, 1/2),\ (\pm 1, -1, -1/2)$$

The points $(x, y) = (\pm 1, 1)$ give $f(\pm 1, 1) = 1$ and is the best candidate for max.

7.4 The Lagrangian is given by $L = xyz - \lambda_1(x^2 + y^2) - \lambda_2(x + z)$, and we have FOC's

$$L'_x = yz - \lambda_1 \cdot 2x - \lambda_2 = 0 \quad L'_y = xz - \lambda_1 \cdot 2y = 0 \quad L'_z = xy - \lambda_2 = 0$$

The last two equations give $\lambda_2 = xy$ and $2\lambda_1 = xz/y$. When we substitute this in the first equation, and multiply it by y, we get

$$y(yz - 2x\lambda_1 - \lambda_2) = y^2z - x^2z - xy^2 = 0$$

From the constraints, we have $y^2 = 1 - x^2$ and $z = 1 - x$, and when we substitute this into the equation above we get

$$(1 - x^2)(1 - x) - x^2(1 - x) - x(1 - x^2) = 0$$

We see that $1 - x$ is a common factor, so this gives

$$(1 - x)(1 - x^2 - x^2 - x(1 + x)) = (1 - x)(-3x^2 - x + 1) = 0$$

The solutions are therefore $x = 1$ and $x = (1 \pm \sqrt{13})/6$. We get candidate points

$$\begin{aligned}
&(x, y, z) = (1, 0, 0), \\
&(x, y, z) \approx (0.4343, \pm 0.9008, 0.5657), \\
&(x, y, z) \approx (-0.7676, \pm 0.6409, 1.7676)
\end{aligned}$$

Since we have divided by y, we also have to check if there are any solutions with $y = 0$: This gives either $x = 1$ and $z = 0$ or $x = -1$ and $z = 2$ from the constraints, and $\lambda_1 = \lambda_2 = 0$ from the first and last FOC. The second FOC is the satisfied if $z = 0$. The only solution with $y = 0$ is therefore $x = 1, y = 0, z = 0$, which we already obtained above. The best candidate for max is $(x,y,z) \approx (-0.7676, -0.6409, 1.7676)$.

7.5 We use that $y = 0$ and reduce this problem to the Lagrange problem

$$\max / \min f(x,z) = x + z^2 \text{ subject to } x^2 + z^2 = 1$$

The Lagrangian is given by $L = x + z^2 - \lambda(x^2 + z^2)$, and we have FOC's

$$L'_x = 1 - \lambda \cdot 2x = 0 \quad L'_z = 2z - \lambda \cdot 2z = 0$$

We solve the FOC's, and the second equation gives $2z(1-\lambda) = 0$, and therefore that $z = 0$ or $\lambda = 1$. If $z = 0$, then the constraint gives $x = \pm 1$ and $\lambda = \pm 1/2$ from the first FOC. If $\lambda = 1$, then the first FOC gives $x = 1/2$, and the constraint gives $z = \pm\sqrt{3}/2$. Since all points have $y = 0$, $(x,y,z) = (-1,0,0)$ is the best candidate for min (with function value $f = -1$), and $(x,y,z) = (1/2, 0, \pm\sqrt{3}/2)$ are the best candidate for max (with function value $f = 5/4$).

7.6 We use that $xz = 3$ and reduce this problem to the Lagrange problem

$$\max f(y,z) = 3 + yz \text{ subject to } y^2 + z^2 = 1$$

The Lagrangian is given by $L = 3 + yz - \lambda(y^2 + z^2)$, and we have FOC's

$$L'_y = z - \lambda \cdot 2y = 0 \quad L'_z = y - \lambda \cdot 2z = 0$$

We solve the FOC's, and get $z = 2y\lambda$ and $y - 4\lambda^2 y = 0$. This gives $y = 0$ or $4\lambda^2 = 1$. If $y = 0$ then $z = 0$ and this point is not admissible since $y^2 + z^2 \neq 1$. In the second case, $\lambda = \pm 1/2$. If $\lambda = 1/2$, then $y = z = \pm 1/\sqrt{2}$ and $x = 3/z$, and if $\lambda = -1/2$ then $y = -z = \pm 1/\sqrt{2}$ and $x = 3/z$. The points $(x,y,z) = (3\sqrt{2}, 1/\sqrt{2}, 1/\sqrt{2})$ and $(x,y,z) = (-3\sqrt{2}, -1/\sqrt{2}, -1/\sqrt{2})$ are the best candidates for max (with function value $f = 7/2$).

7.7 We consider the Lagrangian

$$\mathscr{L}(x,y,z,\lambda_1,\lambda_2) = x + 4y + z - \lambda_1(x^2 + y^2 + z^2) - \lambda_2(x + 2y + 3z)$$

and solve the first order conditions

$$\begin{aligned}
\mathscr{L}'_x &= 1 - \lambda_1 \cdot 2x - \lambda_2 = 0 \\
\mathscr{L}'_y &= 4 - \lambda_1 \cdot 2y - \lambda_2 \cdot 2 = 0 \\
\mathscr{L}'_z &= 1 - \lambda_1 \cdot 2z - \lambda_2 \cdot 3 = 0
\end{aligned}$$

together with $x^2 + y^2 + z^2 = 216$ and $x + 2y + 3z = 0$. From the first order conditions, we get

$$2x\lambda_1 = 1 - \lambda_2,\ 2y\lambda_1 = 4 - 2\lambda_2,\ 2z\lambda_1 = 1 - 3\lambda_2$$

We see from these equations that we cannot have $\lambda_1 = 0$, and multiply the last constraint with $2\lambda_1$. We get

$$2\lambda_1(x+2y+3z) = 0 \quad \Rightarrow \quad (1-\lambda_2)+2(4-2\lambda_2)+3(1-3\lambda_2) = 0$$

This gives $12-14\lambda_2 = 0$, or $\lambda_2 = 12/14 = 6/7$. We use this and solve for x, y, z, and get

$$x = \frac{1}{14\lambda_1},\ y = \frac{8}{7\lambda_1},\ z = -\frac{11}{14\lambda_1}$$

Then we substitute this in the first constraint, and get

$$\left(\frac{1}{14\lambda_1}\right)^2(1+16^2+(-11)^2) = 216 \quad \Rightarrow \quad 216\cdot 14^2\lambda_1^2 = 378$$

This implies that $\lambda_1 = \pm\frac{\sqrt{7}}{28}$, and we have two solutions to the first order equations and constraints. Therefore, the candidates for solutions are

$$(x^*, y^*, z^*) = (\pm\frac{2}{7}\sqrt{7}, \pm\frac{32}{7}\sqrt{7}, \mp\frac{22}{7}\sqrt{7})$$

corresponding to $\lambda_1 = \pm\frac{\sqrt{7}}{28}$ and $\lambda_2 = 6/7$.

7.8 The Lagrangian is $\mathscr{L}(x,y,\lambda) = xy - \lambda(x^2+y^2)$, and we solve the first order conditions

$$\mathscr{L}'_x = y - \lambda\cdot 2x = 0$$
$$\mathscr{L}'_y = x - \lambda\cdot 2y = 0$$

together with the contraint $x^2+y^2 \leq 1$ and the CSC $\lambda \geq 0$ and $\lambda(x^2+y^2-1) = 0$. From the first order condition, we get $y = 2\lambda x$ and

$$x - 2\lambda\cdot 2\lambda x = x(1-4\lambda^2) = 1$$

which gives $x = 0$ or $\lambda = \pm 1/2$. If $x = 0$, then $y = 0$ by the FOC's, and $x^2+y^2 = 0 < 1$ so $\lambda = 0$ by the CSC. We get the candidate $(x,y) = (0,0)$ with $\lambda = 0$ and $f = 0$. If $\lambda = \pm 1/2$, then we must have $\lambda = 1/2$ by the CSC, and $x = y$ by the FOC's. Moreover, $x^2+y^2 = 1$ must be binding by the CSC, so $(x,y) = (1/\sqrt{2}, 1/\sqrt{2})$ or $(x,y) = (-1/\sqrt{2}, -1/\sqrt{2})$. At both these points, $f = 1/2$, so these two points are the best candidates for max.

7.9 We first rewrite the Kuhn-Tucker problem to standard form, and get

$$\max f(x,y,z) = xyz \text{ subject to } \begin{cases} x+y+z \leq 1 \\ -x \leq 0 \\ -y \leq 0 \\ -z \leq 0 \end{cases}$$

The Lagrangian is $\mathscr{L}(x,y,z,\lambda,\mu_1,\mu_2,\mu_3) = xyz - \lambda(x+y+z) + \mu_1 x + \mu_2 y + \mu_3 z$ (where we call the multipliers $\lambda,\mu_1,\mu_2,\mu_3$ instead of $\lambda_1,\dots,\lambda_4$). We solve the first order conditions

$$\begin{aligned}
\mathscr{L}'_x &= yz - \lambda + \mu_1 = 0\\
\mathscr{L}'_y &= xz - \lambda + \mu_2 = 0\\
\mathscr{L}'_z &= xy - \lambda + \mu_3 = 0
\end{aligned}$$

together with the contraints $x+y+z \le 1$ and $x,y,z \ge 0$ and the CSC's $\lambda \ge 0$, $\lambda(x+y+z-1) = 0$, $\mu_1,\mu_2,\mu_3 \ge 0$ and $\mu_1 x = \mu_2 y = \mu_3 z = 0$. From the first order condition, we get

$$\lambda = yz + \mu_1 = xz + \mu_2 = xy + \mu_3$$

We consider the two cases $\lambda = 0$ and $\lambda > 0$. If $\lambda = 0$, then 0 is the sum of $yz \ge 0$ and $\mu_1 \ge 0$, and therefore $yz = \mu_1 = 0$. Similarly, $xy = yz = 0$ and $\mu_2 = \mu_3 = 0$. Since $xy = xz = yz = 0$, two of the variables must be zero and the third must be in the interval $[0,1]$ because of the constraints. We therefore get a lot of candidates of this type when $\lambda = 0$, and all of these candidates have $f = 0$. If $\lambda > 0$, then $x+y+z = 1$ by the CSC. Let us for a minute assume one of the variables are zero, lets say $x = 0$. Then $\lambda = \mu_2 = \mu_3 > 0$, and this implies that $y = z = 0$ by the CSC's. Since $x = 0$ implies that $y = z = 0$, which contradicts $x+y+z = 1$, we conclude that $x \ne 0$. Similar arguments show that $y \ne 0$ and $z \ne 0$. This means that $\mu_1 = \mu_2 = mu_3 = 0$, and therefore that $\lambda = xy = xz = yz$. Since $xy = xz$ and $x \ne 0$, we have $y = z$. A simiar argument shows that $x = y$. Therefore $x = y = z = 1/3$ by the first constraint, and $\lambda = 1/9$. This candidate point has $f = 1/27$, and is the best candidate for max.

7.10 The Lagrangian is $\mathscr{L}(x,y,z,\lambda) = e^x + y + z - \lambda_1(x^2+y^2+z^2) - \lambda_2(x+y+z)$, and we solve the first order conditions

$$\begin{aligned}
\mathscr{L}'_x &= e^x - \lambda_1 \cdot 2x - \lambda_2 = 0\\
\mathscr{L}'_y &= 1 - \lambda_1 \cdot 2y - \lambda_2 = 0\\
\mathscr{L}'_z &= 1 - \lambda_1 \cdot 2z - \lambda_2 = 0
\end{aligned}$$

together with the contraints $x^2+y^2+z^2 = 1$ and $x+y+z = 1$. From the last two first order conditions, we get

$$\lambda_2 = 1 - 2y\lambda_1 = 1 - 2z\lambda_1$$

This means that either $\lambda_1 = 0$ or $y = z$. We first consider the case with $\lambda_1 = 0$, which implies that $\lambda_2 = 1$ and that $x = 0$ from the first of the first order conditions. The constraints give

$$y + z = 1, \quad y^2 + z^2 = 1$$

since $x = 0$, and inserting $y = 1 - z$ in the second equation gives $(1-z)^2 + z^2 = 1$ or $2z^2 - 2z = 0$. This gives $z = 0$ or $z = 1$. We therefore find two solutions with $\lambda_1 = 0$:

$$(x, y, z; \lambda_1, \lambda_2) = (0, 1, 0; 0, 1),\ (0, 0, 1; 0, 1)$$

Both points have $f(x, y, z) = e^0 + 1 = 2$. Secondly, we consider the case with $\lambda_1 \neq 0$, so that $y = z$. Then the constraints are given by

$$x + 2y = 1, \quad x^2 + 2y^2 = 1$$

Inserting $x = 1 - 2y$ in the second equation gives $(1-2y)^2 + 2y^2 = 1$ or $6y^2 - 4y = 0$. This gives $y = 0$ or $y = 2/3$. For $y = z = 0$, we get $x = 1$, $\lambda_2 = 1$ and $e - 2\lambda_1 = 1$, which gives the solution

$$(x, y, z; \lambda_1, \lambda_2) = (1, 0, 0; \frac{e-1}{2}, 1)$$

with $f(x, y, z) = e^1 = e \simeq 2.72$. For $y = z = 2/3$, we get $x = -1/3$, $1 - 4\lambda_1/3 = \lambda_2$ and $e^{-1/3} + 2\lambda_1/3 = \lambda_2$. We solve the last two equations for λ_1, λ_2 and find the solution

$$(x, y, z; \lambda_1, \lambda_2) = (-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}; \frac{1 - e^{-1/3}}{2}, \frac{1 + 2e^{-1/3}}{3})$$

with $f(x, y, z) = e^{-1/3} + 4/3 \simeq 2.05$. The point $(x^*, y^*, z^*) = (1, 0, 0)$ is the best candidate for max.

7.11 Final Exam in GRA6035 on 06/02/2012, Problem 4a

We rewrite the optimization problem in standard form as

$$\max\ -(x^2 + y^2 + z^2) \text{ subject to } -2x^2 - 6y^2 - 3z^2 \leq -36$$

The Lagrangian for this problem is given by $\mathscr{L} = -(x^2 + y^2 + z^2) - \lambda(-2x^2 - 6y^2 - 3z^2)$, and the first order conditions are

$$\begin{aligned} \mathscr{L}'_x &= -2x + 4x\lambda = 0 \\ \mathscr{L}'_y &= -2y + 12y\lambda = 0 \\ \mathscr{L}'_z &= -2z + 6z\lambda = 0 \end{aligned}$$

We solve the first order conditions, and get $x = 0$ or $\lambda = \frac{1}{2}$ from the first equation, $y = 0$ or $\lambda = \frac{1}{6}$ from the second, and $z = 0$ or $\lambda = \frac{1}{3}$ from the third. The constraint is $-2x^2 - 6y^2 - 3z^2 \leq -36$, and the complementary slackness conditions are that $\lambda \geq 0$, and moreover that $\lambda = 0$ if the constraint is not binding (that is, if $-2x^2 - 6y^2 - 3z^2 < -36$). We shall find all admissible points that satisfy the first order condition and the complementary slackness condition. In the case where the constraint is not binding (that is, $-2x^2 - 6y^2 - 3z^2 < -36$), we have $\lambda = 0$ and therefore $x = y = z = 0$ from the first order conditions. This point does not satisfy $-2x^2 - 6y^2 - 3z^2 < -36$, and it is therefore not a solution. In the case where the constraint is binding ($-2x^2 -$

$6y^2 - 3z^2 = -36$), we have that either $x = y = z = 0$, or that at least one of these variables are non-zero. In the first case, $x = y = z = 0$ does not satisfy $-2x^2 - 6y^2 - 3z^2 = -36$, so this is not a solution. In the second case, we see that exactly one of the variables is non-zero (otherwise λ would take two different values), so we have the following possibilities:

$$\begin{cases} x = \pm\sqrt{18}, y = z = 0, \lambda = \frac{1}{2} \\ x = 0, y = \pm\sqrt{6}, z = 0, \lambda = \frac{1}{6} \\ x = y = 0, z = \pm\sqrt{12}, \lambda = \frac{1}{3} \end{cases}$$

These six points are the admissible points that satisfies the first order conditions and the complementary slackness conditions.

Lecture 8
Constrained Optimization and Second Order Conditions

8.1 Main concepts

A Lagrange problem can be written in the standard form as

$$\max / \min f(\mathbf{x}) \quad \text{when} \begin{cases} g_1(\mathbf{x}) = a_1 \\ g_2(\mathbf{x}) = a_2 \\ \vdots \\ g_m(\mathbf{x}) = a_m \end{cases}$$

where $f, g_1, \dots, g_m$ are functions in n variables and $a_1, \dots, a_m$ are given numbers. It can be solved in the following way:

1. Write down all Lagrange conditions (FOC + C) and solve them, to obtain a list of candidates for max/min. Compute the value $f(\mathbf{x})$ of each candidate on the list.
2. Check if there are any admissible points where NDCQ fails. If there are any such points, these are also candidates for max/min, and we compute $f(\mathbf{x})$ for these points as well.
3. There are no other candidates for max/min than the ones found above, so by comparing values we can find the best candidate for max/min (if there are any candidates).
4. If there is a best candidate for max/min, we try to determine if this point is in fact the max/min using either the Second order condition or the Extreme value theorem.

Proposition 8.1 (Second Order Condition). *Let* $(\mathbf{x}^*; \lambda^*)$ *be a candidate for max or min that satisfies the Lagrange conditions, and consider the function*

$$\mathbf{x} \mapsto \mathscr{L}(\mathbf{x}, \lambda^*)$$

If this function is concave as a function in $\mathbf{x}$*, then* $\mathbf{x}^*$ *is a max, and if it is convex as a function in* $\mathbf{x}$ *then* $\mathbf{x}^*$ *is a min.*

If we use the Second order condition, it is not necessary to check NDCQ. If we use the Extreme value theorem, we must check that the set D of points satisfying all constraints is bounded; if this is the case, there is a max and a min and therefore the best candidate for max/min is in fact max/min.

A Kuhn-Tucker problem can be written in standard form as

$$\max f(\mathbf{x}) \quad \text{when} \begin{cases} g_1(\mathbf{x}) \leq a_1 \\ g_2(\mathbf{x}) \leq a_2 \\ \vdots \\ g_m(\mathbf{x}) \leq a_m \end{cases}$$

It can be solved in the following way:

1. Write down all Kuhn-Tucker conditions (FOC + C + CSC) and solve them, to obtain a list of candidates for max. Compute the value $f(\mathbf{x})$ of each candidate on the list.
2. Check if there are any admissible points where NDCQ fails. If there are any such points, these are also candidates for max, and we compute $f(\mathbf{x})$ for these points as well.
3. There are no other candidates for max, so by comparing values we can find the best candidate for max (if there are any candidates).
4. If there is a best candidate for max, we try to determine if this point is in fact the max using either the Second order condition or the Extreme value theorem.

Proposition 8.2 (Second Order Condition). *Let* $(\mathbf{x}^*;\lambda^*)$ *be a candidate for max that satisfies the Kuhn-Tucker conditions, and consider the function*

$$\mathbf{x} \mapsto \mathscr{L}(\mathbf{x},\lambda^*)$$

If this function is concave as a function in $\mathbf{x}$*, then* $\mathbf{x}^*$ *is a max.*

If we use the Second order condition, it is not necessary to check NDCQ. If we use the Extreme value theorem, we must check that the set D of points satisfying all constraints is bounded; if this is the case, there is a max and therefore the best candidate for max is in fact max.

8.2 Problems

8.1. Solve the Lagrange problem

$$\max f(x,y) = xy \text{ subject to } x+4y = 16$$

8.2. Solve the Lagrange problem

$$\max f(x,y) = x^2y \text{ subject to } 2x^2+y^2 = 3$$

8.3. Solve the Lagrange problem

$$\max f(x,y,z) = xyz \text{ subject to } \begin{cases} x^2+y^2=1 \\ x+z=1 \end{cases}$$

8.4. Solve the Lagrange problem

$$\max / \min f(x,y,z) = x+y+z^2 \text{ subject to } \begin{cases} x^2+y^2+z^2=1 \\ y=0 \end{cases}$$

8.5. Solve the Lagrange problem

$$\max f(x,y,z) = xz+yz \text{ subject to } \begin{cases} y^2+z^2=1 \\ xz=3 \end{cases}$$

8.6. Solve the Lagrange problem

$$\max f(x,y,z) = x+4y+z \text{ subject to } \begin{cases} x^2+y^2+z^2=216 \\ x+2y+3z=0 \end{cases}$$

8.7. Solve the Kuhn-Tucker problem

$$\max f(x,y) = xy \text{ subject to } x^2+y^2 \leq 1$$

8.8. Solve the Kuhn-Tucker problem

$$\max f(x,y,z) = xyz \text{ subject to } \begin{cases} x+y+z \leq 1 \\ x \geq 0 \\ y \geq 0 \\ z \geq 0 \end{cases}$$

8.9. Solve the Lagrange problem

$$\max f(x,y,z) = e^x+y+z \text{ subject to } \begin{cases} x^2+y^2+z^2=1 \\ x+y+z=1 \end{cases}$$

8.10. Final Exam in GRA6035 on 06/02/2012, Problem 4b
We consider the optimization problem

$$\min\ x^2+y^2+z^2 \text{ subject to } 2x^2+6y^2+3z^2 \geq 36$$

Solve the optimization problem and compute the minimum value. Give an argument that proves that your solution is a minimum.

8.11. Final Exam in GRA6035 on 30/05/2011, Problem 4
We consider the function $f(x,y) = xye^{x+y}$ defined on $D_f = \{(x,y) : (x+1)^2 + (y+1)^2 \leq 1\}$.

1. Compute the Hessian of f. Is f a convex function? Is f a concave function?
2. Find the maximum and minimum values of f.

8.12. Final Exam in GRA6035 on 06/06/2012, Problem 4
We consider the optimization problem

$$\max x^2yz \text{ subject to } x^2 + 2y^2 - 2z^2 \leq 32$$

1. Write down the first order conditions and the complementary slackness conditions for the maximum problem, and find all admissible points that satisfy these conditions.
2. Does the maximum problem have a solution?

8.13. Final Exam in GRA6035 on 13/12/2012, Problem 5
We consider the following optimization problem:

$$\max\ \ln(x^2y) - x - y \text{ subject to } \begin{cases} x + y \geq 4 \\ x \geq 1 \\ y \geq 1 \end{cases}$$

Sketch the set of admissible points, and solve the optimization problem.

8.3 Solutions

8.1 From Problem 7.2 it follows that the only solutions to the Lagrange conditions is $x = 8, y = 2$ with $\lambda = 2$. Moreover, NDCQ is satisfied at all points since

$$\text{rk}\begin{pmatrix} 1 & 4 \end{pmatrix} = 1$$

To see that $(8,2)$ is a max with $f(8,2) = 16$, we cannot use the SOC (second order condition) since the Lagrangian is not concave, or the EVT (extreme value theorem) since the domain is not bounded. We solve the constraint for x, to get $x = 16 - 4y$ (with y free), and substitute this into

$$f(x,y) = xy = (16-4y)y = 16y - 4y^2$$

So we can consider $f(y) = 16y - 4y^2$ as a function in one variable, with y arbitrary; this is an unconstrained problem. The stationary point is given by $f'(y) = 16 - 8y = 0$, and is at $y = 2$, and it is a max since $f''(y) = -2$ so that f is concave in y. Hence $(x,y) = (8,2)$ is a max with $f(8,2) = 16$.

8.2 From Problem 7.3 it follows that there are several solutions to the Lagrange conditions, but $(x,y) = (\pm 1, 1)$ with $\lambda = 1/2$ are the best candidates for max with $f = 1$. Moreover, NDCQ is satisfied at all admissible points since

$$\operatorname{rk}\begin{pmatrix} 4x & 2y \end{pmatrix} = 1$$

for all points except $(x,y) = (0,0)$, and this point does not satisfy the constraint $2x^2 + y^2 = 3$. Since $D : 2x^2 + y^2 = 3$ is bounded, there is a max by the EVT, and $(x,y) = (\pm 1, 1)$ with $f = 1$ are max points.

8.3 From Problem 7.4 it follows that there are several solutions to the Lagrange conditions, but $(x,y,z) \simeq (-0.7676, -0.6409, 1.7676)$ is the best candidates for max. Moreover, NDCQ is satisfied at all admissible points since

$$\operatorname{rk}\begin{pmatrix} 2x & 2y & 0 \\ 1 & 0 & 1 \end{pmatrix} = 2 \quad \Leftrightarrow \quad -2y = 2x = 2y = 0$$

for all points except $(x,y,z) = (0,0,z)$ (with z any value). The equations come from the fact that the rank is less than two if and only if all 2-minors are zero. The points $(0,0,z)$ do not satisfy the constraint $x^2 + y^2 = 1$. Since $D : x^2 + y^2 = 1$ and $x + z = 1$ is bounded (the first condition means that $-1 \leq x, y \leq 1$ and the second condition means that $0 \leq z \leq 2$ since $z = 1 - x$), there is a max by the EVT, and $(x,y,z) \simeq (-0.7676, -0.6409, 1.7676)$ is the max.

8.4 First transform the problem into $\max / \min f(x,z) = x + z^2$ when $x^2 + z^2 = 1$. The domain $D : x^2 + z^2 = 1$ is bounded, so there is a max/min by EVT. By Problem 7.5, there are several solutions to the Lagrange conditions, but the candidates $(x,z) = (1/2, \pm\sqrt{3}/2)$ are the best candidates for max with $f = 5/4$, and $(x,z) = (\pm 1, 0)$ are best candidates for min with $f = -1$. There are no admissible points where NDCQ fails, since

$$\operatorname{rk}\begin{pmatrix} 2x & 2z \end{pmatrix} = 1$$

for all points except $(x,z) = (0,0)$, which is not admissible. Therefore, the best candidates above are the max and min points.

8.5 First transform the problem into $\max f(y,z) = 3 + yz$ when $y^2 + z^2 = 1$. The domain $D : y^2 + z^2 = 1$ is bounded, so there is a max/min by EVT. By Problem 7.6, there are several solutions to the Lagrange conditions, but the candidates $(y,z) = \pm(1/\sqrt{2}, 1/\sqrt{2})$ are the best candidates for max with $f = 7/2$. There are no admissible points where NDCQ fails, since

$$\operatorname{rk}\begin{pmatrix} 2y & 2z \end{pmatrix} = 1$$

for all points except $(y,z) = (0,0)$, which is not admissible. Therefore, the best candidates above are the max points.

8.6 The domain D is bounded since the points in D satisfies $x^2 + y^2 + z^2 = 216$, so there is a max by EVT. By Problem 7.7, there are several solutions to the Lagrange conditions, but the candidate $(x,y,z) = (\frac{2}{7}\sqrt{7}, \frac{32}{7}\sqrt{7}, -\frac{22}{7}\sqrt{7})$ is the best candidate for max with $f = 108/7\sqrt{7}$. There are no admissible points where NDCQ fails, since

$$\text{rk}\begin{pmatrix} 2x & 2y & 2z \\ 1 & 2 & 3 \end{pmatrix} = 2$$

for all points except those points where the 2-minors $4x - 2y = 0$, $6y - 4z = 0$ and $6x - 2z = 0$, which gives $y = 2x$ and $z = 3x$ with x free. None of these points are admissible. Therefore, the best candidate above is the max point. We could also have used the SOC to conclude this.

8.7 From Problem 7.8 it follows that there are several solutions to the Kuhn-Tucker conditions, but the solutions

$$(x,y) = \pm(1/\sqrt{2}, 1/\sqrt{2})$$

with $\lambda = 1/2$ and $f = 1/2$ are the best candidates for max. The constraint set D is bounded, and there are no admissible points where NDCQ fails since

$$\text{rk}\begin{pmatrix} 2x & 2y \end{pmatrix} = 1$$

for all points except $(x,y) = (0,0)$, and this point is not admissible. Therefore, the best candidate above is the max.

8.8 From Problem 7.9 it follows that there are several solutions to the Kuhn-Tucker conditions, but the solution

$$(x,y,z) = (1/3, 1/3, 1/3)$$

with $\lambda = 1/9$ and $f = 1/27$ is the best candidate for max. The constraint set D is bounded since $0 \le x,y,z \le 1$ for all admissible points. We must check if there are admissible points where NDCQ fails. The matrix of partial derivatives of the g_i's are given by

$$J = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

It is not possible to have all four constraints binding at the same time (the set of admissible points form a solid pyramid). If three of the constraints are binding (one of the corners), NDCQ is satisfied since all 3-minors of J are non-zero. If two of the constraints are binding (one of the edges), NDCQ is satified since any submatrix of J consisting of two of its rows has a non-zero 2-minor. If only one constraint is binding (one side), NDCQ is satisfied since any row of J is non-zero. It follows that NDCQ is satisfied for all points, Therefore, the best candidate above is the max.

8.9 From Problem 7.10 it follows that there are several solutions to the Lagrange conditions, but the solution

$$(x,y,z) = (1,0,0)$$

with $\lambda_1 = (e-1)/2$, $\lambda_2 = 1$ and $f = e$ is the best candidate for max. The constraint set D is bounded since $-1 \leq x, y, z \leq 1$ for all admissible points. We must check if there are admissible points where NDCQ fails. The NDCQ condition

$$\text{rk}\begin{pmatrix} 2x & 2y & 2z \\ 1 & 1 & 1 \end{pmatrix} = 2$$

is satisfied for all points except where the 2-minors $2x-2y = 2x-2z = 2y-2z = 0$, or $x = y = z$. These are no admissible points with $x = y = z$, so the best candidate above is the max.

8.10 Final Exam in GRA6035 on 06/02/2012, Problem 4b
We rewrite the optimization problem in standard form as

$$\max\ -(x^2+y^2+z^2) \text{ subject to } -2x^2-6y^2-3z^2 \leq -36$$

By problem 7.10, the best candidate for max is the points $(x,y,z) = (0, \pm\sqrt{6}, 0)$ with $\lambda = 1/6$ and $f = -(x^2+y^2+z^2) = -6$. We consider the Lagrangian

$$\mathscr{L}(x,y,z,1/6) = -(x^2+y^2+z^2) - 1/6(-2x^2-6y^2-3z^2) = -\frac{2}{3}x^2 - \frac{1}{2}z^2$$

which is clearly concave in (x,y,z). Hence $(x,y,z) = (0, \pm\sqrt{6}, 0)$ are max points with max value $f = -6$ by the SOC (second order condition). This means that the points $(x,y,z) = (0, \pm\sqrt{6}, 0)$ are min points for the original optimization problem with minimum value 6.

8.11 Final Exam in GRA6035 on 30/05/2011, Problem 4

1. We compute the partial derivatives $f(x,y) = xye^{x+y}$, and find that

$$f'_x = ye^{x+y} + xye^{x+y} = (y+xy)e^{xy}, \quad f'_y = xe^{x+y} + xye^{x+y} = (x+xy)e^{xy}$$

   The first among the second order partial derivative is given by

$$f''_{xx} = ye^{x+y} + (y+xy)e^{x+y} = (2y+xy)e^{x+y} = (x+2)ye^{x+y}$$

   The other second order partial derivatives can be computed in a similar way, and we find that the Hessian is given by

$$H(f)(x,y) = \begin{pmatrix} (x+2)ye^{x+y} & (x+1)(y+1)e^{x+y} \\ (x+1)(y+1)e^{x+y} & x(y+2)e^{x+y} \end{pmatrix}$$

   We compute the principal minors of the Hessian, and we find that the first order principal minors are $\Delta_1 = (x+2)ye^{x+y}$ and $\Delta_1 = x(y+2)e^{x+y}$. The second order principal minor is

$$D_2 = (x+2)ye^{x+y} \cdot x(y+2)e^{x+y} - (x+1)^2(y+1)^2(e^{x+y})^2$$
$$= (1-(x+1)^2-(y+1)^2)e^{2(x+y)}$$

since the expressions in D_2 are $(x+2)xy(y+2) = x^2y^2+2xy^2+2x^2y+4xy$ and $(x+1)^2(y+1)^2 = (x^2+2x+1)(y^2+2y+1) = x^2y^2+2xy^2+2x^2y+x^2+4xy+y^2+2x+2y+1$, so that their difference is

$$(x^2y^2+2xy^2+2x^2y+4xy)-(x^2y^2+2xy^2+2x^2y+x^2+4xy+y^2+2x+2y+1)$$
$$= -x^2-2x-y^2-2y-1 = -(x^2+2x+1)-(y^2+2y+1)+1$$
$$= 1-(x+1)^2-(y+1)^2$$

Finally, we have to determine the signs of the principal minors when (x,y) is in the domain of definition D_f, given by $(x+1)^2+(y+1)^2 \leq 1$. We see that $D_2 \geq 0$. Moreover, we notice that D_f is the disc inside a circle with radius 1 and with center $(-1,-1)$. Therefore $-2 \leq x,y \leq 0$ for all points in D_f, and $\Delta_1 \leq 0$. It follows that f is a concave function on D_f. It is not convex since $\Delta_1 < 0$ for some points in D_f (in fact, all points except $(-1,0)$ and $(0,-1)$).

2. Since D_f is closed and bounded, f has maximum and minimum values on D_f. We compute the stationary points of f: We have

$$f'_x = (x+1)ye^{x+y} = 0, \quad f'_y = x(y+1)e^{x+y} = 0$$

and $(x,y) = (0,0)$ and $(x,y) = (-1,-1)$ are the solutions. Hence there is only one stationary point $(x,y) = (-1,-1)$ in D_f, and $f(-1,-1) = \mathbf{e^{-2}}$ is the maximum value of f since f is concave on D_f. The minimum value most occur for (x,y) on the boundary of D_f. We see that $f(x,y) \geq 0$ for all $(x,y) \in D_f$ while $f(-1,0) = f(0,-1) = 0$. Hence $\mathbf{f(-1,0) = f(0,-1) = 0}$ is the minimum value of f.

8.12 Final Exam in GRA6035 on 06/06/2012, Problem 4

1. The Lagrangian for this problem is given by $\mathscr{L} = x^2yz - \lambda(x^2+2y^2-2z^2)$, and the first order conditions are

$$\mathscr{L}'_x = 2xyz - 2x\lambda = 0$$
$$\mathscr{L}'_y = x^2z - 4y\lambda = 0$$
$$\mathscr{L}'_z = x^2y + 4z\lambda = 0$$

The complementary slackness conditions are given by $\lambda \geq 0$, and $\lambda = 0$ if $x^2+2y^2-2z^2 < 32$. Let us find all admissible points satisfying these conditions. We solve the first order conditions, and get $x=0$ or $\lambda = yz$ from the first equation. If $x=0$, then $y\lambda = z\lambda = 0$, so either $\lambda = 0$ or $\lambda \neq 0 \Rightarrow y = z = 0$. In the first case, the constraint gives $2y^2-2z^2 \leq 32 \Rightarrow y^2-z^2 \leq 16$. This gives the solution

$$\boxed{x = 0,\ y^2-z^2 \leq 16,\ \lambda = 0}$$

In the second case, $x = y = z = 0$, $\lambda \neq 0$. Since the constraint is not binding, this is not a solution. If $x \neq 0$, then $\lambda = yz$, and the last two first order conditions give

$$x^2 z - 4y \cdot yz = 0 \Rightarrow z(x^2 - 4y^2) = 0$$
$$x^2 y + 4z \cdot yz = 0 \Rightarrow y(x^2 + 4z^2) = 0$$

If $z = 0$, then $x^2 + 4z^2 \neq 0 \Rightarrow y = 0$ and $\lambda = yz = 0$ give a solution

$$\boxed{0 < x^2 \leq 32,\ y = z = 0,\ \lambda = 0}$$

If both $x \neq 0$ and $z \neq 0$, then $x^2 - 4y^2 = 0 \Rightarrow y \neq 0$, and $x^2 = 4y^2 = -4z^2$. This is not possible. Therefore, there are no more solutions.

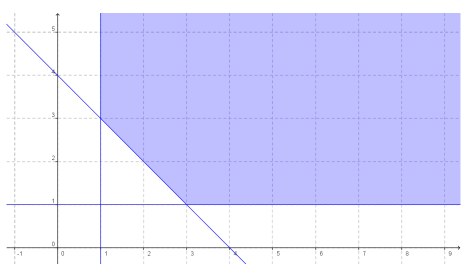
2. For any number a, we have that $x = \sqrt{32}, y = a, z = a$ is an admissible point for any value of a, since

$$x^2 + 2y^2 - 2z^2 = 32 + 2a^2 - 2a^2 = 32$$

The value of the function $f(x,y,z) = x^2yz$ at this point is $f(\sqrt{32}, a, a) = 32a^2$. When $a \to \infty$, we see that $f(\sqrt{32}, a, a) = 32a^2 \to \infty$, and this means that there is no maximum value. Hence the maximum problem has no solution.

8.13 Final Exam in GRA6035 on 13/12/2012, Problem 5

For the sketch, see the figure below. Since $\ln(ab) = \ln(a) + \ln(b)$, we can rewrite the



function as $f(x,y) = 2\ln x + \ln y - x - y$ and the constraints as $-x - y \leq -4, -x \leq -1, -y \leq -1$. We write the Lagrangian for this problem as

$$\begin{aligned}\mathscr{L} &= 2\ln x + \ln y - x - y - \lambda(-x - y) - \nu_1(-x) - \nu_2(-y) \\ &= 2\ln x + \ln y - x - y + \lambda(x + y) + \nu_1 x + \nu_2 y\end{aligned}$$

The Kuhn-Tucker conditions for this problem are the first order conditions

$$\mathscr{L}'_x = \frac{2}{x} - 1 + \lambda + \nu_1 = 0$$
$$\mathscr{L}'_y = \frac{1}{y} - 1 + \lambda + \nu_2 = 0$$

the constraints $x+y \geq 4$ and $x,y \geq 1$, and the complementary slackness conditions $\lambda, \nu_1, \nu_2 \geq 0$ and

$$\lambda(x+y-4) = 0, \quad \nu_1(x-1) = 0, \quad \nu_2(y-1) = 0$$

Let us find all solutions of the Kuhn-Tucker conditions: If $x = 1$, then $1+\lambda+\nu_1 = 0$ by the first FOC and this is not possible (since $\lambda, \nu_1 \geq 0$). So we must have $x > 1$ and $\nu_1 = 0$. If $y = 1$, then $\lambda + \nu_2 = 0$ by the second FOC, and this implies that $\lambda = \nu_2 = 0$ (since $\lambda, \nu_2 \geq 0$). Then the first FOC implies that $x = 2$, and this is not possible since $x+y \geq 4$. Hence we must also have $y > 1$ and $\nu_2 = 0$. Using the FOC's, we get

$$\lambda = 1 - \frac{2}{x} = 1 - \frac{1}{y}$$

which gives $2/x = 1/y$ or $x = 2y$ and $\lambda = 1 - 1/y > 0$ since $y > 1$. This implies that $x+y = 4$, which gives $3y = 4$ or $y = 4/3$, $x = 8/3$ and $\lambda = 1/4$. We conclude that there is exactly one solution of the Kuhn-Tucker conditions:

$$(x, y; \lambda, \nu_1, \nu_2) = (8/3, 4/3; 1/4, 0, 0)$$

The Lagrangian $\mathscr{L} = \mathscr{L}(x, y; 1/4, 0, 0) = 2\ln x + \ln y - x - y + (x+y)/4$ has Hessian

$$\mathscr{L}'' = \begin{pmatrix} -\frac{2}{x^2} & 0 \\ 0 & -\frac{1}{y^2} \end{pmatrix}$$

so $\mathscr{L}$ is a concave function, since $D_1 = -2/x^2 < 0$ and $D_2 = 2/(x^2y^2) > 0$ ($\mathscr{L}$ is only defined for $x, y \neq 0$). Therefore $(x,y) = (\mathbf{8/3}, \mathbf{4/3})$ is the maximum point.

Lecture 9
Envelope Theorems and Bordered Hessians

9.1 Main concepts

Let $f(\mathbf{x};a)$ be a function in n variables that depend on a parameter a, and consider the unconstrained optimization problem

$$\max / \min f(\mathbf{x};a)$$

for a given value of a.

Theorem 9.1 (Envelope Theorem, Unconstrained case). *Assume that the problem has a solution $\mathbf{x}^*(a)$, and consider the* optimal value function $f^*(a) = f(\mathbf{x}^*(a))$. *Then we have*

$$\frac{df^*(a)}{da} = \frac{\partial f}{\partial a}\bigg|_{\mathbf{x}=\mathbf{x}^*(a)}$$

Let $f(\mathbf{x};a)$ be a function in n variables that depend on a parameter a, and consider the Lagrange problem

$$\max / \min f(\mathbf{x};a) \quad \text{when} \begin{cases} g_1(\mathbf{x};a) = 0 \\ g_2(\mathbf{x};a) = 0 \\ \vdots \\ g_m(\mathbf{x};a) = 0 \end{cases}$$

or the Kuhn-Tucker problem

$$\max f(\mathbf{x};a) \quad \text{when} \begin{cases} g_1(\mathbf{x};a) \leq 0 \\ g_2(\mathbf{x};a) \leq 0 \\ \vdots \\ g_m(\mathbf{x};a) \leq 0 \end{cases}$$

for a given value of a, where the functions $g_1, g_2, \dots, g_m$ defining the constraints depend on the parameter a.

Theorem 9.2 (Envelope Theorem, Constrained case). *Assume that the problem has a solution* $\mathbf{x}^*(a)$*, and consider the* optimal value function $f^*(a) = f(\mathbf{x}^*(a))$. *Then we have*

$$\frac{df^*(a)}{da} = \frac{\partial \mathscr{L}}{\partial a}\Big|_{\mathbf{x}=\mathbf{x}^*(a);\lambda=\lambda^*(a)}$$

where $\mathscr{L}$ *is the Lagrangian of the problem and* $\lambda^*(a)$ *are the Lagrange multipliers such that* $(\mathbf{x}^*(a);\lambda^*(a))$ *satisfy FOC (and CSC in case of a Kuhn-Tucker problem).*

Consider a Lagrange or Kuhn-Tucker problem, and assume that $(\mathbf{x}^*;\lambda^*)$ is a solution of the Lagrange conditions (FOC + C) or the Kuhn-Tucker conditions (FOC + C + CSC). To find out if $\mathbf{x}^*$ is a local max or local min of the given optimization problem, we may use the $(m+n)\times(m+n)$ *bordered Hessian matrix*

$$B = \begin{pmatrix} 0 & J \\ J^T & H(\mathscr{L}) \end{pmatrix}\Big|_{\mathbf{x}=\mathbf{x}^*;\lambda=\lambda^*}$$

where J is the matrix of all partial derivatives of the functions $g_1, g_2, \dots, g_m$ defining the constraints (in the Lagrange case), or only those g_i's that are binding at $\mathbf{x}^*$ (in the Kuhn-Tucker case), and $H(\mathscr{L})$ is the Hessian matrix of the Lagrangian $\mathscr{L}$ as a function of $\mathbf{x}$ (that is, the matrix of the second order derivatives with respect to $x_1, x_2, \dots, x_n$):

1. Compute the $n-m$ last leading principal minors of B
2. If they have alternating signs, with the last sign equal to the sign of $(-1)^n$, then $\mathbf{x}^*$ is a local max.
3. If they have the same sign, equal to the sign of $(-1)^m$, then $\mathbf{x}^*$ is a local min.

9.2 Problems

9.1. Consider the unconstrained optimization problem

$$\max f(x,y;r) = -x^2 - xy - 2y^2 + 2rx + 2ry$$

where r is a parameter. Find the solution $\mathbf{x}^*(r) = (x^*(r), y^*(r))$ of the optimization problem, and verify the Envelope Theorem.

9.2. Consider the unconstrained optimization problem

$$\max f(x,y;r,s) = r^2x + 3s^2y - x^2 - 8y^2$$

where r,s are parameters. Find the solution $\mathbf{x}^*(r,s) = (x^*(r,s), y^*(r,s))$ of the optimization problem, and verify the Envelope Theorem.

9.3. Consider the Lagrange problem

$$\max f(x,y,z) = 100 - x^2 - y^2 - z^2 \text{ subject to } x + 2y + z = a$$

with parameter a. Find the solution $\mathbf{x}^*(a) = (x^*(a), y^*(a), z^*(a))$ of the Lagrange problem and let $\lambda^*(a)$ be the corresponding Lagrange multiplier. Show that

$$\lambda^*(a) = \frac{df^*(a)}{da}$$

where $f^*(a) = f(x^*(a), y^*(a), z^*(a), \lambda^*(a))$ is the optimal value function.

9.4. Solve the Lagrange problem

$$\max f(x,y,z) = x + 4y + z \text{ subject to } \begin{cases} x^2 + y^2 + z^2 = 216 \\ x + 2y + 3z = 0 \end{cases}$$

Use the Lagrange multiplier to estimate the new maximum value when the constraints are changed to $x^2 + y^2 + z^2 = 215$ and $x + 2y + 3z = 0.1$.

9.5. Consider the Lagrange problem

$$\max U(x,y) = \frac{1}{2}\ln(1+x) + \frac{1}{4}\ln(1+y) \quad \text{subject to} \quad 2x + 3y = m$$

with parameter $m \geq 4$.

1. Solve the optimization problem
2. Show that $U^*(m)$, the optimal value function of the optimization problem, satisfies $dU^*(m)/dm = \lambda$, where $\lambda = \lambda^*(m)$ is the Lagrange multiplier.

9.6. Determine the definiteness of the following constrained quadratic forms using bordered Hessians:

1. $Q(x,y) = x^2 + 2xy - y^2$ subject to $x + y = 0$
2. $Q(\mathbf{x}) = x_1^2 + x_2^2 - x_3^2 + 4x_1x_3 - 2x_1x_2$ subject to $x_1 + x_2 + x_3 = 0,\ x_1 + x_2 - x_3 = 0$

9.7. Determine the definiteness of the following constrained quadratic forms without using bordered Hessians:

1. $Q(x,y) = x^2 + 2xy - y^2$ subject to $x + y = 0$
2. $Q(x_1,x_2,x_3) = x_1^2 + x_2^2 - x_3^2 + 4x_1x_3 - 2x_1x_2$ subject to $x_1 + x_2 + x_3 = 0$ and $x_1 + x_2 - x_3 = 0$

9.8. Consider the Lagrange problem

$$\max x^2y^2z^2 \quad \text{subject to} \quad x^2 + y^2 + z^2 = 1$$

Find all solutions to the Lagrange conditions, and use the Bordered Hessian to determine which of the solutions are local maxima. What is the solution to the Lagrange problem?

9.9. Final Exam in GRA6035 on 13/12/2012, Problem 2
We consider the function f with parameter h, given by $f(x,y,z;h) = 12 - x^4 - hx^2 - 3y^2 + 6xz - 6z^2 + h^2$. The function f is defined for all points $(x,y,z) \in \mathbb{R}^3$.

1. Compute the Hessian matrix of f, and show that f is concave if and only if $h \geq H$ for a constant H. What is the value of H?
2. Find the global maximum point $(x^*(h), y^*(h), z^*(h))$ of f when $h \geq H$.
3. Will the global maximum value $f^*(h)$ increase or decrease when the value of the parameter h increases? We assume that the initial value of h satisfies $h \geq H$.

9.10. Final Exam in GRA6035 on 11/06/2013, Problem 2
We consider the function $f(x,y;a) = xy^2 + 5x^3y - a^2xy$ with parameter a defined for all points $(x,y) \in \mathbb{R}^2$. We assume that $a > 0$.

1. Compute the partial derivatives and the Hessian matrix of f.
2. Compute all stationary points of f. Show that there is exactly one stationary point $(x^*(a), y^*(a))$ that is a local maximum, and find it.
3. Will the local maximum value $f^*(a) = f(x^*(a), y^*(a))$ increase or decrease when the value of the parameter a increases?

9.11. Final Exam in GRA6035 on 12/12/2011, Problem 4
We consider the optimization problem

$$\min\ 2x^2 + y^2 + 3z^2 \text{ subject to } \begin{cases} x - y + 2z & = 3 \\ x + y & = 3 \end{cases}$$

1. Write down the first order conditions for this optimization problem and show that there is exactly one admissible point that satisfy the first order conditions, the point $(x,y,z) = (2,1,1)$.
2. Use the bordered Hessian at $(x,y,z) = (2,1,1)$ to show that this point is a local minimum for $2x^2 + y^2 + 3z^2$ among the admissible points. What is the local minimum value?
3. Prove that $(x,y,z) = (2,1,1)$ solves the above optimization problem with equality constraints. What is the solution of the Kuhn-Tucker problem

$$\min\ 2x^2 + y^2 + 3z^2 \text{ subject to } \begin{cases} x - y + 2z & \geq 3 \\ x + y & \geq 3 \end{cases}$$

   with inequality constraints?

9.3 Advanced Optimization Problems

The advanced problems are challenging and optional, and are meant for advanced students. It is recommended that you work through the ordinary problems and exam problems and make sure that you master them before you attempt these problems.

9.12. Consider the following Kuhn-Tucker problem:

$$\max\ e^x(1+z) \quad \text{subject to} \quad \begin{cases} x^2+y^2 \le 1 \\ x+y+z \le 1 \end{cases}$$

Write down the Kuhn-Tucker conditions for this problem and solve them. Use the result to solve the optimization problem. (Hint: Even if the set of admissible points is not bounded, you could still find another argument to show that the problem must have a solution).

9.13. Consider the following Kuhn-Tucker problem:

$$\max\ 3xy-x^3 \quad \text{subject to} \quad \begin{cases} 2x-y \ge -5 \\ 5x+2y \le 37 \\ x,y \ge 0 \end{cases}$$

1. Sketch the region in the xy-coordinate plane that satisfy all constraints, and use this to show that the region is bounded.
2. Write down the Kuhn-Tucker conditions, and solve the problem.
3. We replace the constraint $2x-y \ge -5$ with $2x-y=-5$, so that the optimization problem has *mixed constraints* — both equality and inequality constraints. Describe the changes we must make to the Kuhn-Tucker conditions to solve this new problem and explain why. Use this to solve the new problem.

9.14. Solve the optimization problem

$$\max 4z-x^2-y^2-z^2 \text{ subject to } \begin{cases} z \le xy \\ x^2+y^2+z^2 \le 3 \end{cases}$$

9.4 Solutions

9.1 The stationary points are given by $f'_x = -2x-y+2r = 0$, $f'_y = -x-4y+2r = 0$. The equations are linear, and there is a unique solution:

$$\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2r \\ 2r \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2r \\ 2r \end{pmatrix} = \begin{pmatrix} 6r/7 \\ 2r/7 \end{pmatrix}$$

The function f is concave, since the Hessian of f is given by

$$f'' = \begin{pmatrix} -2 & -1 \\ -1 & -4 \end{pmatrix}$$

with $D_1 = -2$ and $D_2 = 7$. The stationary point $x^*(r) = 6r/7$ and $y^*(r) = 2r/7$ is therefore the unique maximum of f. The optimal value function is given by

$$f^*(r) = -(6r/7)^2 - (6r/7)(2r/7) - 2(2r/7)^2 + 2r(6r/7) + 2r(2r/7) = \frac{8}{7}r^2$$

The Envelope Theorem states that the derivative of $f^*(r)$ is given by

$$\frac{d}{dr}\, f^*(r) = \frac{\partial f}{\partial r}\,(x = x^*(r), y = y^*(r)) = 2x^*(r) + 2y^*(r) = \frac{16}{7}r$$

and we see that this is the derivative of $f^*(r)$ computed above.

9.2 The stationary points are given by $f'_x = r^2 - 2x = 0,\ f'_y = 3s^2 - 16y = 0$. The unique stationary point is therefore given by $x = r^2/2$ and $y = 3s^2/16$. The function f is clearly concave, so $x^*(r,s) = r^2/2$ and $y^*(r,s) = 3s^2/16$ is the unique maximum of f. The optimal value function is given by

$$f^*(r,s) = r^2(r^2/2) + 3s^2(3s^2/16) - (r^2/2)^2 - 8(3s^2/16)^2 = \frac{1}{4}r^4 + \frac{9}{32}s^4$$

The Envelope Theorem states that the partial derivatives of $f^*(r,s)$ are given by

$$\frac{\partial}{\partial r}\, f^*(r,s) = \frac{\partial f}{\partial r}\,(x = x^*(r,s), y = y^*(r,s)) = 2rx^*(r,s) = r^3$$

and

$$\frac{\partial}{\partial s}\, f^*(r,s) = \frac{\partial f}{\partial s}\,(x = x^*(r,s), y = y^*(r,s)) = 6sy^*(r,s) = \frac{9}{8}s^3$$

We see that these expressions are the partial derivatives of $f^*(r,s)$ computed above.

9.3 We consider the Lagrangian $\mathscr{L}(x,y,z,\lambda) = 100 - x^2 - y^2 - z^2 - \lambda(x + 2y + z)$, and solve the first order conditions

$$\begin{aligned}
\mathscr{L}'_x &= -2x - \lambda = 0\\
\mathscr{L}'_y &= -2y - \lambda \cdot 2 = 0\\
\mathscr{L}'_z &= -2z - \lambda = 0
\end{aligned}$$

together with $x + 2y + z = a$. We get $2x = -\lambda,\ 2y = -2\lambda,\ 2z = -\lambda$ and (after multiplying the constraint by 2)

$$-\lambda - 4\lambda - \lambda = 2a \quad \Rightarrow \quad \lambda = -a/3$$

The unique solution of the equations is $(x,y,z;\lambda) = (a/6, a/3, a/6; -a/3)$. Since $\mathscr{L}(x,y,z;-a/3)$ is a concave function in (x,y,z), we have that this solution solves the maximum problem. The optimal value function is given by

$$f^*(a) = f(a/6, a/3, a/6) = 100 - \frac{a^2}{36} - \frac{a^2}{9} - \frac{a^2}{36} = 100 - \frac{a^2}{6}$$

We see that the derivative of the optimal value function is $-2a/6 = -a/3 = \lambda(a)$.

9.4 We consider the Lagrangian

$$\mathscr{L}(x,y,z,\lambda_1,\lambda_2) = x+4y+z-\lambda_1(x^2+y^2+z^2)-\lambda_2(x+2y+3z)$$

and solve the first order conditions

$$\begin{aligned} \mathscr{L}'_x &= 1-\lambda_1\cdot 2x-\lambda_2 = 0 \\ \mathscr{L}'_y &= 4-\lambda_1\cdot 2y-\lambda_2\cdot 2 = 0 \\ \mathscr{L}'_z &= 1-\lambda_1\cdot 2z-\lambda_2\cdot 3 = 0 \end{aligned}$$

together with $x^2+y^2+z^2 = 216$ and $x+2y+3z = 0$. From the first order conditions, we get

$$2x\lambda_1 = 1-\lambda_2,\ 2y\lambda_1 = 4-2\lambda_2,\ 2z\lambda_1 = 1-3\lambda_2$$

We see from these equations that we cannot have $\lambda_1 = 0$, and multiply the last constraint with $2\lambda_1$. We get

$$2\lambda_1(x+2y+3z) = 0 \quad \Rightarrow \quad (1-\lambda_2)+2(4-2\lambda_2)+3(1-3\lambda_2) = 0$$

This gives $12-14\lambda_2 = 0$, or $\lambda_2 = 12/14 = 6/7$. We use this and solve for x,y,z, and get

$$x = \frac{1}{14\lambda_1},\ y = \frac{8}{7\lambda_1},\ z = -\frac{11}{14\lambda_1}$$

Then we substitute this in the first constraint, and get

$$\left(\frac{1}{14\lambda_1}\right)^2(1+16^2+(-11)^2) = 216 \quad \Rightarrow \quad 216\cdot 14^2\lambda_1^2 = 378$$

This implies that $\lambda_1 = \pm\frac{\sqrt{7}}{28}$, and we have two solutions to the first order equations and constraints. Moreover, we see that $\mathscr{L}(x,y,z,\pm\frac{\sqrt{7}}{28},\frac{6}{7})$ is a concave function in (x,y,z) when $\lambda_1 > 0$, and convex when $\lambda_1 < 0$. Therefore, the solution

$$(x^*,y^*,z^*) = (\frac{2}{7}\sqrt{7}, \frac{32}{7}\sqrt{7}, -\frac{22}{7}\sqrt{7})$$

corresponding to $\lambda_1 = \frac{\sqrt{7}}{28}$ solves the maximum problem, and the maximum value is

$$f(x^*,y^*,z^*) = x^*+4y^*+z^* = \frac{2+128-22}{7}\sqrt{7} = \frac{108}{7}\sqrt{7} \simeq 40.820$$

When $b_1 = 216$ is changed to 215 and $b_2 = 0$ is changed to 0.1, the approximate change in the the maximum value is given by

$$\lambda_1(215-216)+\lambda_2(0.1-0) = (-1)\frac{\sqrt{7}}{28}+(0.1)\frac{6}{7} \simeq -0.009$$

The estimate for the new maximum value is therefore $\simeq 40.811$.

9.5 The Lagrangian is $\mathscr{L}(x,y,z,\lambda) = \frac{1}{2}\ln(1+x) + \frac{1}{4}\ln(1+y) - \lambda(2x+3y)$, and we solve the first order conditions

$$\mathscr{L}'_x = \frac{1}{2(x+1)} - \lambda \cdot 2 = 0$$
$$\mathscr{L}'_y = \frac{1}{4(y+1)} - \lambda \cdot 3 = 0$$

together with the contraints $2x+3y = m$. We solve the first order conditions for x and y, and find

$$x = \frac{1}{4\lambda} - 1, \quad y = \frac{1}{12\lambda} - 1$$

The constraint $2x+3y = m$ gives the equation

$$2(\frac{1}{4\lambda} - 1) + 3(\frac{1}{12\lambda} - 1) = m \quad \Rightarrow \quad \frac{3}{4\lambda} = (m+5)$$

This gives the solution

$$\lambda^*(m) = \frac{3}{4(m+5)}, \quad x^*(m) = \frac{m+5}{3} - 1 = \frac{m+2}{3}, \quad y^*(m) = \frac{m+5}{9} - 1 = \frac{m-4}{9}$$

This is the maximum, since the Hessian of $\mathscr{L}(x,y;\lambda^*(m))$ is

$$\begin{pmatrix} -\frac{1}{2(x+1)^2} & 0 \\ 0 & -\frac{1}{4(y+1)^2} \end{pmatrix}$$

and therefore negative semidefinite for all (x,y) in the domain of definition of U (U is defined for all points (x,y) such that $x > -1$ and $y > -1$). The optimal value function $U^*(m)$ is given by

$$U^*(m) = \frac{1}{2}\ln(x^*(m)+1) + \frac{1}{4}\ln(y^*(m)+1) = \frac{1}{2}\ln(\frac{m+5}{3}) + \frac{1}{4}\ln(\frac{m+5}{9})$$

We see that this can be simplified to

$$U^*(m) = \ln\left(\left(\frac{m+5}{3}\right)^{1/2} \cdot \left(\frac{m+5}{9}\right)^{1/4}\right) = \ln\left(\frac{(m+5)^{3/4}}{3}\right)$$

The derivative of the optimal value function is

$$\frac{d}{dm}\,U^*(m) = \frac{3}{(m+5)^{3/4}} \cdot \frac{3}{4}\frac{(m+5)^{-1/4}}{3} = \frac{3}{4(m+5)} = \lambda^*(m)$$

9.6 In a) we have $n = 2$ and $m = 1$, so $n - m = 1$. We consider the determinant of the bordered Hessian

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -1(-2) + 1(0) = 2$$

Since it has the same sign as $(-1)^n = (-1)^2 = 1$, we see that the constrained quadratic form is negative definite. In b) we have $n = 3$ and $m = 2$, so $n - m = 1$. We consider the determinant of the bordered Hessian

$$\begin{vmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 2 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -2 \\ 1 & 1 & 1 & -1 & 2 \\ 0 & 0 & -2 & 2 & -2 \\ 0 & -2 & 1 & 1 & -3 \end{vmatrix} = 2 \begin{vmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 0 & 0 & -2 & 2 \\ 0 & -2 & 1 & 1 \end{vmatrix} = 2(-1)(-2)(4) = 16$$

Since it has the same sign as $(-1)^m = (-1)^2 = 1$, we see that the constrained quadratic form is positive definite.

9.7 In a) we solve the constraint and get $y = -x$. Substitution in the quadratic form gives $Q(x,y) = x^2 + 2xy - y^2 = -2x^2$, which is negative definite. Therefore, the constrained quadratic form in a) is negative definite. In b) we solve the constraint using Gaussian elimination and get one free variable x_2 and solution $x_1 = -x_2$ and $x_3 = 0$. Substitution in the quadratic form gives $Q(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2 + 4x_1x_3 - 2x_1x_2 = 4x_2^2$, which is positive definite. Therefore, the constrained quadratic form in b) is positive definite.

9.8 The Lagrangian is $\mathscr{L}(x,y,z,\lambda) = x^2y^2z^2 - \lambda(x^2 + y^2 + z^2)$, and we solve the first order conditions

$$\begin{aligned} \mathscr{L}'_x &= 2xy^2z^2 - \lambda \cdot 2x = 0 \\ \mathscr{L}'_y &= 2x^2yz^2 - \lambda \cdot 2y = 0 \\ \mathscr{L}'_z &= 2x^2y^2z - \lambda \cdot 2z = 0 \end{aligned}$$

together with the contraint $x^2 + y^2 + z^2 = 1$. The first order conditions can be reduced to

$$\begin{aligned} x = 0 \quad &\text{or} \quad y^2z^2 = \lambda \\ y = 0 \quad &\text{or} \quad x^2z^2 = \lambda \\ z = 0 \quad &\text{or} \quad x^2y^2 = \lambda \end{aligned}$$

If $x = 0$ or $y = 0$ or $z = 0$, then $\lambda = 0$, and we obtain the solutions

$$(x,y,0) \text{ with } x^2 + y^2 = 1, \quad (x,0,z) \text{ with } x^2 + z^2 = 1, \quad (0,y,z) \text{ with } y^2 + z^2 = 1$$

which all satisfy $f = 0$. These points are clearly local minima, since $x^2y^2z^2 \geq 0$. If $x \neq 0, y \neq 0, z \neq 0$, then we have

$$x^2y^2 = x^2z^2 = y^2z^2 = \lambda$$

and this implies that $x^2 = y^2 = z^2 = 1/3$. The solutions are therefore the eight points

$$(x,y,z) = (\pm\sqrt{3}/3, \pm\sqrt{3}/3, \pm\sqrt{3}/3)$$

with $f = 1/27$ and $\lambda = 1/9$. The Bordered Hessian matrix at one of the solutions $(x^*, y^*, z^*; \lambda^*) = (\sqrt{3}/3, \pm\sqrt{3}/3, \pm\sqrt{3}/3; 1/3)$ is given by

$$B = \begin{pmatrix} 0 & 2x & 2y & 2z \\ 2x & 2y^2z^2 - 2\lambda & 4xyz^2 & 4xy^2z \\ 2y & 4xyz^2 & 2x^2z^2 - 2\lambda & 4x^2yz \\ 2z & 4xy^2z & 4x^2yz & 2x^2y^2 - 2\lambda \end{pmatrix} = \begin{pmatrix} 0 & 2x^* & 2y^* & 2z^* \\ 2x^* & 0 & 4/3x^*y^* & 4/3x^*z^* \\ 2y^* & 4/3x^*y^* & 0 & 4/3y^*z^* \\ 2z^* & 4/3x^*z^* & 4/3y^*z^* & 0 \end{pmatrix}$$

We need to compute the $n - m = 3 - 1 = 2$ last leading principal minors, that is D_3 and D_4. We have

$$D_3 = \frac{32}{27}, \quad D_4 = -\frac{64}{81}$$

Since the sign is alternating and the last sign is negative, and therefore equal to the sign of $(-1)^n = (-1)^3 = -1$, it follows that all eight points are local maxima. Since the set given by $x^2 + y^2 + z^2 = 1$ is bounded and NDCQ is satisfied for all admissible points, it follows that these eight points are maxima, and therefore solutions to the Lagrange problem.

9.9 Final Exam in GRA6035 on 13/12/2012, Problem 2

1. We compute the partial derivatives and the Hessian matrix of f:

$$\begin{pmatrix} f'_x \\ f'_y \\ f'_z \end{pmatrix} = \begin{pmatrix} -4x^3 - 2hx + 6z \\ -6y \\ 6x - 12z \end{pmatrix}, \quad f'' = \begin{pmatrix} -12x^2 - 2h & 0 & 6 \\ 0 & -6 & 0 \\ 6 & 0 & -12 \end{pmatrix}$$

   We see that the leading principal minors are given by $D_1 = -12x^2 - 2h$, $D_2 = -6D_1$ and $D_3 = -6(144x^2 + 24h - 36)$. Hence $D_1 \leq 0$ for all (x,y,z) if and only if $h \geq 0$, and if this is the case then $D_2 = -6D_1 \geq 0$. Moreover, $D_3 \leq 0$ for all (x,y,z) if and only if $h \geq 3/2$. This means that $D_1 \leq 0$, $D_2 \geq 0$, $D_3 \leq 0$ if and only if $h \geq 3/2$, and the equalities are strict if $h > 3/2$. If $h = 3/2$, then $D_3 = 0$, and we compute the remaining principal minors. We find that $\Delta_1 = -6, -12 \leq 0$ and that $\Delta_2 = 144x^2, 72 \geq 0$. We conclude that f is concave if and only if $h \geq 3/2$, and $H = \mathbf{3/2}$.
2. We compute the stationary points, which are given by the equations

$$-4x^3 - 2hx + 6z = 0, \quad -6y = 0, \quad 6x - 12z = 0$$

The last two equations give $y = 0$ and $z = x/2$, and the first equations becomes

$$-4x^3 - 2hx + 3x = x(-4x^2 + 3 - 2h) = 0 \quad \Leftrightarrow \quad x = 0$$

since $x^2 = (3-2h)/4$ has no solutions when $h > 3/2$ and the solution $x = 0$ when $h = 3/2$. The stationary points are therefore given by $(x^*(h), y^*(h), z^*(h)) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$ when $h \geq 3/2$, and this is the global maximum since f is concave.

3. Let $h \geq 3/2$. By the Envelope Theorem, we have that

$$\frac{d}{dh} f^*(h) = \frac{\partial f}{\partial h}\Big|_{(x,y,z)=(0,0,0)} = (-x^2 + 2h)\Big|_{(x,y,z)=(0,0,0)} = 2h \geq 3$$

Since the derivative is positive, the maximal value will **increase** when h increases. We could also compute $f^*(h) = f(0,0,0) = 12 + h^2$ explicitly for $h \geq 3/2$, and use this to see that $f^*(h)$ increases when h increases.

9.10 Final Exam in GRA6035 on 11/06/2013, Problem 2

1. We compute the partial derivatives and the Hessian matrix of f:

$$\begin{pmatrix} f'_x \\ f'_y \end{pmatrix} = \begin{pmatrix} y^2 + 15x^2y - a^2y \\ 2xy + 5x^3 - a^2x \end{pmatrix}, \quad f'' = \begin{pmatrix} 30xy & 2y + 15x^2 - a^2 \\ 2y + 15x^2 - a^2 & 2x \end{pmatrix}$$

2. We compute the stationary points, which are given by the equations

$$y^2 + 15x^2y - a^2y = 0, \quad 2xy + 5x^3 - a^2x = 0$$

The first equation give $y = 0$ or $y + 15x^2 - a^2 = 0$. If $y = 0$, then the second equation gives $5x^3 - a^2x = 0$, which means that $x = 0$ or $x = \pm a/\sqrt{5}$. This gives stationary points

$$(0,0), (a/\sqrt{5}, 0), (-a/\sqrt{5}, 0)$$

If $y \neq 0$, then $y = a^2 - 15x^2$, and the second equation gives $x = 0$ or $2y + 5x^2 - a^2 = 0$. In the first case, $x = 0$ and $y = a^2$. In the second case, $2(a^2 - 15x^2) + 5x^2 - a^2 = a^2 - 25x^2 = 0$, or $x = \pm a/5$ and $y = 2a^2/5$. We get stationary points with $y \neq 0$ given by

$$(0, a^2), (\pm a/5, 2a^2/5)$$

To find the local maximum, we look at the leading principal minors of $f''(x^*, y^*)$ for each stationary point (x^*, y^*). We see that all the stationary points with $x = 0$ or $y = 0$ are saddle points, since $D_2 < 0$ when $a > 0$. For $(x^*(a), y^*(a)) = (\pm a/5, 2a^2/5)$, we have

$$D_2 = \frac{24}{25}a^4 - (\frac{2}{5}a^2)^2 = \frac{20}{25}a^4 > 0$$

and $D_1 = 30xy = \pm 12/5\, a^3$. This means that there is exactly one local maximum point for a given $a > 0$, given by $(x^*(a), y^*(a)) = (-a/5, 2a^2/5)$. The point $(a/5, 2a^2/5)$ is a local minimum point.

3. Let $a > 0$. By the Envelope Theorem, we have that

$$\frac{d}{da} f^*(a) = \frac{\partial f}{\partial a}\Big|_{(x,y)=(x^*(a),y^*(a))} = (-2axy)\big|_{(x,y)=(x^*(a),y^*(a))} = \frac{4}{25}\, a^4 > 0$$

Since the derivative is positive, the local maximal value will **increase** when a increases. We could also compute $f^*(a) = f(x^*(a), y^*(a)) = 4/125\, a^5$ explicitly for $a > 0$, and use this to see that $f^*(a)$ increases when a increases.

9.11 Final Exam in GRA6035 on 12/12/2011, Problem 4

1. The Lagrangian for this problem is given by $\mathscr{L} = 2x^2 + y^2 + 3z^2 - \lambda_1(x - y + 2z) - \lambda_2(x + y)$, and the first order conditions are

$$\begin{aligned} \mathscr{L}'_x &= 4x - \lambda_1 - \lambda_2 = 0 \\ \mathscr{L}'_y &= 2y + \lambda_1 - \lambda_2 = 0 \\ \mathscr{L}'_z &= 6z - 2\lambda_1 = 0 \end{aligned}$$

We solve the first order conditions for x, y, z and get

$$x = \frac{\lambda_1 + \lambda_2}{4}, \quad y = \frac{\lambda_2 - \lambda_1}{2}, \quad z = \frac{\lambda_1}{3}$$

When we substitute these expressions into the two constraints $x - y + 2z = 3$ and $x + y = 3$, we get the equations

$$17\lambda_1 - 3\lambda_2 = 36, \quad -\lambda_1 + 3\lambda_2 = 12$$

Adding the two equations, we get $16\lambda_1 = 48$, or $\lambda_1 = 3$, and the last equation gives $\lambda_2 = 5$. When we substitute this into the expressions for x, y, z we get $(x, y, z) = (2, 1, 1)$. This means that $(x, y, z; \lambda_1, \lambda_2) = (2, 1, 1; 3, 5)$ **is the unique point that satisfies the first order conditions and the constraints**. Alternatively, one may observe that the first order conditions and the constraints form a 5×5 linear system. If we substitute $(x, y, z) = (2, 1, 1)$ in this system, we find that $\lambda_1 = 3$ and $\lambda_2 = 5$; hence $(x, y, z; \lambda_1, \lambda_2) = (2, 1, 1; 3, 5)$ is one solution of the system. To show that this is the only solution, we may check that the determinant of the coefficient matrix is non-zero. We first use some elementary row operations that preserve the determinant:

$$\begin{vmatrix} 4 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 6 & -2 & 0 \\ 1 & -1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 4 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 6 & -2 & 0 \\ 0 & 0 & 0 & 17/12 & -1/4 \\ 0 & 0 & 0 & -1/4 & 3/4 \end{vmatrix}$$

Then we see that the determinant is given by $4 \cdot 2 \cdot 6 \cdot (17/4 \cdot 3/4 - 1/4 \cdot 1/4) = 48 \neq 0$.

2. The bordered Hessian at $(x, y, z; \lambda_1, \lambda_2) = (2, 1, 1; 3, 5)$ is the matrix

$$B = \begin{pmatrix} 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 \\ -1 & 1 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 & 6 \end{pmatrix}$$

Since there are $n = 3$ variables and $m = 2$ constratints, we have to compute the $n - m = 1$ last principal minors; that is, just the determinant $D_5 = |B|$. We first use an elementary row operation to simplify the computation, then develop the determinant along the last column:

$$|B| = \begin{vmatrix} 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 \\ -1 & 1 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 & 6 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 \\ -1 & 1 & 0 & 2 & 0 \\ 2 & 0 & -3 & 3 & 0 \end{vmatrix} = 2 \begin{vmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 4 & 0 \\ -1 & 1 & 0 & 2 \\ 2 & 0 & -3 & 3 \end{vmatrix}$$

Then we develop the last determinant along the first row, and get

$$|B| = 2 \begin{vmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 4 & 0 \\ -1 & 1 & 0 & 2 \\ 2 & 0 & -3 & 3 \end{vmatrix} = 2 \left(\begin{vmatrix} 1 & 1 & 0 \\ -1 & 1 & 2 \\ 2 & 0 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 4 \\ -1 & 1 & 0 \\ 2 & 0 & -3 \end{vmatrix} \right) = 2(10 + 14) = 48$$

Since $|B| = 48 > 0$ has the same sign as $(-1)^m = (-1)^2 = 1$, we conclude that **the point** $(x, y, z) = (2, 1, 1)$ **is a local minimum for** $2x^2 + y^2 + 3z^2$ (among the admissible points). The local minimum value is $f(2, 1, 1) = 8 + 1 + 3 = \mathbf{12}$.

3. We fix $\lambda_1 = 3$ and $\lambda_2 = 5$, and consider the Lagrangian

$$\mathscr{L}(x, y, z) = 2x^2 + y^2 + 3z^2 - 3(x - y + 2z) - 5(x + y)$$

This function is clearly convex, since the Hessian matrix

$$\mathscr{L}'' = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

is positive definite (with eigenvalues $4, 2, 6$). Therefore, the point $(x, y, z) = (2, 1, 1)$ **solves the minimum problem**. The Kuhn-Tucker problem can be reformulated in standard form as

$$\max -(2x^2 + y^2 + 3z^2) \text{ subject to } \begin{cases} -(x - y + 2z) & \leq -3 \\ -(x + y) & \leq -3 \end{cases}$$

Therefore, we see that the Lagrangian of the Kuhn-Tucker problem is

$$-(2x^2+y^2+3z^2)+\lambda_1(x-y+2z)+\lambda_2(x+y)=-\mathscr{L}$$

and the first order conditions of the Kuhn-Tucker problem are the same as in the original problem. Hence $(x,y,z;\lambda_1,\lambda_2)=(2,1,1;3,5)$ is still a solution of the first order conditions and the constraints, and $\lambda_1,\lambda_2\geq 0$ also solves the complementary slackness conditions. When we fix $\lambda_1=3$ and $\lambda_2=5$, $-\mathscr{L}$ is concave since $\mathscr{L}$ is convex, and this means that $(x,y,z)=(2,1,1)$ **also solves the Kuhn-Tucker problem**.

9.14 We rewrite the first constraint as $z-xy\leq 0$, and write the Lagrangian for this problem as

$$\mathscr{L}=4z-x^2-y^2-z^2-\lambda_1(z-xy)-\lambda_2(x^2+y^2+z^2)$$

The Kuhn-Tucker conditions for this problem are the first order conditions

$$\begin{aligned}\mathscr{L}'_x&=-2x-\lambda_1 y-\lambda_2\cdot 2x=0\\ \mathscr{L}'_y&=-2y-\lambda_1 x-\lambda_2\cdot 2y=0\\ \mathscr{L}'_x&=4-2z-\lambda_1-\lambda_2\cdot 2z=0\end{aligned}$$

the constraints $z\leq xy$ and $x^2+y^2+z^2\leq 3$, and the complementary slackness conditions $\lambda_1,\lambda_2\geq 0$ and

$$z<xy\Rightarrow\lambda_1=0,\quad x^2+y^2+z^2<3\Rightarrow\lambda_2=0$$

Let us find all solutions of the Kuhn-Tucker conditions in the case $z=xy$ and $x^2+y^2+z^2=3$. We solve the last of the FOC for λ_1 and obtain $\lambda_1=4-2z(1+\lambda_2)$. Then we substitute this for λ_1 in the first two FOC's and get

$$\begin{aligned}-2x-(4-2z(1+\lambda_2))y-2\lambda_2 x=0\quad&\Rightarrow\quad -2x(1+\lambda_2)+4y-2z(1+\lambda_2)y=0\\ -2y-(4-2z(1+\lambda_2))x-2\lambda_2 y=0\quad&\Rightarrow\quad -2y(1+\lambda_2)+4x-2z(1+\lambda_2)x=0\end{aligned}$$

We see that $\lambda_2=-1$ gives $x=y=0$ and $z=xy=0$. But $x^2+y^2+z^2=0\neq 3$ so this is not possible, and we may assume that $\lambda_2\neq -1$. Subtraction of the two equations gives

$$-2(1+\lambda_2)(x-y)-4(x-y)+2z(1+\lambda_2)(x-y)=0$$

This implies that $x=y$ or that $(2z-2)(1+\lambda_2)=4$. If $x=y$ then $z=x^2$ and the second constraint gives $2x^2+x^4=3$. This is a quadratic equation $u^2+2u-3=0$ in $u=x^2$, with solution $x^2=1$ (since $x^2=-3$ is not possible). Therefore we get $(x,y,z)=(1,1,1)$ or $(x,y,z)=(-1,-1,1)$. In both cases, we see that the FOC's are satisfied with $\lambda_1=2$ and $\lambda_2=0$. If $x\neq y$, then $z=2/(1+\lambda_2)+1=(3+\lambda_2)/(1+\lambda_2)$. We substitute this for z in the first FOC, and get

$$-2x(1+\lambda_2)+4y-(6+2\lambda_2)(1+\lambda_2)y=0 \quad \Rightarrow \quad x=\frac{\lambda_2-1}{\lambda_2+1}y$$

When we substitute this for x in the second FOC, we get

$$-2y(1+\lambda_2)+4\frac{\lambda_2-1}{\lambda_2+1}y-(6+2\lambda_2)\frac{\lambda_2-1}{\lambda_2+1}y=0$$

Multiplication with $(1+\lambda_2)$ gives $y=0$ or $-4\lambda_2^2-4\lambda_2=0$. If $y=0$, then $x=0$, and this is not possible since $z=xy=0$. If $y\neq 0$, then we must have $\lambda_2=0$ (since ($\lambda_2\neq -1$), and this implies that $z=3$, which is impossible. We conclude that the only solutions to the Kuhn-Tucker conditions with both constraints binding are $(x,y,z;\lambda_1,\lambda_2)=(1,1,1;2,0),(-1,-1,1;2,0)$ with $f=1$ in both cases. If $z<xy$ and $x^2+y^2+z^2=3$, then $\lambda_1=0$ and the first two FOC's give $x=y=0$, so $z=-\sqrt{3}<xy=0$. But $\lambda_2<0$ from the last FOC, so this is not a solution. If $xy=z$ and $x^2+y^2+z^2<3$, then $\lambda_2=0$ and the first two FOC's imply that $x=y$ and that $\lambda_1=2$. The third FOC give that $z^2=1$. This implies that $x=y=z=1$, and this is not possible since $x^2+y^2+z^2<3$. Finally, if $z<xy$ and $x^2+y^2+z^2<3$, then $\lambda_1=\lambda_2=0$. Then the FOC's imply that $x=y=0, z=2$ and this is impossible since $x^2+y^2+z^2<3$. The only solutions occur when both constraints are binding. The set of admissible points is bounded since $x^2+y^2+z^2\leq 3$ implies that $x,y,z\in(-\sqrt{3},\sqrt{3})$. Therefore the problem has a solution. We consider the NDCQ, and look at the matrix

$$\begin{pmatrix} -y & -x & 1 \\ 2x & 2y & 2z \end{pmatrix}$$

When $z=xy$ and $x^2+y^2+z^2=3$, the full matrix has rank two since the first minor $-2y^2-2x^2=0$ if and only if $x=y=0$ and this is not possible. If $z<xy$ and $x^2+y^2+z^2=3$, the submatrix consisting of the last row has rank one, since $x=y=z=0$ is not possible. If $xy=z$ and $x^2+y^2+z^2<3$, then the submatrix consisting of the first row clearly has rank one. So there are no admissible points where NDCQ is not satisfied, and there must be a maximum at one of the points satisfying the Kuhn-Tucker conditions. The conclusion is that $(x,y,z)=(1,1,1)$ and $(x,y,z)=(-1,-1,1)$ are the maximum points, with maximum value $f=1$.

Lecture 10
First Order Differential Equations

10.1 Main concepts

A *differential equation* in the function $y = y(t)$ is an equation relating y and its derivatives $y' = y'(t) = \dot{y}$, $y'' = y''(t) = \ddot{y}$, and other higher derivatives. A *solution* is a function $y = y(t)$ that fits in the differential equation. There are usually many solutions. The general form of a solution is called the *general solution*.

A *first order differential equation* involves only y', y and t since the *order* of a differential equation is the highest order of the derivatives involved. In general, a first order differential equation has the form

$$y' = F(y,t)$$

where $F(y,t)$ is some expression in y and t. The general solution of a first order differential equation will depend on *one* free variable. An *initial value problem* consists of the differential equation and an initial condition, and has a unique solution, called the *particular solution*.

Separable differential equations. A first order differential equation is *separable* if it can be written in the form

$$y' = f(y) \cdot g(t)$$

for some expressions $f(y)$ in y and $g(t)$ in t. It can be solved using *separation of the variables*:

$$y' = f(y)g(t) \quad \Leftrightarrow \quad \frac{1}{f(y)}y' = g(t) \quad \Leftrightarrow \quad \int \frac{1}{f(y)}\,dy = \int g(t)\,dt$$

After the integrals are computed, one gets a solution in implicit form. To find an explicit form $y = y(t)$, one must solve for y.

Linear differential equations. A first order differential equation is *linear* if it can be written in the form

$$y' + a(t)y = b(t) \quad \Leftrightarrow \quad y' = b(t) - a(t)y$$

for some expressions $a(t), b(t)$ in t. It can be solved using *integrating factor*. The integrating factor is given by

$$u = e^{\int a(t)\,dt}$$

and multiplying with $u = u(t)$ in the differential equtation $y' + a(t)y = b(t)$ gives

$$uy' + a(t)uy = b(t)u \quad \Leftrightarrow \quad (uy)' = b(t)u \quad \Leftrightarrow \quad uy = \int b(t)u\,dt$$

The solution in explicit form is therefore given by

$$y = \frac{1}{u} \cdot \int b(t)u\,dt$$

If $a(t) = a$ and $b(t) = b$ are constants, then the solution is $y = b/a + Ce^{-at}$.

Exact differential equations. A first order differential equation is *exact* if it can be written in the form

$$p(y,t) \cdot y' + q(y,t) = 0$$

for some expressions $p(y,t), q(y,t)$ in y and t that satisfies the *exactness condition* $p'_t = q'_y$. This is the case if and only if there is a function $h = h(y,t)$ such that the equations

$$h'_y = p(y,t) \quad \text{and} \quad h'_t = q(y,t)$$

and then the solution of the differential equation is given by $h(y,t) = C$. To find an explicit form $y = y(t)$, one must solve for y.

10.2 Problems

10.1. Find the derivative $\dot{y}$ of the following functions:

1. $y = \frac{1}{2}t - \frac{3}{2}t^2 + 5t^3$
2. $y = (2t^2 - 1)(t^4 - 1)$
3. $y = (\ln t)^2 - 5\ln t + 6$
4. $y = \ln(3t)$
5. $y = 5e^{-3t^2+t}$
6. $y = 5t^2e^{-3t}$

10.2. Compute the following integrals:

1. $\int t^3 dt$
2. $\int_0^1 (t^3 + t^5 + \frac{1}{3})dt$
3. $\int \frac{1}{t} dt$
4. $\int te^{t^2} dt$

5. $\int \ln t\,dt$

10.3. Find the general solution, and the particular solution satisfying $y(0) = 1$ in the following differential equations:

1. $\dot{y} = 2t$.
2. $\dot{y} = e^{2t}$
3. $\dot{y} = (2t+1)e^{t^2+t}$
4. $\dot{y} = \frac{2t+1}{t^2+t+1}$

10.4. We consider the differential equation $\dot{y} + y = e^t$. Show that $y(t) = Ce^{-t} + \frac{1}{2}e^t$ is a solution of the differential equation for all values of the constant C.

10.5. Show that $y = Ct^2$ is a solution of $t\dot{y} = 2y$ for all choices of the constant C, and find the particular solution satisfying $y(1) = 2$.

10.6. Solve the differential equation $y^2\dot{y} = t + 1$, and find the integral curve that goes through the point $(t, y) = (1, 1)$.

10.7. Solve the following differential equations:

1. $\dot{y} = t^3 - 1$
2. $\dot{y} = te^t - t$
3. $e^y\dot{y} = t + 1$

10.8. Solve the following differential equations with initial conditions:

1. $t\dot{y} = y(1-t)$, with $(t_0, y_0) = (1, \frac{1}{e})$
2. $(1+t^3)\dot{y} = t^2 y$, with $(t_0, y_0) = (0, 2)$
3. $y\dot{y} = t$, with $(t_0, y_0) = (\sqrt{2}, 1)$
4. $e^{2t}\dot{y} - y^2 - 2y = 1$, with $(t_0, y_0) = (0, 0)$

10.9. Find the general solution of the differential equation $\dot{y} + \frac{1}{2}y = \frac{1}{4}$. Determine the equilibrium state of the equation. Is it stable? Draw some typical solutions.

10.10. Find the general solution of the following differential equations:

1. $\dot{y} + y = 10$
2. $\dot{y} - 3y = 27$
3. $4\dot{y} + 5y = 100$

10.11. Find the general solution of the following differential equations, and in each case, find the particular solution satisfying $y(0) = 1$:

1. $\dot{y} - 3y = 5$
2. $3\dot{y} + 2y + 16 = 0$
3. $\dot{y} + 2y = t^2$

10.12. Find the general solution of the following differential equations:

1. $t\dot{y} + 2y + t = 0 \quad (t \neq 0)$

2. $\dot{y} - y/t = t \quad (t > 0)$
3. $\dot{y} - \dfrac{t}{t^2-1}y = t \quad (t > 1)$
4. $\dot{y} - \dfrac{2}{t}y + \dfrac{2a^2}{t^2} = 0 \quad (t > 0)$

10.13. Determine which of the following differential equations are exact, and find the general solution in those cases where it is exact:

1. $(2y+t)\dot{y} + 2 + y = 0$
2. $y^2\dot{y} + 2t + y = 0$
3. $(t^5 + 6y^2)\dot{y} + 5yt^4 + 2 = 0$

10.14. Solve the differential equation $2t + 3y^2\dot{y} = 0$, first as a separable differential equation and then as an exact differential equation.

10.15. Final Exam in GRA6035 30/05/2011, 3c
Solve the initial value problem $(2t+y) - (4y-t)y' = 0$, $y(0) = 0$.

10.16. Final Exam in GRA6035 10/12/2010, 3c
Solve the initial value problem

$$\frac{t}{y^2}y' = \frac{1}{y} - 3t^2, \quad y(1) = \frac{1}{3}$$

10.3 Solutions

10.1

1. $\dot{y} = \frac{1}{2} - 3t + 15t^2$
2. $\dot{y} = 4t(t^4-1) + (2t^2-1)4t^3 = 12t^5 - 4t^3 - 4t$
3. $\dot{y} = 2(\ln t)\frac{1}{t} - 5\frac{1}{t}$
4. $\dot{y} = \frac{1}{t}$
5. $\dot{y} = 5e^{-3t^2+t}(-6t+1)$
6. $\dot{y} = 10te^{-3t} - 15t^2e^{-3t}$

10.2

1. $\int t^3\,dt = \frac{1}{4}t^4 + C$
2. $\int_0^1 (t^3 + t^5 + \frac{1}{3})\,dt = \frac{3}{4}$
3. $\int \frac{1}{t}\,dt = \ln|t| + C$
4. To find the integral $\int te^{t^2}\,dt$ we substitute $u = t^2$. This gives $\frac{du}{dt} = 2t$ or $\frac{du}{2} = t\,dt$. We get

$$\int te^{t^2}\,dt = \int e^u \frac{du}{2} = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{t^2} + C$$

5. We use integration by parts
$$\int uv'dt = uv - \int u'vdt.$$
We write $\int \ln t dt$ as $\int (\ln t)\cdot 1dt$ and let $u = \ln t$ and $v' = 1$. Thus $u' = \frac{1}{t}$ and $v = t$, and
$$\begin{aligned}\int \ln t dt &= (\ln t)t - \int \frac{1}{t}t dt\\ &= t\ln t - \int 1dt\\ &= t\ln t - t + C\end{aligned}$$

10.3

1. $y = \int 2tdt = t^2 + C$. The general solution is $y = t^2 + C$. We get $y(0) = C = 1$, so $y = t^2 + 1$ is the particular solution satisfying $y(0) = 1$.
2. $y = \frac{1}{2}e^{2t} + C$ is the general solution. We get $y(0) = \frac{1}{2}e^{2\cdot 0} + C = \frac{1}{2} + C = 1 \implies C = \frac{1}{2}$. Thus $y(t) = \frac{1}{2}e^{2t} + \frac{1}{2}$ is the particular solution.
3. To find the integral $\int (2t+1)e^{t^2+t}dt$, we substitute $u = t^2 + t$. We get $\frac{du}{dt} = 2t + 1 \implies du = (2t+1)dt$, so
$$\int (2t+1)e^{t^2+t}dt = \int e^u du = e^u + C = e^{t^2+t} + C.$$
The general solution is $y = e^{t^2+t} + C$. This gives $y(0) = 1 + C = 1 \implies C = 0$. The particular solution is $y = e^{t^2+t}$.
4. We substitute $u = t^2 + t + 1$ in $\int \frac{2t+1}{t^2+t+1}dt$ to find the general solution $y = \ln(t^2 + t + 1) + C$. We get $y(0) = \ln 1 + C = C = 1$. The particular solution is $y(t) = \ln(t^2 + t + 1) + 1$.

10.4 $y(t) = Ce^{-t} + \frac{1}{2}e^t \implies \dot{y} = -Ce^{-t} + \frac{1}{2}e^t$. From this we get
$$\dot{y} + y = -Ce^{-t} + \frac{1}{2}e^t + Ce^{-t} + \frac{1}{2}e^t = e^t$$
so we see that $\dot{y} + y = e^t$ is satisfied when $y = Ce^{-t} + \frac{1}{2}e^t$.

10.5 $y = Ct^2 \implies \dot{y} = 2Ct$. We have
$$t\dot{y} = t\cdot 2Ct = 2Ct^2 = 2y$$

10.6 The equation $y^2\dot{y} = t + 1$ is separable:
$$y^2\frac{dy}{dt} = t + 1$$
gives

$$\int y^2 dy = \int (t+1)dt$$
$$\frac{1}{3}y^3 = \frac{1}{2}t^2 + t + C$$
$$y^3 = \frac{3}{2}t^2 + 3t + 3C$$

Taking third root and renaming the constant

$$y(t) = \sqrt[3]{\frac{3}{2}t^2 + 3t + K}$$

We want the particular solution with $y(1) = 1$. We have

$$y(1) = \sqrt[3]{\frac{3}{2}1^2 + 3 + K}$$
$$= \sqrt[3]{K + \frac{9}{2}} = 1 \implies K + \frac{9}{2} = 1$$

We get $K = -\frac{7}{2}$. Thus

$$y(t) = \sqrt[3]{\frac{3}{2}t^2 + 3t - \frac{7}{2}}$$

is the particular solution.

10.7

1. $\dot{y} = t^3 - 1$ gives
$$y = \int (t^3 - 1)dt$$
We get
$$y = \frac{1}{4}t^4 - t + C.$$

2. We must evaluate the integral $\int (te^t - t)dt$. To evaluate $\int te^t dt$ we use integration by parts
$$\int uv' dt = uv - \int u'v dt.$$
with $v' = e^t$ and $u = t$. We get $u' = 1$ and $v = e^t$. Thus
$$\int te^t dt = te^t - \int e^t dt = te^t - e^t + C$$
We get
$$y = \int (te^t - t)dt = te^t - e^t - \frac{1}{2}t^2 + C$$

3. $e^y \dot{y} = t + 1$ is separated as

$$e^y dy = (t+1)dt \implies \int e^y dy = \int (t+1)dt$$

Thus we get

$$e^y = \frac{1}{2}t^2 + t + C.$$

Taking the natural logarithm on each side, we get

$$y(t) = \ln(\frac{1}{2}t^2 + t + C).$$

10.8

1. $t\dot{y} = y(1-t)$ is separated as

$$\frac{dy}{y} = \frac{1-t}{t}dt \implies \int \frac{dy}{y} = \int \frac{1-t}{t}dt$$

Note that $\frac{1-t}{t} = \frac{1}{t} - 1$, so

$$\ln|y| = \ln|t| - t + C$$

From this we get

$$e^{\ln|y|} = e^{\ln|t|-t+C} = e^{\ln|t|}e^{-t}e^C \implies |y| = |t|e^{-t}e^C$$

From this we deduce that

$$y(t) = te^{-t}K$$

where K is a constant as the general solution. We will find the particular solution with $y(1) = \frac{1}{e}$. We get

$$y(1) = e^{-1}K = e^{-1} \implies K = 1.$$

The particular solution is

$$y(t) = te^{-t}.$$

2. The equation $(1+t^3)\dot{y} = t^2 y$ is separated as

$$\frac{dy}{y} = \frac{t^2}{1+t^3}dt \implies \int \frac{dy}{y} = \int \frac{t^2}{1+t^3}dt$$

We get

$$\ln|y| = \frac{1}{3}\ln|1+t^3| + C = \ln|1+t^3|^{\frac{1}{3}} + C$$

This gives

$$e^{\ln|y|} = e^{\ln|1+t^3|^{\frac{1}{3}}+C}$$

This gives

$$|y| = |1+t^3|^{\frac{1}{3}} e^C$$

from which we deduce the general solution

$$y(t) = K(1+t^3)^{\frac{1}{3}}$$

where K is a constant. We which to find the particular solution with $y(0) = 2$. We get

$$y(0) = K = 2.$$

Thus the particular solution is

$$y(t) = 2(1+t^3)^{\frac{1}{3}}.$$

3. $y\dot{y} = t$ is separated as

$$ydy = tdt \implies \int ydy = \int tdt$$

The general solution is

$$y^2 = t^2 + C$$

where y is define implicitly. We want the particular solution where $y(\sqrt{2}) = 1$. We get

$$1^2 = (\sqrt{2})^2 + C \implies 1 = 2 + C \implies C = -1$$

We have

$$y^2 = t^2 - 1 \implies y = \pm\sqrt{t^2-1}$$

since $y(\sqrt{2}) < 0$ we have

$$y(t) = \sqrt{t^2-1}$$

as the particular solution.

4. $e^{2t}\frac{dy}{dt} - y^2 - 2y = 1$, is separated as follows:

$$e^{2t}\dot{y} - y^2 - 2y = 1 \implies e^{2t}\dot{y} = 1 + y^2 + 2y = (y+1)^2 \implies$$
$$\frac{dy}{(y+1)^2} = e^{-2t}dt \implies \int \frac{dy}{(y+1)^2} = \int e^{-2t}dt$$

To solve the integral

$$\int \frac{dy}{(y+1)^2}$$

we substitute $u = y+1$. We get $\frac{du}{dy} = 1 \implies dy = du$. Thus

$$\int \frac{dy}{(y+1)^2} = \int \frac{1}{u^2}du = \int u^{-2}du = \frac{1}{-1}u^{-2+1} + C = -u^{-1} + C = -\frac{1}{(y+1)} + C$$

Thus we get

$$-\frac{1}{(y+1)} = \frac{1}{-2}e^{-2t} + C = -\frac{1}{2}e^{-2t} + C \implies -y-1 = \frac{1}{-\frac{1}{2}e^{-2t} + C}$$

From this we get

$$y(t) = \frac{-1}{-\frac{1}{2}e^{-2t} + C} - 1$$

as the general solution. We want the particular solution with $y(0) = 0$. We get

$$y(0) = \frac{-1}{-\frac{1}{2}e^{0} + C} - 1 = 0$$

From this we get $C = -\frac{1}{2}$. Thus the particular solution is

$$\begin{aligned} y(t) &= \frac{-1}{-\frac{1}{2}e^{-2t} - \frac{1}{2}} - 1 \\ &= \frac{1 - e^{-2t}}{1 + e^{-2t}}. \end{aligned}$$

10.9 Since the differential equation is linear first order with constant coeffients $a = 1/2$ and $b = 1/4$, it has general solution

$$y = \frac{b}{a} + Ce^{-at} = \frac{1}{2} + Ce^{-t/2}$$

Since $y \to 1/2$ when $t \to \infty$ for all values of C, the equation is stable (and also globally asymptotically stable), with equilibrium state $y = 1/2$.

10.10 Since the differential equations are linear first order, they have general solutions

$$y = \frac{b}{a} + Ce^{-at}$$

This gives

1. $y = 10 + Ce^{-t}$
2. $y = -9 + Ce^{3t}$
3. $y = 20 + Ce^{-5t/4}$

10.11 Since the first two differential equation are linear first order, they have general solutions

$$y = \frac{b}{a} + Ce^{-at}$$

This gives

1. $y = -5/3 + Ce^{3t}$, and $y(0) = 1$ gives $C = 8/3$
2. $y = -8 + Ce^{-2t/3}$, and $y(0) = 1$ gives $C = 9$

The last equation has integration factor $u = e^{2t}$, and this gives $ye^{2t} = \int t^2 e^{2t}\,dt$. We solve the integral by using integration by parts twice, and get

$$ye^{2t} = (t^2/2 - t/2 + 1/4)e^{2t} + \mathscr{C} \quad \Rightarrow \quad y = t^2/2 - t/2 + 1/4 + \mathscr{C}e^{-2t}$$

The condition $y(0) = 1$ gives $\mathscr{C} = 3/4$.

10.12

1. The differential equation $t\dot{y} + 2y + t = 0$ is linear, and can be written as

$$\dot{y} + \frac{2}{t}y = -1$$

It has integrating factor $e^{2\ln t} = t^2$, and therefore

$$t^2 y = \int -t^2\,dt = -\frac{1}{3}t^3 + C \quad \Leftrightarrow \quad y = -\frac{1}{3}t + \frac{C}{t^2}$$

2. The differential equation $\dot{y} - y/t = t$ is linear with integrating factor $e^{-\ln t} = t^{-1}$, and therefore

$$t^{-1}y = \int 1\,dt = t + C \quad \Leftrightarrow \quad y = t^2 + Ct$$

3. The differential equation $\dot{y} - \dfrac{t}{t^2-1}y = t$ is linear with integrating factor

$$e^{-\ln(t^2-1)/2} = (t^2-1)^{-1/2} \quad \text{since} \int -\frac{t}{t^2-1}\,dt = -\frac{1}{2}\ln(t^2-1)$$

Therefore we obtain

$$(t^2-1)^{-1/2}y = \int t(t^2-1)^{-1/2}\,dt = (t^2-1)^{1/2} + C$$

and this gives

$$y = t^2 - 1 + C(t^2-1)^{1/2}$$

4. The differential equation $\dot{y} - \dfrac{2}{t}y + \dfrac{2a^2}{t^2} = 0$ is linear since it can be written

$$\dot{y} - \frac{2}{t}y = -\frac{2a^2}{t^2}$$

It has integrating factor $e^{-2\ln t} = t^{-2}$, and therefore

$$t^{-2}y = \int -\frac{2a^2}{t^4}\,dt = \frac{2}{3}a^2 t^{-3} + C \quad \Leftrightarrow \quad y = \frac{2a^2}{3t} + Ct^2$$

10.13

1. We consider $(2+y) + (2y+t)\dot{y} = 0$ with $f = 2+y$ and $g = 2y+t$. Since $f'_y = 1 = g'_t$, the equation is exact. We find $h = y^2 + yt + 2t$ satisfy $h'_t = f$ and $h'_y = g$, so the general solution is

$$y^2 + yt + 2t = C \quad \Rightarrow \quad y = \frac{-t \pm \sqrt{t^2 - 8t + 4C}}{2}$$

2. $(2t + y) + y^2\dot{y} = 0$ with $f = 2t + y$ and $g = y^2$. Since $f'_y = 1$ and $g'_t = 0$, the equation is not exact.
3. $(5yt^4 + 2) + (t^5 + 6y^2)\dot{y} = 0$ with $f = 5yt^4 + 2$ and $g = t^5 + 6y^2$. Since $f'_y = 5t^4 = g'_t$, the equation is exact. We find $h = t^5y + 2y^3 + 2t$ satisfy $h'_t = f$ and $h'_y = g$, so the general solution (in implicit form) is

$$t^5y + 2y^3 + 2t = C$$

It is difficult to find the solution in explicit form in this problem (it is a third degree equation in y).

10.14 The differential equation $2t + 3y^2\dot{y} = 0$ can be written as $3y^2\dot{y} = -2t$, and is therefore separable with solution $y^3 = -t^2 + C$, which gives $y = \sqrt[3]{C - t^2}$. We write $f = 2t$ and $g = 3y^2$; then $h = t^2 + y^3$ has the property that $h'_t = f$ and $h'_y = g$, so the equation is exact with solution $t^2 + y^3 = C$, which again gives $y = \sqrt[3]{C - t^2}$.

10.15 Final Exam in GRA6035 30/05/2011, 3c
The differential equation can be written in the form

$$(2t + y) + (t - 4y)y' = 0$$

and we see that it is exact. Hence its solution can be written in the form $u(y,t) = C$, where $u(y,t)$ is a function that satisfies

$$\frac{\partial u}{\partial t} = 2t + y \quad \text{and} \quad \frac{\partial u}{\partial y} = t - 4y$$

One solution is $u(y,t) = t^2 + ty - 2y^2$, and the initial condition $y(0) = 0$ gives $C = 0$. Hence

$$t^2 + ty - 2y^2 = 0 \quad \Leftrightarrow \quad y = \frac{-t \pm 3t}{-4}$$

The solution to the initial value problem is therefore

$$y = -\frac{1}{2}t \text{ or } y = t$$

10.16 Final Exam in GRA6035 10/12/2010, 3c
The differential equation can be written in the form

$$\left(3t^2 - \frac{1}{y}\right) + \frac{t}{y^2}y' = 0$$

and we see that it is exact. Hence it can be written of the form $u(y,t) = C$, where $u(y,t)$ is a function that satisfies

$$\frac{\partial u}{\partial t} = 3t^2 - \frac{1}{y} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{t}{y^2}$$

One solution is $u(y,t) = t^3 - t/y$, and this gives

$$t^3 - \frac{t}{y} = C \quad \Leftrightarrow \quad y = \frac{t}{t^3 - C}$$

The initial condition gives $1/(1-C) = 1/3$ or $C = -2$. The solution to the initial value problem is therefore

$$y = \frac{t}{t^3 + 2}$$

Lecture 11
Second Order Differential Equations

11.1 Main concepts

A *second order differential equation* may contain y'', y', y and t. The general solution of a second order differential equation will depend on *two* free variables. An *initial value problem* consists of the differential equation and two initial conditions, and has a unique solution, called the *particular solution*.

A second order differential equation is *linear* (with constant coefficients) if it can be written in the form

$$y'' + ay' + by = f(t)$$

for some numbers a, b and an expression $f(t)$ in t. It is *homogeneous* if it can be written $y'' + ay' + by = 0$ (the case where $f = 0$), and *inhomogeneous* otherwise.

Proposition 11.1 (Superposition principle). *If y_1 is a solution of the differential equation $y'' + ay' + by = f_1(t)$ and y_2 is a solution of $y'' + ay' + by = f_2(t)$, then the linear combination $c_1y_1 + c_2y_2$ is a solution of $y'' + ay' + by = c_1f_1(t) + c_2f_2(t)$ for any given numbers c_1, c_2.*

Homogeneous case. To solve the differential equation $y'' + ay' + by = 0$, we find the roots of the characteristic equation $r^2 + ar + b = 0$, given by

$$r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

since $y = e^{rt}$ is a solution if and only if r is a root. There are three subcases to consider:

1. If $a^2 - 4b > 0$, there are two distinct roots $r_1 \neq r_2$, and the general solution is

$$y = C_1e^{r_1t} + C_2e^{r_2t}$$

2. If $a^2 - 4b = 0$, there is one double root $r = -a/2$, and the general solution is

$$y = C_1e^{rt} + C_2te^{rt} = (C_1 + C_2t)e^{rt}$$

3. If $a^2 - 4b < 0$, there are no (real) roots (but two complex roots $r = \alpha \pm \beta\sqrt{-1}$). The general solution is

$$y = e^{\alpha t}(C_1 \cos(\beta t) + C_2 \sin(\beta t))$$

where $\alpha = -a/2$ and $\beta = \sqrt{4b - a^2}/2$.

Inhomogeneous case. By the superposition principle, the differential equation $y'' + ay' + by = f(t)$ has general solution of the form $y = y_h + y_p$, where y_h is the general solution of the corresponding *homogeneous* equation $y'' + ay' + by = 0$, and y_p is a *particular* solution of $y'' + ay' + by = f(t)$.

To find a particular solution y_p of $y'' + ay' + by = f(t)$, we may consider $f(t), f'(t), f''(t)$ and guess a solution $y = y_p$ such that

- The guess for y contains parameter that we may later adjust
- The guess for y has the same form as $f(t), f'(t), f''(t)$

When a guess for y is given, we compute y', y'' and insert it in $y'' + ay' + by = f(t)$ to see if it is a solution for any choices of the parameters. In that case, we have found a particular solution y_p. If the initial guess for y does not work, we may try to multiply it by t (and thereafter by t^2, t^3, etc if necessary).

Superposition principle for first order linear differential equations. We may also use the superposition principle to solve the first order linear differential equation $y' + ay = b(t)$, where a is a number and $b(t)$ is an expression in t. This is an alternative to the method using integrating factor used in Lecture 10. We obtain the solution $y = y_h + y_p$, and $y_h = Ce^{-at}$ since the characteristic equation in this case is $r + a = 0$ with root $r = -a$. The general solution is therefore

$$y = Ce^{-at} + y_p$$

where y_p is a particular solution of $y' + ay = b(t)$.

11.2 Problems

11.1. Find the general solution of the following differential equations:

1. $\ddot{y} = t$
2. $\ddot{y} = e^t + t^2$

11.2. Solve the initial value problem $\ddot{y} = t^2 - t, \quad y(0) = 1,\ \dot{y}(0) = 2.$

11.3. Solve the initial value problem $\ddot{y} = \dot{y} + t, \quad y(0) = 1,\ y(1) = 2.$

11.4. Find the general solution of the following differential equations:

1. $\ddot{y} - 3\dot{y} = 0$
2. $\ddot{y} + 4\dot{y} + 8y = 0$

3. $3\ddot{y}+8\dot{y}=0$
4. $4\ddot{y}+4\dot{y}+y=0$
5. $\ddot{y}+\dot{y}-6y=0$
6. $\ddot{y}+3\dot{y}+2y=0$

11.5. Consider the equation $\ddot{y}+a\dot{y}+by=0$ when $a^2/4-b=0$, so that that the characteristic equation has a double root $r=-a/2$. Let $y(t)=u(t)e^{rt}$ and prove that this function is a solution if and only if $\ddot{u}=0$. Conclude that the general solution is $y=(A+Bt)e^{rt}$.

11.6. Find the general solutions

1. $\ddot{x}-x=e^{-t}$
2. $3\ddot{x}-30\dot{x}+75x=2t+1$

11.7. Solve

1. $\ddot{x}+2\dot{x}+x=t^2$, $x(0)=0$, $\dot{x}(0)=1$
2. $\ddot{x}+4x=4t+1$, $x(\frac{\pi}{2})=0$, $\dot{x}(\frac{\pi}{2})=0$

11.8. Find the general solutions of the following equations for $t>0$. Hint: Insert $x=t^r$ into the differential equation and determine for which values of r this is a solution.

1. $t^2\ddot{x}+5t\dot{x}+3x=0$
2. $t^2\ddot{x}-3t\dot{x}+3x=t^2$

11.9. Solve the differential equation $\ddot{x}+2a\dot{x}-3a^2x=100e^{bt}$ for all values of the constants a and b.

11.10. Final Exam in GRA6035 30/05/2011, 3b
Find the general solution of the differential equation $y''+2y'-35y=11e^t-5$.

11.11. Final Exam in GRA6035 10/12/2010, 3b
Find the general solution of the differential equation $y''+y'-6y=te^t$.

11.12. Final Exam in GRA6035 10/12/2007, Problem 3

1. Find the solution of $\dot{x}=(t-2)x^2$ that satisfies $x(0)=1$.
2. Find the general solution of the differential equation $\ddot{x}-5\dot{x}+6x=e^{7t}$.
3. Find the general solution of the differential equation $\dot{x}+2tx=te^{-t^2+t}$.
4. Find the solution of $3x^2e^{x^3+3t}\dot{x}+3e^{x^3+3t}-2e^{2t}=0$ with $x(1)=-1$.

11.13. Mock Final Exam in GRA6035 12/2010, Problem 3ab

1. Find the solution of $y'=y(1-y)$ that satisfies $y(0)=1/2$.

2. Find the general solution of the differential equation

$$\big(\ln(t^2+1)-2\big)\,y' = 2t - \frac{2ty}{t^2+1}$$

11.14. Final Exam in GRA6035 12/12/2011, Problem 3
We consider the differential equation $(x+1)t\dot{x}+(t+1)x=0$ with initial condition $x(1)=1$.

1. Show that the differential equation is separable, and use this to find an implicit expression for $x=x(t)$. In other words, find an equation of the form

$$F(x,t)=A$$

 that defines $x=x(t)$ implicitly. It is not necessary to solve this equation for x.
2. Show that the differential equation becomes exact after multiplication with e^{x+t}. Use this to find an implicit expression for $x=x(t)$. In other words, find an equation of the form

$$G(x,t)=B$$

 that defines $x=x(t)$ implicitly. It is not necessary to solve this equation for x.

11.15. Final Exam in GRA6035 06/02/2012, Problem 3

1. Find the general solution $y=y(t)$ of the differential equation $y''+3y'-10y=2t$.
2. Find the general solution $y=y(t)$ of the differential equation $2t+3y^2y'=2te^{t^2}$.
3. Find the solution $y=y(t)$ of the differential equation $y^2+2ty\cdot y'=1$ that satisfies $y(1)=3$.

11.16. Final Exam in GRA6035 06/06/2012, Problem 2
Find the general solution $y=y(t)$ of the following differential equations:

1. $y''-7y'+12y=t-3$
2. $1-3y^2y'=te^t$
3. $(t/y)\cdot y'+\ln y=1$

11.3 Solutions

11.1 We solve the differential equation by direct integration:

1. $\ddot{y}=t \implies \dot{y}=\frac{1}{2}t^2+C_1 \implies y=\frac{1}{6}t^3+C_1t+C_2$
2. $\ddot{y}=e^t+t^2 \implies \dot{y}=e^t+\frac{1}{3}t^3+C_1 \implies y=e^t+\frac{1}{12}t^4+C_1t+C_2$

11.2 We have $\ddot{y}=t^2-t \implies \dot{y}=\frac{1}{3}t^3-\frac{1}{2}t^2+C_1 \implies y=\frac{1}{12}t^4-\frac{1}{6}t^3+C_1t+C_2$. The initial condition $y(0)=1$ gives $\frac{1}{12}0^4-\frac{1}{6}0^3+C_10+C_2=C_2=1$, and $\dot{y}(0)=2$ gives $\frac{1}{3}0^3-\frac{1}{2}0+C_1=C_1=2$. The particular solution is therefore

$$y(t) = \frac{1}{12}t^4 - \frac{1}{6}t^3 + 2t + 1$$

11.3 Substitute $u = \dot{y}$. Then $\ddot{y} = \dot{y} + t \Leftrightarrow \dot{u} = u + t \Leftrightarrow \dot{u} - u = t$. The integrating factor is e^{-t}, and we get

$$ue^{-t} = \int te^{-t}dt = -e^{-t} - te^{-t} + C_1$$

From this we obtain $u = (-e^{-t} - te^{-t} + C_1)e^t = C_1e^t - t - 1$ and we integrate to find y from $u = \dot{y}$, and get $y = \int(C_1e^t - t - 1)dt = C_1e^t - t - \frac{1}{2}t^2 + C_2$. The initial condition $y(0) = 1$ gives $C_1 + C_2 = 1 \implies C_2 = 1 - C_1$, and the condition $y(1) = 2$ gives $C_1e - 1 - \frac{1}{2} + C_2 = C_1e - 3/2 + 1 - C_1 = 2$. This gives $C_1(e-1) = 5/2$, or

$$C_1 = \frac{5}{2(e-1)}, \quad C_2 = 1 - \frac{5}{2(e-1)} = \frac{2e-7}{2(e-1)}$$

The particular solution is therefore

$$y(t) = \frac{5}{2(e-1)} \cdot e^t - t - \frac{1}{2}t^2 + \frac{2e-7}{2(e-1)}$$

11.4

1. The characteristic equation is $r^2 - 3r = 0 \implies r = 0, 3 \implies y(t) = C_1 + C_2e^{3t}$.
2. Characteristic equation is $r^2 + 4r + 8 = 0$. This has no real solutions. Thus we put $\alpha = -\frac{1}{2}a = -\frac{1}{2}4 = -2, \beta = \sqrt{b - \frac{1}{4}a^2} = \sqrt{8 - \frac{1}{4}4^2} = 2$. From this the general solution is $y(t) = e^{\alpha t}(A\cos\beta t + B\sin\beta t) = e^{-2t}(A\cos 2t + B\sin 2t)$.
3. $3\ddot{y} + 8\dot{y} = 0 \iff \ddot{y} + \frac{8}{3}\dot{y} = 0$. The characteristic equation is $r^2 + \frac{8}{3}r = 0 \implies r = 0$ or $r = -\frac{8}{3}$. The general solution is $y(t) = C_1e^{0t} + C_2e^{-\frac{8}{3}t} = C_1 + C_2e^{-\frac{8}{3}t}$.
4. $4\ddot{y} + 4\dot{y} + y = 0$ has characteristic equation $4r^2 + 4r + 1 = 0$. There is one solution $r = -\frac{1}{2}$. The general solution is $y(t) = (C_1 + C_2t)e^{-\frac{1}{2}t}$.
5. $\ddot{y} + \dot{y} - 6y = 8$ has characteristic equation $r^2 + r - 6 = 0$. It has the solutions $r = -3$ and $r = 2$. The general solution is thus
$$y(t) = C_1e^{-3t} + C_2e^{2t}$$
6. $\ddot{y} + 3\dot{y} + 2y = 0$ has characteristic equation $r^2 + 3r + 2 = 0$. The solutions are $r = -1$ and $r = -2$. The general solution is thus
$$y(t) = C_1e^{-t} + C_2e^{-2t}$$

11.5 $y = ue^{rt} \implies \dot{y} = \dot{u}e^{rt} + ure^{rt} = e^{rt}(\dot{u} + ur) \implies \ddot{y} = \ddot{u}e^{rt} + \dot{u}re^{rt} + r(\dot{u}e^{rt} + ure^{rt}) = e^{rt}(\ddot{u} + 2r\dot{u} + ur^2)$. From this we get

$$\begin{aligned}\ddot{y} + a\dot{y} + by &= e^{rt}[(\ddot{u} + 2r\dot{u} + ur^2) + a(\dot{u} + ur) + bu] \\ &= e^{rt}[\ddot{u} + (2r + a)\dot{u} + (r^2 + ar + b)u]\end{aligned}$$

The characteristic equation is assumed to have one solution $r = \frac{-a}{2}$. Putting $r = \frac{-a}{2}$ into the eypression we get

$$\ddot{y} + a\dot{y} + by = e^{rt}\ddot{u}$$

So $y = ue^{rt}$ is a solution if and only if $e^{rt}\ddot{u} = 0 \Leftrightarrow \ddot{u} = 0$. The differential equation $\ddot{u} = 0$ has the general solution $u = A + Bt$. Thus $y = (A+Bt)e^{rt}$ is the general solution of $\ddot{y} + a\dot{y} + by = 0$.

11.6

1. We first solve $\ddot{y} - y = 0$. The characteristic equation is $r^2 - 1 = 0$. We get $y_h = C_1e^{-t} + C_2e^t$. To find a solution of $\ddot{y} - y = e^{-t}$, we guess on solution of the form $y_p = Ae^{-t}$. We have $\dot{y}_p = -Ae^{-t}$ and $\ddot{y}_p = Ae^{-t}$. Putting this into the left hand side of the equation, we get

$$Ae^{-t} - (Ae^{-t}) = 0$$

So this does not work. The reason is that e^{-t} is a solution of the homogenous equation. We try something else: $y_p = Ate^{-t}$. This gives

$$\begin{aligned} \dot{y}_p &= A(e^{-t} - te^{-t}) \\ \ddot{y}_p &= A(-e^{-t} - (e^{-t} - te^{-t})) \\ &= Ae^{-t}(t-2) \end{aligned}$$

Putting this into the left hand side of the equation, we obtain

$$\begin{aligned} \ddot{y}_p - y_p &= Ae^{-t}(t-2) - Ate^{-t} \\ &= -2Ae^{-t} \end{aligned}$$

We get a solution for $A = -\frac{1}{2}$. Thus the general solution is

$$y(t) = -\frac{1}{2}te^{-t} + C_1e^{-t} + C_2e^t$$

2. The equation is equivalent to

$$\ddot{y} - 10\dot{y} + 25y = \frac{2}{3}t + \frac{1}{3}$$

We first solve the homogenous equation for which the characteristic equation is

$$r^2 - 10r + 25 = 0$$

This has one solution $r = 5$. The general homogenous solution is thus

$$y_h = (C_1 + C_2t)e^{5t}$$

To find a particular solution, we try

$$y_p = At + B$$

We have $\dot{y}_p = A$ and $\ddot{y}_p = 0$. Putting this into the equation, we obtain

$$0 - 10A + 25(At + B) = \frac{2}{3}t + \frac{1}{3}$$

We obtain $25A = \frac{2}{3}$ and $-10A + 25B = \frac{1}{3}$. From this we get $A = \frac{2}{75}$ and $-\frac{20}{75} + 25B = \frac{25}{75} \implies B = \frac{45}{25 \cdot 75} = \frac{3}{125}$. Thus

$$y(t) = \frac{2}{75}t + \frac{3}{125} + (C_1 + C_2 t)e^{5t}$$

11.7

1. We first solve the homogenous equation $\ddot{y} + 2\dot{y} + y = 0$. The characteristic equation is $r^2 + 2r + 1 = 0$ which has the one solution, $r = -1$. We get

$$y_h(t) = (C_1 + C_2 t)e^{-t}.$$

To find a particular solution we try with $y_p = At^2 + Bt + C$. We get $\dot{y}_p = 2At + B$ and $\ddot{y}_p = 2A$. Substituting this into the left hand side of the equation, we get

$$\begin{aligned} &2A + 2(2At + B) + (At^2 + Bt + C) \\ &= 2A + 2B + C + (4A + B)t + At^2 \end{aligned}$$

We get $A = 1$, $(4A + B) = 0$ and $2A + 2B + C = 0$. We obtain $A = 1$, $B = -4$ and $C = -2A - 2B = -2 + 8 = 6$. Thus the general solution is

$$y(t) = t^2 - 4t + 6 + (C_1 + C_2 t)e^{-t}.$$

We get $\dot{y} = 2t - 4 + C_2 e^{-t} + (C_1 + C_2 t)e^{-t}(-1) = 2t - C_1 e^{-t} + C_2 e^{-t} - tC_2 e^{-t} - 4$. From $y(0) = 0$ we get $6 + C_1 = 0 \implies C_1 = -6$. From $\dot{y}(0) = 1$, we get $-C_1 + C_2 - 4 = 1 \implies C_2 = 5 + C_1 = 5 - 6 = -1$. Thus we have

$$y(t) = t^2 - 4t + 6 - (6 + t)e^{-t}.$$

2. We first solve the homogenous equation $\ddot{y} + 4y = 0$. The characteristic equation $r^2 + 4 = 0$ has no solutions, so we put $\alpha = -\frac{1}{2}0 = 0$ and $\beta = \sqrt{4 - \frac{1}{2}0} = 2$. This gives $y_h = e^{\alpha t}(C_1 \cos\beta t + C_2 \sin\beta t) = C_1 \cos 2t + C_2 \sin 2t$. To find a solution of $\ddot{y} + 4y = 4t + 1$ we try $y_p = A + Bt$. This gives $\dot{y}_p = B$ and $\ddot{y}_p = 0$. Putting this into the equation, we find that

$$\ddot{y}_p + 4y_p = 0 + 4(A + Bt) = 4A + 4Bt = 4t + 1.$$

This implies that $B = 1$ and $A = \frac{1}{4}$. Thus the general solution is

$$y(t) = C_1 \cos 2t + C_2 \sin 2t + \frac{1}{4} + t$$

We compute $y'(t) = -2C_1 \sin(2t) + 2C_2 \cos(2t) + 1$, and the particular solution is given by $y(\pi/2) = C_1 \cdot (-1) + C_2 \cdot 0 + 1/4 + \pi/2 = 0$ and $y'(\pi/2) = -2 \cdot 0 + 2C_2 \cdot (-1) + 1 = 0$. This gives $C_1 = \pi/2 + 1/4$ and $C_2 = 1/2$, and particular solution

$$y(t) = \frac{2\pi + 1}{4} \cos(2t) + \frac{1}{2} \sin(2t) + \frac{1}{4} + t$$

11.8 We have the following solutions:

1. Inserting $x = t^r$ into $t^2\ddot{x} + 5t\dot{x} + 3x = 0$ gives

$$r(r-1)t^r + 5rt^r + 3t^r = (r^2 - r + 5r + 3)t^r = 0$$

   or $r^2 + 4r + 3 = 0$, which has roots $r = -1$ and $r = -3$. Hence the general solution is $x = C_1 t^{-1} + C_2 t^{-3}$.
2. The general solution is given by $x = x_h + x_p$. To find the homogeneous solution x_h, we insert $x = t^r$ into $t^2\ddot{x} - 3t\dot{x} + 3x = 0$ and get

$$r(r-1)t^r - 3rt^r + 3t^r = (r^2 - r - 3r + 3)t^r = 0$$

   or $r^2 - 4r + 3 = 0$, which has roots $r = 1$ and $r = 3$. Hence the homogeneous solution is $x = C_1 t + C_2 t^3$. To find a particular solution of $t^2\ddot{x} - 3t\dot{x} + 3x = t^2$, we try $x_p = At^2$, which gives

$$A \cdot 2t^2 - 3A \cdot 2t^2 + 3A \cdot t^2 = t^2(A - 3A + 3A) = t^2$$

   Hence $2A - 6A + 3A = 1$, or $A = -1$. Therefore $x_p = t^2$ and $x = C_1 t + C_2 t^3 - t^2$.

Alternative solution:

1. Substituting $t = e^s$ transforms the equation into $y''(s) + 4y'(s) + 3y'(s) = 0$. The characteristic equation is $r^2 + 4r + 3 = 0$. The solutions are $r = -3, -1$. Thus $y(s) = C_1 e^{-3t} + C_2 e^{-t}$. Substituting $s = \ln t$ gives $y(t) = C_1 t^{-3} + C_2 t^{-1}$.
2. Substituting $t = e^s$ transforms the equation into $y''(s) - 4y'(s) + 3y'(s) = (e^s)^2 = e^{2s}$. First we solve the homogenous equation $y''(s) - 5y'(s) + 3y'(s) = 0$. The characteristic equation is $r^2 - 4r + 3 = 0$, and has the solutions $r = 1$ and $r = 3$. Thus $y_h = C_1 e^s + C_2 e^{3s}$. To find a particular solution of $y''(s) - 4y'(s) + 3y(s) = (e^s)^2 = e^{2s}$ we try $y_p = Ae^{2s}$. We have $y_p' = 2Ae^{2s}$ and $y_p'' = 4Ae^{2s}$. Substituting this into the equation, gives

$$\begin{aligned} y''(s) - 4y'(s) + 3y(s) &= 4Ae^{2s} - 4 \cdot 2Ae^{2s} + 3 \cdot Ae^{2s} \\ &= -Ae^{2s} \end{aligned}$$

   Thus we get $A = -1$, and

$$y(s) = C_1 e^s + C_2 e^{3s} - e^{2s}$$

Substituting $s = \ln t$ gives

$$y(t) = C_1 t + C_2 t^3 - t^2.$$

11.9 If $a \neq 0$ we get the general solution

$$y = 100\frac{e^{bt}}{2ab - 3a^2 + b^2} + C_1 e^{at} + C_2 e^{-3at}$$

provided that $2ab - 3a^2 + b^2 \neq 0$. When $a = 0$ and $b \neq 0$ we get the general solution

$$y = C_1 + \frac{100}{b^2} e^{bt} + C_2 t$$

There are also some other cases to consider, see answers in FMEA ey.6.3.9.

11.10 Final Exam in GRA6035 30/05/2011, 3b

The homogeneous equation $y'' + 2y' - 35y = 0$ has characteristic equation $r^2 + 2r - 35 = 0$ and roots $r = 5$ and $r = -7$, so $y_h = C_1 e^{5t} + C_2 e^{-7t}$. We try to find a particular solution of the form $y = Ae^t + B$, which gives

$$y' = y'' = Ae^t$$

Substitution in the differential equation gives

$$Ae^t + 2Ae^t - 35(Ae^t + B) = 11e^t - 5 \Leftrightarrow -32A = 11 \text{ and } -35B = -5$$

This gives $A = -11/32$ and $B = 1/7$. Hence the general solution of the differential equation is $y = y_h + y_p = C_1 e^{5t} + C_2 e^{-7t} - \frac{11}{32} e^t + \frac{1}{7}$

11.11 Final Exam in GRA6035 10/12/2010, 3b

The homogeneous equation $y'' + y' - 6y = 0$ has characteristic equation $r^2 + r - 6 = 0$ and roots $r = 2$ and $r = -3$, so $y_h = C_1 e^{2t} + C_2 e^{-3t}$. We try to find a particular solution of the form $y = (At + B)e^t$, which gives

$$y' = (At + A + B)e^t, \quad y'' = (At + 2A + B)e^t$$

Substitution in the differential equation gives

$$(At + 2A + B)e^t + (At + A + B)e^t - 6(At + B)e^t = te^t \Leftrightarrow -4A = 1 \text{ and } 3A - 4B = 0$$

This gives $A = -1/4$ and $B = -3/16$. Hence the general solution of the differential equation is $y = y_h + y_p = C_1 e^{2t} + C_2 e^{-3t} - (\frac{1}{4}t + \frac{3}{16})e^t$

11.12 Final Exam in GRA6035 10/12/2007, Problem 3

1. We have $\dot{x} = (t-2)x^2 \implies \frac{1}{x^2}\dot{x} = t-2 \implies \int \frac{1}{x^2}dx = \int(t-2)dt \implies -\frac{1}{x} = \frac{1}{2}t^2 - 2t + C \implies x = \frac{-2}{t^2-4t+2C}$. The initial condition $x(0) = \frac{-2}{2C} = \frac{-1}{C} = 1 \implies C = -1 \implies x(t) = \frac{-2}{t^2-4t-2}$.
2. We have $\ddot{x} - 5\dot{x} + 6x = 0, r^2 - 5r + 6 = 0 \implies r = 3, r = 2 \implies x_h(t) = Ae^{2t} + Be^{3t}$, and $x_p = Ce^{7t} \implies \dot{x}_p = 7Ce^{7t} \implies \ddot{x}_p = 49Ce^{7t}$ gives $\ddot{x}_p - 5\dot{x}_p + 6x_p = Ce^{7t}(49 - 5\cdot 7 + 6) = 20Ce^{7t} = 1 \implies C = \frac{1}{20}$. Hence $x(t) = Ae^{2t} + Be^{3t} + \frac{1}{20}e^{7t}$.
3. Integrating factor $e^{t^2} \implies xe^{t^2} = \int te^{-t^2+t}e^{t^2}dt = \int te^t dt = te^t - e^t + C \implies x(t) = (te^t - e^t + C)e^{-t^2}$.
4. We have $\frac{\partial}{\partial t}(3x^2e^{x^3+3t}) = 9x^2e^{3t+x^3}$ and $\frac{\partial}{\partial x}(3e^{x^3+3t} - 2e^{2t}) = 9x^2e^{3t+x^3}$, so the differential equation is exact. We look for h with $h'_x = 3x^2e^{x^3+3t} \implies h = e^{x^3+3t} + \alpha(t) \implies h'_t = 3e^{x^3+3t} + \alpha'(t)$. But $h'_t = 3e^{x^3+3t} + \alpha'(t) = 3e^{x^3+3t} - 2e^{2t} \implies \alpha'(t) = -2e^{2t} \implies \alpha(t) = -e^{2t} + C \implies h = e^{x^3+3t} - e^{2t} + C$. This gives solution in implicit form
$$h = e^{x^3+3t} - e^{2t} = K$$
The initial condition $x(1) = -1 \implies e^{(-1)^3+3} - e^2 = K \implies K = 0 \implies e^{x^3+3t} - e^{2t} = 0 \implies e^{x^3+3t} = e^{2t} \implies x^3 + 3t = 2t \implies x^3 = -t \implies x(t) = \sqrt[3]{-t}$.

11.13 Mock Final Exam in GRA6035 12/2010, Problem 3ab

1. We have that $y' = y(1-y)$ is separable, and we after separating the variables we get the equation
$$\int \frac{1}{y(1-y)}\,dy = \int 1\,dt = t + C$$
To solve the first integral, we rewrite the integrand as
$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y} \quad \Leftrightarrow \quad 1 = A(1-y) + By$$
which has solution $A = 1, B = 1$. Hence we obtain the implicit solution
$$\int \frac{1}{y} + \frac{1}{1-y}\,dy = \ln|y| - \ln|1-y| = t + C$$
This means that
$$\ln\left|\frac{y}{1-y}\right| = t + C \quad \Leftrightarrow \quad \frac{y}{1-y} = \pm e^{t+C} = Ke^t$$
with $K = \pm e^C$. The initial condition $y(0) = 1/2$ gives $1 = Ke^0$, or $K = 1$. We solve for y and obtain the solution
$$y = e^t(1-y) \quad \Leftrightarrow \quad y(1+e^t) = e^t \quad \Leftrightarrow \quad y = \frac{e^t}{1+e^t}$$
2. We chech if the differential equation is exact by writing it in the form

$$\big(\ln(t^2+1)-2\big)\,y' + \frac{2ty}{t^2+1} - 2t = 0$$

and try to find a function $h = h(y,t)$ such that $h'_y = \ln(t^2+1) - 2$ and $h'_t = 2ty/(t^2+1) - 2t$. From the first equation, we obtain that h must have the form

$$h(y,t) = y\ln(t^2+1) - 2y + \phi(t)$$

and therefore

$$h'_t = y\frac{2t}{t^2+1} + \phi'(t) = 2ty/(t^2+1) - 2t$$

We see that the equation is exact since we may choose $\phi(t) = -t^2$, and we find the implicit solution

$$y\ln(t^2+1) - 2y - t^2 = C \quad \Leftrightarrow \quad y = \frac{C+t^2}{\ln(t^2+1)-2}$$

11.14 Final Exam in GRA6035 12/12/2011, Problem 3

1. We re-write the differential equation as

$$(x+1)t\dot{x} + (t+1)x = 0 \quad \Rightarrow \quad (x+1)\dot{x} = -\frac{(t+1)x}{t} \quad \Rightarrow \quad \frac{x+1}{x}\dot{x} = -\frac{t+1}{t}$$

This differential equation is separated, so the original difference equation is **separable**. We integrate on both sides:

$$\int (1+\frac{1}{x})\,dx = -\int (1+\frac{1}{t})\,dt \quad \Rightarrow \quad x + \ln(|x|) = -(t+\ln(|t|)) + \mathscr{C}$$

The initial condition $x(1) = 1$ gives $1 + \ln 1 = -1 - \ln 1 + \mathscr{C}$, or $\mathscr{C} = 2$. This solution can therefore be described implicitly by the equation

$$\mathbf{x} + \mathbf{t} + \ln|\mathbf{x}| + \ln|\mathbf{t}| = \mathbf{2}$$

It is not necessary (or possible) to solve this equation for x.

2. We try to multiply the differential equation by e^{x+t} and get the new differential equation

$$(x+1)te^{x+t}\dot{x} + (t+1)xe^{x+t} = P(x,t)\dot{x} + Q(x,t) = 0$$

with $P(x,t) = (x+1)te^{x+t}$ and $Q(x,t) = (t+1)xe^{x+t}$. We have

$$P'_t = (x+1)e^{x+t} + t(x+1)e^{x+t} = (t+1)(x+1)e^{x+t}$$

and

$$Q'_x = (t+1)e^{x+t} + x(t+1)e^{x+t} = (t+1)(x+1)e^{x+t}$$

We see that $P'_t = Q'_x$, and it follows that the new differential equation is **exact**. To solve it, we find a function $h(x,t)$ such that $h'_x = P(x,t)$ and $h'_t = Q(x,t)$. The first equation gives

$$h'_x = P(x,t) = (x+1)te^{x+t} \quad \Rightarrow \quad h = \int (x+1)te^{x+t}\,dx = te^t \int (x+1)e^x\,dx$$

Using integration by parts, we find

$$\int (x+1)e^x\,dx = (x+1)e^x - \int 1 \cdot e^x\,dx = (x+1)e^x - e^x + \mathscr{C} = xe^x + \mathscr{C}$$

This implies that

$$h = te^t \int (x+1)e^x\,dx = te^t xe^x + \mathscr{C}(t) = txe^{x+t} + \mathscr{C}(t)$$

where $\mathscr{C}(t)$ is a function of t (or a constant considered as a function in x). The second equation is $h'_t = Q(x,t)$, and we use the expression above for h:

$$h'_t = Q(x,t) \quad \Rightarrow \quad xe^{x+t} + txe^{x+t} + \mathscr{C}'(t) = (t+1)xe^{x+t} + \mathscr{C}'(t) = (t+1)xe^{x+t}$$

We see that this condition holds if and only if $\mathscr{C}'(t) = 0$, or if $\mathscr{C} = C_1$ is a constant. In conclusion, we may choose $h = txe^{x+t} + C_1$, and the general solution of the exact differential equation is $h = C_2$, where C_2 is another constant. This gives

$$txe^{x+t} = B$$

where $B = C_2 - C_1$ is a new constant. The initial condition is $x(1) = 1$, and this gives $1 \cdot e^2 = B$, or $B = e^2$. The solution can therefore be written in implicit form as

$$\mathbf{txe^{x+t} = e^2}$$

It is not necessary (or possible) to solve this equation for x. (If we first take absolute values on both sides of the equation, and then the natural logarithm, we obtain the equation from question a).

11.15 Final Exam in GRA6035 06/02/2012, Problem 3

1. The homogeneous equation $y'' + 3y' - 10y = 0$ has characteristic equation $r^2 + 3r - 10 = 0$, and therefore roots $r = 2, -5$. Hence the homogeneous solution is $y_h(t) = C_1e^{2t} + C_2e^{-5t}$. To find a particular solution of $y'' + 3y' - 10y = 2t$, we try $y = At + B$. This gives $y' = A$ and $y'' = 0$, and substitution in the equation gives $3A - 10(At + B) = 2t$. Hence $A = -1/5$ and $B = -3/50$ is a solution, and $y_p(t) = -\frac{1}{5}t - \frac{3}{50}$ is a particular solution. This gives general solution

$$y(t) = \mathbf{C_1e^{2t} + C_2e^{-5t} - \frac{1}{5}t - \frac{3}{50}}$$

2. We re-write the differential equation as

$$3y^2y' = 2te^{t^2} - 2t$$

This differential equation is separable, and we integrate on both sides to solve it:

$$\int 3y^2\,dy = \int (2te^{t^2} - 2t)\,dt \quad \Rightarrow \quad y^3 = e^{t^2} - t^2 + \mathscr{C} \quad \Rightarrow \quad y = \sqrt[3]{e^{t^2} - t^2 + \mathscr{C}}$$

3. We rewrite the differential equation as $y^2 - 1 + 2ty \cdot y' = 0$, and try to find a function $u = u(y,t)$ such that $u'_t = y^2 - 1$ and $u'_y = 2ty$ to find out if the equation is exact. We see that $u = y^2t - t$ is a solution, so the differential equation is exact, with solution $y^2t - t = \mathscr{C}$. The initial condition $y(1) = 3$ gives $9 - 1 = \mathscr{C}$, or $\mathscr{C} = 8$. The solution is therefore

$$t(y^2 - 1) = 8 \quad \Rightarrow \quad y = \sqrt{\frac{\mathbf{8}}{\mathbf{t}} + \mathbf{1}}$$

11.16 Final Exam in GRA6035 06/06/2012, Problem 2

1. The homogeneous equation $y'' - 7y' + 12y = 0$ has characteristic equation $r^2 - 7r + 12 = 0$, and therefore roots $r = 3, 4$. Hence the homogeneous solution is $y_h(t) = C_1e^{3t} + C_2e^{4t}$. To find a particular solution of $y'' - 7y' + 12y = t - 3$, we try $y = At + B$. This gives $y' = A$ and $y'' = 0$, and substitution in the equation gives $-7A + 12(At + B) = t - 3$. Hence $A = 1/12$ and $B = -29/144$ is a solution, and $y_p(t) = \frac{1}{12}t - \frac{29}{144}$ is a particular solution. This gives general solution

$$y(t) = \mathbf{C_1e^{3t}} + \mathbf{C_2e^{4t}} + \frac{\mathbf{1}}{\mathbf{12}}\mathbf{t} - \frac{\mathbf{29}}{\mathbf{144}}$$

2. We rewrite the differential equation as $3y^2y' = 1 - te^t$. This differential equation is separable, and we integrate on both sides to solve it:

$$\int 3y^2\,dy = \int 1 - te^t\,dt \quad \Rightarrow \quad y^3 = t - \int te^t\,dt = t - (te^t - e^t) + \mathscr{C} = t - te^t + e^t + \mathscr{C}$$

This gives

$$y = \sqrt[3]{t - te^t + e^t + \mathscr{C}}$$

3. We rewrite the differential equation as $(t/y) \cdot y' + (\ln y - 1) = 0$, and try to find a function $u = u(y,t)$ such that $u'_t = \ln y - 1$ and $u'_y = t/y$ to find out if the equation is exact. We see that $u = t\ln y - t$ is a solution, so the differential equation is exact, with solution $t\ln y - t = \mathscr{C}$ or $\ln y - 1 = \mathscr{C}/t$. The solution is therefore

$$\ln y = \frac{\mathscr{C}}{t} + 1 \quad \Rightarrow \quad y = \mathbf{e}^{\mathscr{C}/\mathbf{t}+\mathbf{1}}$$

Lecture 12
Difference Equations

12.1 Main concepts

A *difference equation* is an equation relating the general term y_t in the sequence $\{y_t : t = 0, 1, 2, \dots\}$ with one or more of the term preceeding y_t. In general, it has the form

$$y_t = F(t, y_{t-1}, y_{t-2}, \dots, y_{t-d})$$

for some positive integer $d \geq 1$ called the *order* of the difference equation. A difference equation is also called a *recurrence relation*.

Notice that we write y_t instead of $y(t)$ for sequences (that is, when y is a function of t, where t only takes a discrete set of values $t = 0, 1, 2, \dots$ instead of a continuous set of values). Also notice that any difference equation can be transformed to such a form that y_t is expressed in terms of $y_{t-1}, \dots, y_{t-d}$. For instance, the difference equation $y_{t+1} - 2y_t = t$ can be tranformed to $y_t - 2y_{t-1} = t - 1$ be replacing t with $t - 1$.

A first order difference equation is a difference equation that relates two consequtive terms in the sequence. It can be written as

$$y_t = F(t, y_{t-1})$$

The general solution will depend on one parameter, and we therefore need one initial condition to determine a unique solution. A second order difference equation has the form

$$y_t = F(t, y_{t-1}, y_{t-2})$$

The general solution will depend on two parameters, and we therefore need two initial condition to determine a unique solution.

A *first order linear* difference equation (with constant coefficients) is a first order difference equation that has the form

$$y_{t+1} + a y_t = f_t$$

where a is a given number and f_t is an expression in t. It is called homogenous if $f_t = 0$, in which case it has solution

$$y_{t+1} + ay_t = 0 \quad \Leftrightarrow \quad y_t = C(-a)^t$$

Otherwise, it is called inhomogeneous, and has solution

$$y_t = y_t^h + y_t^p = C(-a)^t + y_t^p$$

where y_t^p is a particular solution of $y_{t+1} + ay_t = f_t$.

A *second order linear* difference equation (with constant coefficients) has the form

$$y_{t+2} + ay_{t+1} + by_t = f_t$$

where a, b are given numbers and f_t is an expression in t. It is called homogenous if $f_t = 0$. In this case, we consider the characteristic equation $r^2 + ar + b = 0$, and notice that r is a root in this equation if and only if r^t is a solution of the difference equation. There are three cases to consider:

1. If $a^2 - 4b > 0$, there are two distinct roots $r_1 \neq r_2$, and the general solution is

$$y = C_1 r_1^t + C_2 r_2^t$$

2. If $a^2 - 4b = 0$, there is one double root $r = -a/2$, and the general solution is

$$y = C_1 r^t + C_2 t r^t = (C_1 + C_2 t) r^t$$

3. If $a^2 - 4b < 0$, there are no (real) roots (but two complex roots). The general solution is

$$y = (\sqrt{b})^t (C_1 \cos(\theta t) + C_2 \sin(\theta t))$$

   where θ is a number such that $\cos(\theta) = -\frac{a}{2\sqrt{b}}$.

If f_t is non-zero, the difference equation is called inhomogeneous, and it has solution

$$y_t = y_t^h + y_t^p$$

where y_t^h is the general solution of the homogeneous equation $y_{t+2} + ay_{t+1} + by_t = 0$, and y_t^p is a particular solution of $y_{t+2} + ay_{t+1} + by_t = f_t$.

To find a particular solution y_t^p of a linear difference equation, we may consider f_t, f_{t+1}, f_{t+2} and guess a solution $y_t = y_t^p$ such that

- The guess for y contains parameter that we may later adjust
- The guess for y has the same form as f_t, f_{t+1}, f_{t+2}

When a guess for y_t is given, we compute y_{t+1}, y_{t+2} and insert it in the difference equation to see if it is a solution for any choices of the parameters. In that case, we have found a particular solution y^p. If the initial guess for y does not work, we may try to multiply it by t (and thereafter by t^2, t^3, etc if necessary).

A differential or difference equation is *globally asymptotically stable* if its general solution y (with $y = y(t)$ for a differential equation or $y = y_t$ for a difference equation) satisfy

$$\overline{y} = \lim_{t\to\infty} y$$

exists (that is, $\overline{y}$ it is finite) and is independent of the undetermined coefficients. The interpretation of globally asymptotically stable equations is that there is a long term equilibrium $\overline{y}$ that is independent of the initial conditions.

12.2 Problems

12.1. Find the solution of the difference equation $x_{t+1} = 2x_t + 4$ with $x_0 = 1$.

12.2. Find the solution of the difference equation $w_{t+1} = (1+r)w_t + y_{t+1} - c_{t+1}$ when $r = 0.2$, $w_0 = 1000$, $y_t = 100$ and $c_t = 50$.

12.3. Prove by direct substitution that the following sequences in t are solutions of the associated difference equations when A, B are constants:

1. $x_t = A + B \cdot 2^t$ is a solution of $x_{t+2} - 3x_{t+1} + 2x_t = 0$
2. $x_t = A \cdot 3^t + B \cdot 4^t$ is a solution of $x_{t+2} - 7x_{t+1} + 12x_t = 0$

12.4. Find the general solution of the difference equation $x_{t+2} - 2x_{t+1} + x_t = 0$.

12.5. Find the general solution of the difference equation $3x_{t+2} - 12x_t = 4$.

12.6. Find the general solution of the following difference equations:

1. $x_{t+2} - 6x_{t+1} + 8x_t = 0$
2. $x_{t+2} - 8x_{t+1} + 16x_t = 0$
3. $x_{t+2} + 2x_{t+1} + 3x_t = 0$

12.7. Find the general solution of the difference equation $x_{t+2} + 2x_{t+1} + x_t = 9 \cdot 2^t$.

12.8. A model for location uses the difference equation

$$D_{t+2} - 4(ab+1)D_{t+1} + 4a^2b^2D_t = 0$$

where a, b are constants and D_t is the unknown sequence. Find the solution of this equation assuming that $1 + 2ab > 0$.

12.9. Is the difference equation $x_{t+2} - x_{t+1} - x_t = 0$ globally asymptotically stable?

12.10. Final Exam in GRA6035 10/12/2010, Problem 3a
You borrow an amount K. The interest rate per period is r. The repayment is 500 in the first period, and increases with 10 for each subsequent period. Show that the outstanding balance b_t after period t satisfies the difference equation

$$b_{t+1} = (1+r)b_t - (500+10t), \quad b_0 = K$$

and solve this difference equation.

12.11. Mock Final Exam in GRA6035 12/2010, Problem 3c
Solve the difference equation

$$p_{t+2} = \frac{2}{3}p_{t+1} + \frac{1}{3}p_t, \quad p_0 = 100, \quad p_1 = 102$$

12.12. Final Exam in GRA6035 30/05/2011, Problem 3a
Solve the difference equation $x_{t+1} = 3x_t + 4$, $x_0 = 2$ and compute x_5.

12.13. Mock Final Exam in GRA6035 12/2012, Problem 3a
Solve the difference equation

$$y_{t+2} - 5y_{t+1} + 4y_t = 2^t$$

12.3 Solutions

12.1 We write the difference equation $x_{t+1} - 2x_t = 4$, and see that it is a first order linear inhomogeneous equation. The homogeneous solution is $x_t^h = C \cdot 2^t$ since the characteristic equation is $r - 2 = 0$, so that $r = 2$. We look for a particular solution of the form $x_t^p = A$ (constant), and see that $A - 2A = 4$, so that $A = -4$ and $x_t^p = -4$. Hence the general solution is

$$x_t = x_t^h + x_t^p = C \cdot 2^t - 4$$

The initial condition $x_0 = 1$ gives $C \cdot 1 - 4 = 1$, or $C = 5$. The solution is therefore $x_t = \mathbf{5 \cdot 2^t - 4}$.

12.2 We write the difference equation $w_{t+1} - 1.2w_t = 50$, and see that it is a first order linear inhomogeneous equation. The homogeneous solution is $w_t^h = C \cdot 1.2^t$ since the characteristic root is 1.2. We look for a particular solution of the form $w_t^p = A$ (constant), and see that $A - 1.2A = 50$, so that $A = -250$ and $x_t^p = -250$. Hence the general solution is

$$w_t = w_t^h + w_t^p = C \cdot 1.2^t - 250$$

The initial condition $w_0 = 1000$ gives $C \cdot 1 - 250 = 1000$, or $C = 1250$. The solution is therefore $w_t = \mathbf{1250 \cdot 1.2^t - 250}$.

12.3 We compute the left hand side of the difference equations to check that the given sequences are solutions:

1. $(A + B \cdot 2^{t+2}) - 3(A + B \cdot 2^{t+1}) + 2(A + B \cdot 2^t) = (A - 3A + 2A) + (4B - 6B + 2B) \cdot 2^t = 0$

2. $(A \cdot 3^{t+2} + B \cdot 4^{t+2}) - 7(A \cdot 3^{t+1} + B \cdot 4^{t+1}) + 12(A \cdot 3^t + B \cdot 4^t) = (9A - 21A + 12A) \cdot 3^t + (16B - 28B + 12B) \cdot 2^t = 0$

We see that the given sequence is a solution in each case.

12.4 The difference equation $x_{t+2} - 2x_{t+1} + x_t = 0$ is a second order linear homogeneous equation. The characteristic equation is $r^2 - 2r + 1 = 0$ and has a double root $r = 1$, and therefore the general solution is

$$x_t = C_1 \cdot 1^t + C_2\, t \cdot 1^t = \mathbf{C_1 + C_2 t}$$

12.5 We write the difference equation $3x_{t+2} - 12x_t = 4$ as $x_{t+2} - 4x_t = 1$. It is a second order linear inhomogeneous equation. We first find the homogeneous solution: The characteristic equation is $r^2 - 4 = 0$ and has roots $r = \pm 2$, and therefore the homogeneous solution is $x_t = C_1 \cdot 2^t + C_2 \cdot (-2)^t$. For the particular solution, we see that $f_t = 4$ in the original difference equation $3x_{t+2} - 12x_t = 4$, so we guess $x_t^p = A$, a constant. This gives $x_t = A$ and $x_{t+2} = A$, so $3A - 12A = 4$, or $A = -4/9$. Hence the particular solution is $x_t^p = -4/9$, and the general solution is

$$x_t = x_t^h + x_t^p = \mathbf{C_1 \cdot 2^t + C_2 \cdot (-2)^t - 4/9}$$

12.6 In each case, we solve the characteristic equation to find the general solution:

1. The characateristic equation of $x_{t+2} - 6x_{t+1} + 8x_t = 0$ is $r^2 - 6r + 8 = 0$, and has roots $r = 2, 4$. Therefore, the general solution is $x_t = \mathbf{C_1 \cdot 2^t + C_2 \cdot 4^t}$.
2. The characateristic equation of $x_{t+2} - 8x_{t+1} + 16x_t = 0$ is $r^2 - 8r + 16 = 0$, and has a double root $r = 4$. Therefore, the general solution is $x_t = \mathbf{C_1 \cdot 4^t + C_2\, t \cdot 4^t}$.
3. The characateristic equation of $x_{t+2} + 2x_{t+1} + 3x_t = 0$ is $r^2 + 2r + 3 = 0$, and has roots given by
$$r = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 3}}{2} = -1 \pm \sqrt{-8}/2$$
Hence there are no real roots. We have $a = 2$ and $b = 3$, so the general solution is $x_t = (\sqrt{\mathbf{3}})^{\mathbf{t}}(\mathbf{C_1} \cos(\mathbf{2.186t}) + \mathbf{C_2} \sin(\mathbf{2.186t}))$ since we have that $\cos(2.186) \simeq -1/\sqrt{3}$.

12.7 The difference equation $x_{t+2} + 2x_{t+1} + x_t = 9 \cdot 2^t$ is a second order linear inhomogeneous equation. We first find the homogeneous solution, and therefore consider the homogeneous equation $x_{t+2} + 2x_{t+1} + x_t = 0$. The characteristic equation is $r^2 + 2r + 1 = 0$ and it has a double root $r = -1$. Therefore the homogeneous solution is $x_t^h = C_1 \cdot (-1)^t + C_2\, t \cdot (-1)^t = (C_1 + C_2\, t)(-1)^t$. We then find a particular solution of the inhomogeneous equation $x_{t+2} + 2x_{t+1} + x_t = 9 \cdot 2^t$, and look for a solution of the form $x_t = A \cdot 2^t$. This gives

$$A \cdot 2^{t+2} + 2(A \cdot 2^{t+1}) + (A \cdot 2^t) = 9 \cdot 2^t \quad \Rightarrow \quad (4A + 4A + A) \cdot 2^t = 9 \cdot 2^t$$

This gives $9A = 9$ or $A = 1$, and the particular solution is $x_t^p = 1 \cdot 2^t = 2^t$. Hence the general solution is

$$x_t = x_t^h + x_t^p = (\mathbf{C_1} + \mathbf{C_2\,t}) \cdot (-\mathbf{1})^{\mathbf{t}} + \mathbf{2}^{\mathbf{t}}$$

12.8 The difference equation $D_{t+2} - 4(ab+1)D_{t+1} + 4a^2b^2D_t = 0$ is a linear second order homogeneous equation. Its characteristic equation is $r^2 - 4(ab+1)r + 4a^2b^2 = 0$, and it has roots given by

$$r = \frac{4(ab+1) \pm \sqrt{16(ab+1)^2 - 4 \cdot 4a^2b^2}}{2} = 2(ab+1) \pm 2\sqrt{2ab+1}$$

Since we assume that $1 + 2ab > 0$, there are distinct characteristic roots $r_1 \neq r_2$ given by

$$r_1 = 2(ab + 1 + \sqrt{2ab+1}), \quad r_2 = 2(ab + 1 - \sqrt{2ab+1})$$

and the general solution is

$$D_t = C_1 \cdot r_1^t + C_2 \cdot r_2^t = \mathbf{2^t}(\mathbf{C_1} \cdot (\mathbf{ab} + \mathbf{1} + \sqrt{\mathbf{2ab+1}})^{\mathbf{t}} + \mathbf{C_2} \cdot (\mathbf{ab} + \mathbf{1} - \sqrt{\mathbf{2ab+1}})^{\mathbf{t}})$$

12.9 The difference equation $x_{t+2} - x_{t+1} - x_t = 0$ is a linear second order homogeneous equation, with characteristic equation $r^2 - r - 1 = 0$ and characteristic roots given by

$$r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Hence it has two distinct characteristic roots $r_1 \neq r_2$ given by

$$r_1 = \frac{1+\sqrt{5}}{2} \simeq 1.618, \quad r_2 = \frac{1-\sqrt{5}}{2} \simeq -0.618$$

and the general solution is $x_t = C_1 \cdot r_1^t + C_2 \cdot r_2^t$. It is globally asymptotically stable if $x_t \to 0$ as $t \to \infty$ for all values of C_1, C_2, and this is not the case since $r_1 > 1$. In fact, $x_t \to \pm\infty$ as $t \to \infty$ if $C_1 \neq 0$. Therefore, the difference equation is **not globally asymptotically stable**.

12.10 Final Exam in GRA6035 10/12/2010, Problem 3a
We have $b_{t+1} - b_t = rb_t - s_{t+1}$, where $s_{t+1} = 500 + 10t$ is the repayment in period $t+1$. Hence we get the difference equation

$$b_{t+1} = (1+r)b_t - (500 + 10t), \quad b_0 = K$$

The homogenous solution is $b_t^h = C(1+r)^t$. We try to find a particular solution of the form $b_t = At + B$, which gives $b_{t+1} = At + A + B$. Substitution in the difference equation gives

$$At + A + B = (1+r)(At+B) - (500+10t) = ((1+r)A - 10)t + (1+r)B - 500$$

and this gives $A = 10/r$ and $B = 10/r^2 + 500/r$. Hence the solution of the difference equation is

$$b_t = b_t^h + b_t^p = C(1+r)^t + \frac{10}{r}t + \frac{10}{r^2} + \frac{500}{r}$$

The initial value condition is $K = C + 10/r^2 + 500/r$, hence we obtain the solution

$$b_t = \left(\mathbf{K} - \frac{\mathbf{10}}{\mathbf{r^2}} - \frac{\mathbf{500}}{\mathbf{r}}\right)(\mathbf{1}+\mathbf{r})^{\mathbf{t}} + \frac{\mathbf{10}}{\mathbf{r}}\mathbf{t} + \frac{\mathbf{10}}{\mathbf{r^2}} + \frac{\mathbf{500}}{\mathbf{r}}$$

12.11 Mock Final Exam in GRA6035 12/2010, Problem 3c

By rewriting it, we see that it is linear second order homogeneous difference equation with constant coefficients

$$p_{t+2} - \frac{2}{3}p_{t+1} - \frac{1}{3}p_t = 0$$

with characteristic equation $r^2 - 2r/3 - 1/3 = 0$, or $3r^2 - 2r - 1 = 0$. It has roots

$$r = \frac{2 \pm \sqrt{4+12}}{6} = \frac{1}{3} \pm \frac{2}{3}$$

and therefore the general solution is $p_t = A + B(-1/3)^t$. The initial conditions $p_0 = 100$ and $p_1 = 102$ gives $100 = A + B$ and $102 = A - B/3$, hence $2 = -4B/3$, or $B = -1.5$ and $A = 101.5$. The particular solution is

$$p_t = 101.5 - 1.5(-1/3)^t$$

12.12 Final Exam in GRA6035 30/05/2011, Problem 3a

We have $x_{t+1} - 3x_t = 4$, and the homogenous solution is $x_t^h = C \cdot 3^t$. We try to find a particular solution of the form $x_t = A$, and substitution in the difference equation gives $A = 3A + 4$, so $A = -2$ is a particular solution. Hence the solution of the difference equation is

$$x_t = x_t^h + x_t^p = C \cdot 3^t - 2$$

The initial value condition is $2 = C - 2$, hence we obtain the solution

$$x_t = \mathbf{4} \cdot \mathbf{3^t} - \mathbf{2}$$

This gives $x_5 = \mathbf{970}$.

12.13 Mock Final Exam in GRA6035 12/2012, Problem 3a

The homogeneous equation $y_{t+2} - 5y_{t+1} + 4y_t = 0$ has characteristic equation $r^2 - 5r + 4 = 0$, and therefore roots $r = 1, 4$. Hence the homogeneous solution is $y_h(t) = C_1 1^t + C_2 \cdot 4^t = C_1 + C_2 \cdot 4^t$. To find a particular solution of $y_{t+2} - 5y_{t+1} + 4y_t = 2^t$, we try $y_t = A \cdot 2^t$. This gives $y_{t+1} = 2A \cdot 2^t$ and $y_{t+2} = 4A \cdot 2^t$, and substitution in the equation gives $(4A - 10A + 4A)2^t = 2^t$, or $-2A = 1$. Hence $A = -1/2$ is a solution, and $y_p(t) = -\frac{1}{2} \cdot 2^t = -2^{t-1}$ is a particular solution. This gives general solution

$$y_t = \mathbf{C_1} + \mathbf{C_2} \cdot \mathbf{4^t} - \mathbf{2^{t-1}}$$

Part II
Exams in GRA6035 Mathematics

Lecture 13
Midterm and Final Exams

Midterm Exams

Mock Midterm Exam on 09/2010	Solutions
Midterm Exam on 24/09/2010	Solutions
Midterm Exam on 24/05/2011	Solutions
Midterm Exam on 30/09/2011	Solutions
Midterm Exam on 07/02/2012	Solutions
Midterm Exam on 19/04/2012	Solutions
Midterm Exam on 12/10/2012	Solutions
Midterm Exam on 19/04/2013	Solutions
Midterm Exam on 11/10/2013	Solutions
Midterm Exam on 30/04/2014	Solutions

Final Exams

Mock Final Exam on 12/2010	Solutions
Mock Final Exam on 12/2012	Solutions
Mock Final Exam on 12/2013	Solutions
Final Exam on 10/12/2010	Solutions
Final Exam on 30/05/2011	Solutions
Final Exam on 12/12/2011	Solutions
Final Exam on 06/02/2012	Solutions
Final Exam on 06/06/2012	Solutions
Final Exam on 13/12/2012	Solutions
Final Exam on 11/06/2013	Solutions
Final Exam on 13/12/2013	Solutions
Final Exam on 12/05/2014	Solutions