LECTURE 1

GRA 6035 — MATHEMATICS

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Use email, not IT'S L. messages

TEXTBOOK:

SIMON, BLUME [ME]

WORKBOOK [WB]

Course pages:

IT'S LEARNING

www.dr-eriksen.no/teaching/GRA6035

PREP. COURSE:

FORK1003 MATH

Plan:

1. Linear systems
2. Gaussian elimination
3. Rank

Reading:

[ME] 6.1, (6.2), 7.1 - 7.4, (7.5)

[LSGE]

FORK1003 Math Lecture 1

① Linear systems

Ex:
$$\begin{aligned} x+y+z &= 6 \\ x+2y+4z &= 17 \\ x+3y+9z &= 34 \end{aligned}$$

3x3 - linear system

↑ ↑

#equations | #variables

Defn: An $m \times n$ linear system (linear system of eqn's) is a system of m equations in n variables where each equation is linear.

Linear equation in $x_1, x_2, x_3, \dots, x_n$:

First order equation, i.e. it can be written as

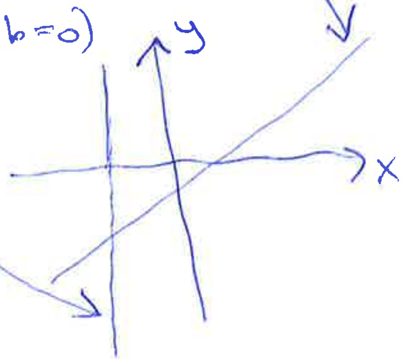
$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b$$

Ex: eqn. that are not linear

$$x^2 + y^2 = 1$$
$$xy = 1$$
$$\ln(x) = x$$

n=2:
$$ax + by = c \qquad (a, b, c \text{ constants})$$
$$by = c - ax$$
$$y = \frac{c}{b} - \frac{a}{b}x \qquad (b \neq 0)$$

(b=0)
$$ax = c$$
$$x = c/a$$

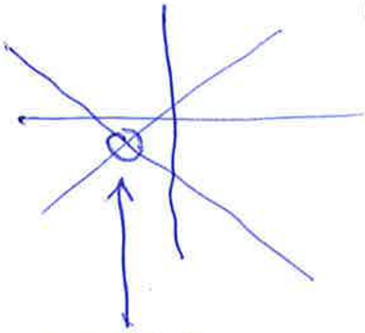


$$\boxed{\text{linear equation} \Updownarrow \text{straight lines}}$$

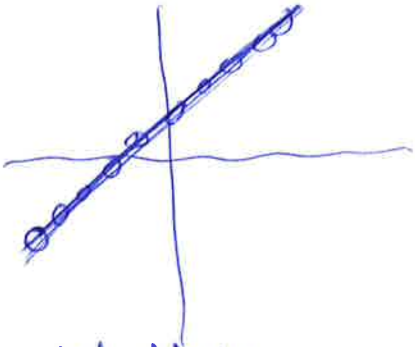
$n=2$: 2x2 linear system

$$a_{11}x + a_{12}y = b_1$$
$$a_{21}x + a_{22}y = b_2$$

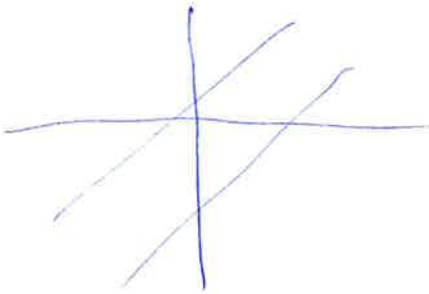
$\longleftrightarrow$



unique solution



infinitely many solutions.



no solutions

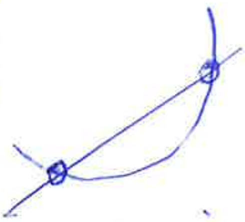
Result:

Any linear system $(m \times n)$ has either

i) no solutions

ii) one unique solution

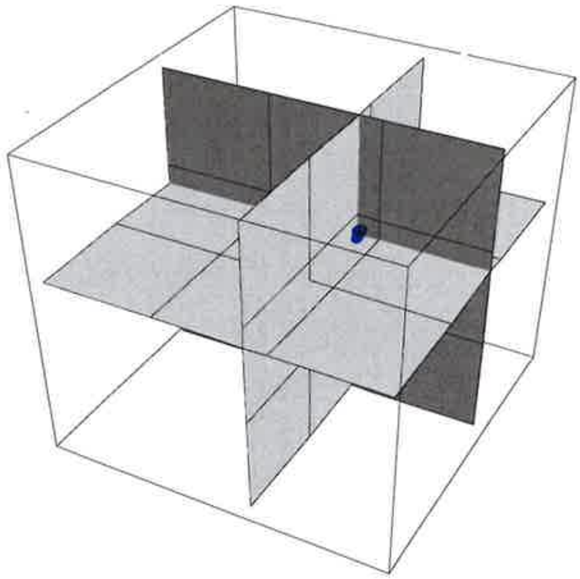
iii) infinitely many solutions



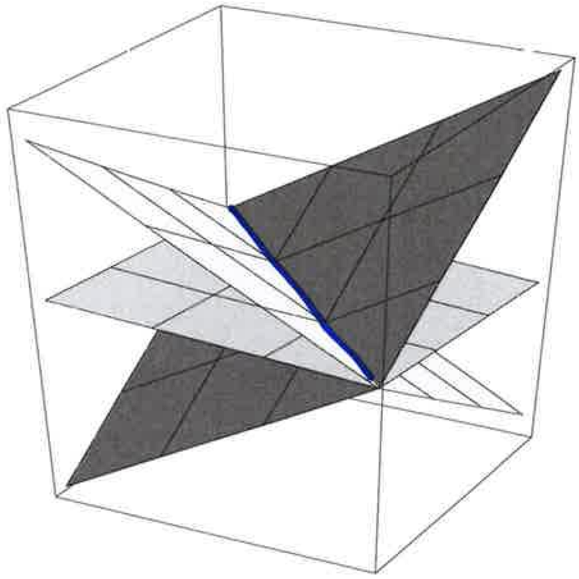
not linear

EXAMPLE: Three equations in three variables. Each equation determines a plane in 3-space.

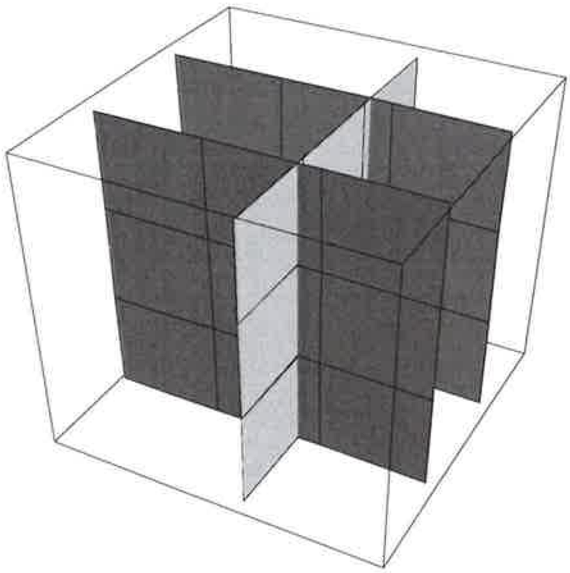
i) The planes intersect in one point. *(one solution)*



ii) The planes intersect in one line. *(infinitely many solutions)*



iii) There is not point in common to all three planes. *(no solution)*



Result Ⓐ:

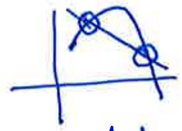
Any $m \times n$ linear system has either

i) one unique solution (consistent)

ii) no solutions (inconsistent)

iii) infinitely many solutions (consistent)

Non-linear case:



two solutions possible

Proof of A:

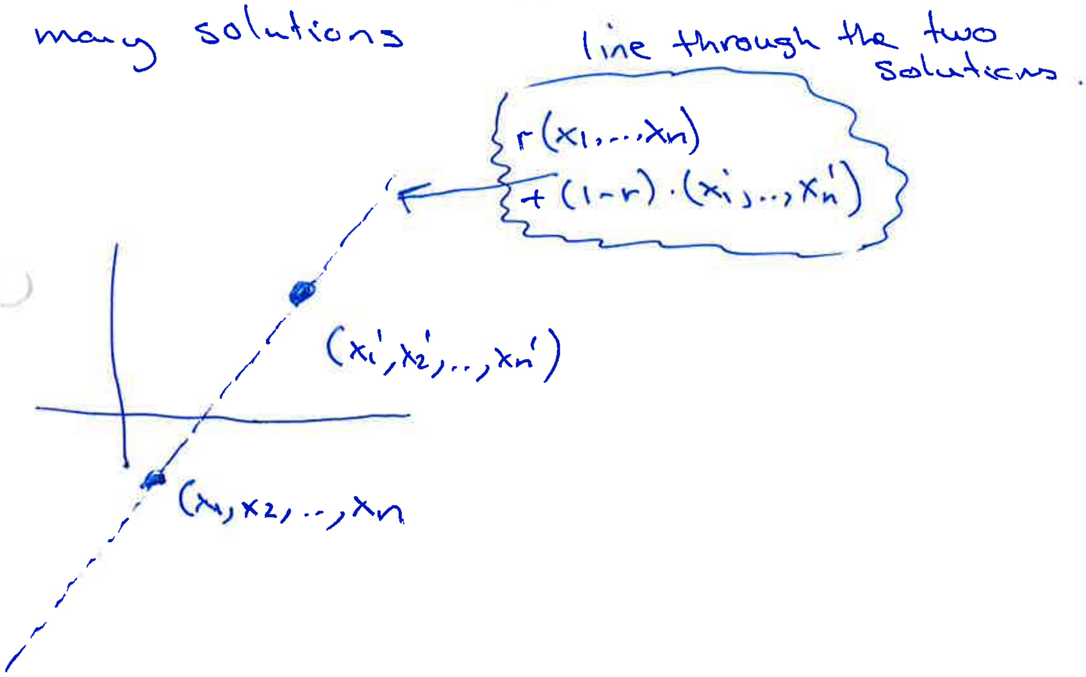
If there are two different solutions

$$(x_1, x_2, \dots, x_n) \neq (x_1', x_2', \dots, x_n')$$

then

$$r\,(x_1, x_2, \dots, x_n) + (1-r)\cdot(x_1', x_2', \dots, x_n')$$
$$= \left(r x_1 + (1-r) x_1', \; r x_2 + (1-r) x_2', \; \dots, \; r x_n + (1-r) x_n' \right)$$

is a solution for any number r. There are therefore infinitely many solutions



This can be checked directly. Equation #i is satisfied since

$$a_{i1}\cdot(r x_1 + (1-r)x_1') + a_{i2}(r x_2 + (1-r)x_2') + \dots + a_{in}\cdot(r x_n + (1-r) x_n')$$
$$= r\cdot(a_{i1}x_1 + \dots + a_{in}x_n) + (1-r)\cdot(a_{i1}x_1' + \dots + a_{in}x_n')$$
$$= r\cdot b_i + (1-r) b_i = b_i$$

since $(x_1, \dots, x_n)$ and $(x_1', \dots, x_n')$ are solutions

So this equation holds for all i. Hence we have solution for all r. □

General linear systems:

An $m \times n$ linear system can be written

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad\qquad & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m
\end{aligned}
$$

where a_{ij}, b_i are given numbers and $x_1, x_2, \dots, x_n$ are variables. A solution is an n-tuple $(x_1, x_2, \dots, x_n)$ that satisfy all m equations simultaneously.

How to solve linear systems

$$
\begin{aligned}
x + y &= 4 \\
x - y &= 2
\end{aligned}
$$

a) Substitution:

$x + y = 4 \Rightarrow y = 4 - x$

$x - y = 2$

$x - (4 - x) = 2$

$2x - 4 = 2$

$2x = 6$

$\underline{x = 3}$ $\quad y = 4 - x = \underline{1}$

One solution: $(x, y) = \underline{\underline{(3, 1)}}$

b) Elimination:

$$
\begin{aligned}
x + y &= 4 \\
x - y &= 2 \\
\hline
2x \quad &= 6
\end{aligned}
$$

$x = 3$

$y = 1$

$(x, y) = \underline{\underline{(3, 1)}}$

2 Gaussian elimination

Ex: $\begin{aligned} x+y+z &= 6 \\ x+2y+4z &= 17 \\ x+3y+9z &= 34 \end{aligned} \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 4 & 17 \\ 1 & 3 & 9 & 34 \end{array}\right)$

augmented matrix of the linear system

Elementary row operations:

1. Interchange two rows
2. Multiply a row with a number $c \neq 0$.
3. Add a multiple of one row to another row

} operations that preserve solutions.

$$R(j) := c \cdot R(i) + R(j)$$

new row #j, number, row #i, row #j

$$R(2) := R(2) + (-1) \cdot R(1)$$

want to get zeros

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 4 & 17 \\ 1 & 3 & 9 & 34 \end{array}\right) \xrightarrow{-1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 11 \\ 1 & 3 & 9 & 34 \end{array}\right) \xrightarrow{-1}$$

want to get zero

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 11 \\ 0 & 2 & 8 & 28 \end{array}\right) \xrightarrow{-2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 11 \\ 0 & 0 & 2 & 6 \end{array}\right)$$

Leading coeff: The first (leftmost) non-zero coeff. in a row.

We want to have zeros under leading coeffs.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 11 \\ 0 & 0 & 2 & 6 \end{array}\right)$$

Echelon form:

i) All rows with only zeros are below other rows

ii) All coeff. under a leading coeff. are zero.

$$\begin{aligned} x + y + z &= 6 \\ y + 3z &= 11 \\ 2z &= 6 \end{aligned}$$

Leading coeffs. in an echelon form: pivots

Pivot positions: Positions where the pivots are.

Back substitution:

$2z = 6 \Rightarrow z = 3$

$y + 3z = 11 \Rightarrow y = 11 - 3 \cdot 3 = 2$

$x + y + z = 6 \Rightarrow x = 6 - 2 - 3 = 1$

One solution: $(x, y, z) = (1, 2, 3)$

No solution and infinitely many solutions

Ex:

$$\begin{aligned} x + y + z &= 3 \\ 2x - y + 4z &= 7 \\ 3x \quad + 5z &= 9 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & -1 & 4 & 7 \\ 3 & 0 & 5 & 9 \end{array}\right) \begin{array}{l} \\ \cdot(-2) \\ \cdot(-3) \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -3 & 2 & 1 \\ 0 & -3 & 2 & 0 \end{array}\right) \cdot(-1) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -3 & 2 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

no solutions

$0x + 0y + 0z = -1$

Fact: Pivot position in last column $\Updownarrow$ no solution (inconsistent)

Ex:

$$\begin{aligned} x+y+z&=1 \\ x+2y+3z&=2 \\ 2x+3y+4z&=3 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 3 & 4 & 3 \end{array}\right) \begin{matrix} -1 \\ -2 \end{matrix}$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{array}\right) -1 \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

x is basic, y is basic, z is free

$\underbrace{x, y}_{\text{pivot positions}}$ $\underbrace{z}_{\text{no pivot position}}$

$$\begin{aligned} \underline{x}+y+z&=1 \\ \underline{y}+2z&=1 \end{aligned}$$

$\underline{y}+2z=1$

$\Rightarrow y=\underline{1-2z}$

$\underline{x}+y+z=1$

$$\begin{aligned} \Rightarrow x &= 1-y-z \\ &= 1-(1-2z)-z \\ &= \underline{z} \end{aligned}$$

$$\boxed{\begin{aligned} x&=z \\ y&=1-2z \\ z&=z \text{ (free)} \end{aligned}}$$

$$\left.\begin{aligned} x&=t \\ y&=1-2t \\ z&=t \end{aligned}\right\}$$ t is a parameter that we can choose freely

degrees of freedom = # free variables
(in the case with no pivot position in the last column)

= dimension of solution space

Result: B i)

Any matrix can be transformed to an echelon form using elementary row operations. The echelon form you get is not unique, but the pivot positions are unique.

Gauss-Jordan elimination

Reduced echelon form: Echelon form with the extra requirements

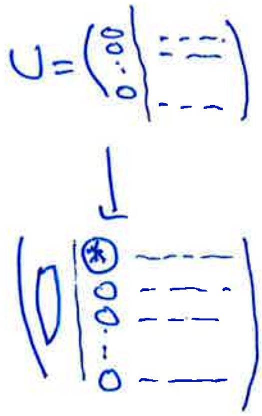
i) All pivots are 1.
ii) All coeff. over a pivot are zero.

Result: B ii)

Any matrix can be transformed to a ~~[illegible]~~ reduced echelon form using elementary row operations, and it is unique.

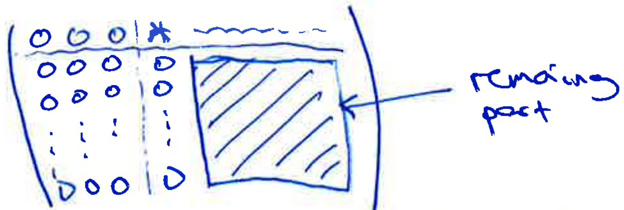
Proof of Result B and more details of Gauss/Gauss-Jordan elimination:

i) Any matrix can be reduced to an echelon form using Elementary row operations.



Start with any matrix U. Move to the right of any columns with only zeros, if any. Look at the first non-zero column, switch two rows if necessary to get a non-zero entry in the top corner. This is a pivot. Use it to get zeros under it.

Now look away from the first row, and look at the remaining part of the matrix.



Repeat the steps above. Since the new matrix is smaller than the original (one row less), we sooner or later get an echelon form this way.

Multiply each row with $\frac{1}{\text{pivot}}$ to get pivot $=1$. Use each pivot to get zeros over it, starting from the rightmost. You get a reduced echelon form.

ii) If a matrix A can be reduced to reduced echelon forms U, V using elementary row operations, $U = V$.

We have $(A \mid 0) \dashrightarrow (U \mid 0)$ and $(A \mid 0) \dashrightarrow (V \mid 0)$, and elementary row operations do not change solutions of linear systems. So $U \cdot \underline{x} = \underline{0}$ and $V \cdot \underline{x} = \underline{0}$ have the same solutions. Write $U = (c_1 | c_2 | \dots | c_n)$ and $V = (c_1' | c_2' | \dots | c_n')$ in terms of their columns. We have

$$c_i = x_1 c_1 + x_2 c_2 + \dots + x_{i-1} c_{i-1} \iff c_i \text{ non-pivot column in } U$$

$$\Updownarrow$$

$$c_i' = x_1 c_1' + x_2 c_2' + \dots + x_{i-1} c_{i-1}' \iff c_i' \text{ non-pivot column in } V$$

So U and V have the same pivot columns; They are in the positions, and the pivot columns are

$$\begin{array}{c}1\\ \\ \\ \\ \\ \end{array}\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix},\quad \begin{array}{c} \\ 2\\ \\ \\ \\ \end{array}\begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix},\quad \begin{array}{c} \\ \\ 3\\ \\ \\ \end{array}\begin{pmatrix}0\\0\\1\\\vdots\\0\end{pmatrix},\ \ldots\ \begin{array}{c} \\ \\ \\ \\ r\\ \\ \\ \end{array}\begin{pmatrix}0\\0\\0\\\vdots\\1\\\vdots\\0\end{pmatrix}$$

To show that $U=V$, we must show that non-pivot columns are equal. But each non-pivot column satisfy

$$\begin{cases} C_i = x_1 C_1 + \cdots + \\ C_i' = x_1 C_1' + \cdots + \end{cases}$$ (linear combination of pivot columns to the right)

and the pivot columns are equal, and the coeff's x_i are equal.

Hence $U=V$.

ii) You can always get from an echelon matrix to a reduced echelon matrix, using elementary row operations, without changing the pivot positions.

This follows from the last steps in i).

$\square$

3 Rank

For any $m \times n$-matrix A, the rank of A is the number of pivot positions in A.

Ex:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \qquad (A|\underline{b}) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 4 & 17 \\ 1 & 3 & 9 & 34 \end{array}\right)$$

Coeff. matrix

augmented matrix

$$x+y+z=6$$
$$x+2y+4z=17$$
$$x+3y+9z=34$$

$$(A|\underline{b}) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 4 & 17 \\ 1 & 3 & 9 & 34 \end{array}\right) \to \cdots \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 11 \\ 0 & 0 & 2 & 6 \end{array}\right)$$

$$rk(A|\underline{b}) = \underline{\underline{3}} \qquad rk(A) = \underline{\underline{3}}$$

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Result:

We consider an $m \times n$ linear system with coeff. matrix A and augmented matrix $(A|b)$, then we have:

i) There are no solutions (inconsistent system) if and only if $\text{rk}\, A \neq \text{rk}\,(A|b)$.

ii) If there are solutions (consistent system), then

$$\#\text{degrees of freedom} = n - \text{rk}(A)$$

In particular:

$\text{rk}(A) = n$: one unique solution

$\text{rk}(A) < n$: infinitely many solutions

Ex: 3×4 linear system

one possibility:

$$\left(\begin{array}{cccc|c} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right)$$

augmented

$\Downarrow$

w free

inf. many solutions

What are the other possibilities?