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LECTURE 12

GRA 6035

EIVIND ERIKSEN

NOV 17, 2016

MATHEMATICS

Plan:

1. Stability of differential equations
2. Linear difference equations

Reading: [MEJ] 23.2

Monday: Exam (final) 12/2015 Review

Tue: TA session

Wed: extra TA-type session (diff. eqns)

Review: Differential equations

First order differential equations: $y' = F(y,t)$

i) Seperable: $y' = f(y)\,g(t) \to \int \frac{1}{f(y)}\,dy = \int g(t)\,dt$

ii) Linear: $y' + a(t)\cdot y = b(t) \to y = \frac{1}{v(t)}\int v(t)\,b(t)\,dt$

with $v(t) = e^{\int a(t)\,dt}$

iii) Exact: $p(y,t) + q(y,t)\cdot y' = 0 \to h(y,t) = C$

s.t. $\boxed{\frac{\partial p}{\partial y} = \frac{\partial q}{\partial t}}$

with $h = h(y,t)$ a function that satisfies

$$\boxed{\begin{aligned} h'_t &= \frac{\partial h}{\partial t} = p \\ h'_y &= \frac{\partial h}{\partial y} = q \end{aligned}}$$

Second order differential equations:

Linear with constant coefficients $y'' + ay' + by = f(t)$

y_h: Char. eqn. $r^2 + ar + b = 0$

$r_1 \neq r_2$: $y_h = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

$r_1 = r_2$: $y_h = (C_1 + C_2 t)e^{r_1 t}$

no roots: something with sin/cos

$$y = y_h + y_p$$

y_p: Guess and check.
- guess a function of the same form as $f(t)$, with undetermined coeffs.

1 Stability of linear differential equations

Ex: $y' + ay = b$ linear, first order diff. eqn.
a constant
b constant

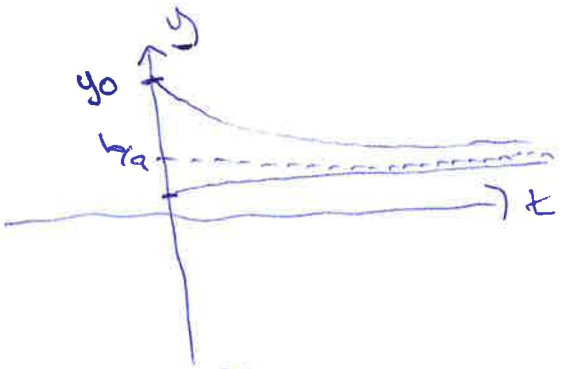
Alt. method: a const.

$$y = y_h + y_p = \underline{\underline{Ce^{-at} + \frac{b}{a}}}$$

y_h: $y' + ay = 0$
Char. eqn. $r + a = 0$
$r = -a$
$y_h = \underline{C \cdot e^{-at}}$

y_p: $y' + ay = b$
$\underline{y = A}$ $f(t) = b$
$y' = 0$ $f'(t) = 0$
$0 + aA = b$
$A = \frac{b}{a}$
$\underline{y_p = \frac{b}{a}}$

Stability of $y' + ay = b$:
Solution: $y = \underline{\frac{b}{a} + C \cdot e^{-at}}$



$y_0 = y(0) = \frac{b}{a} + C$
$C = (y_0 - \frac{b}{a})$

Stability: What happens when $t \to \infty$?

$$\lim_{t\to\infty} \left(\frac{b}{a} + C \cdot e^{-at}\right) = ?$$

$a > 0$: $\lim_{t\to\infty} y(t) = \frac{b}{a} + C \cdot 0 = \underline{\underline{\frac{b}{a}}}$

globally asymptotically stable

$\lim_{t\to\infty} y(t)$ exists and is independent of y_0.

$\underline{a<0}$: $\quad y = \frac{b}{a} + C \cdot e^{-at}$

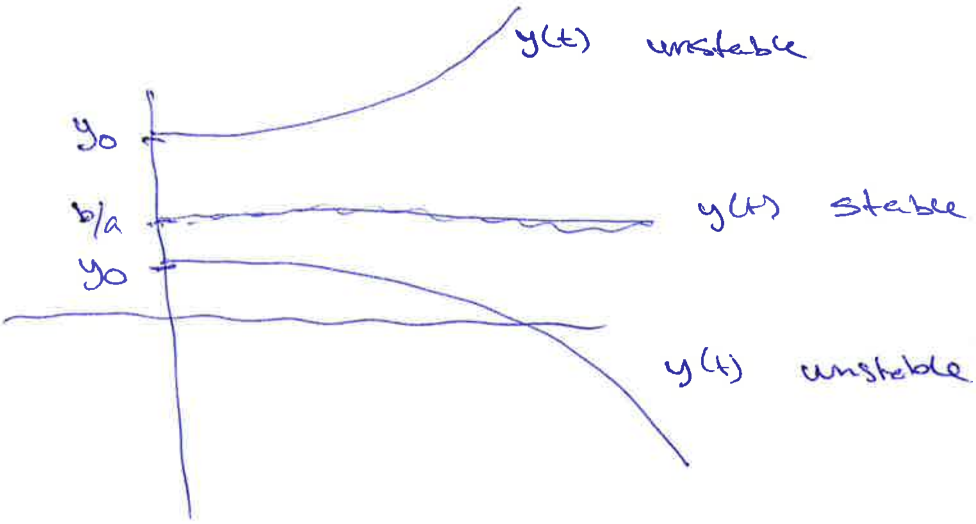
$$\lim_{t\to\infty} y(t) = \frac{b}{a} + C \cdot \lim_{t\to\infty} e^{-at} \text{ "=" } \frac{b}{a} + C \cdot \infty$$

$\underline{C = y_0 - \frac{b}{a}}$; $\quad C>0$ means $y_0 > \frac{b}{a}$

$$= \begin{cases} \infty, & C>0 \\ \frac{b}{a}, & C=0 \\ -\infty, & C<0 \end{cases}$$

$C=0$ means $y_0 = \frac{b}{a}$

$C<0 \quad y_0 < \frac{b}{a}$



not globally asymptotically stable.

$\underline{\text{Ex}}$: $\quad y'' + 3y' + 2y = 8$

$$r^2 + 3r + 2 = 0$$
$$r = -2, -1$$

$$y = A$$
$$y' = 0$$
$$y'' = 0$$
$$0 + 3 \cdot 0 + 2A = 8$$
$$A = 4$$

$$y = y_h + y_p = \underline{C_1 e^{-2t} + C_2 e^{-t} + 4}$$

$$\lim_{t\to\infty} y(t) = 4 + C_1 \cdot \lim_{t\to\infty} e^{-2t} + C_2 \cdot \lim_{t\to\infty} e^{-t}$$

$$= 4 + C_1 \cdot 0 + C_2 \cdot 0 = \underline{4}$$

$\underline{\text{globally asymptotically stable}}$ since $r_1, r_2 < 0$

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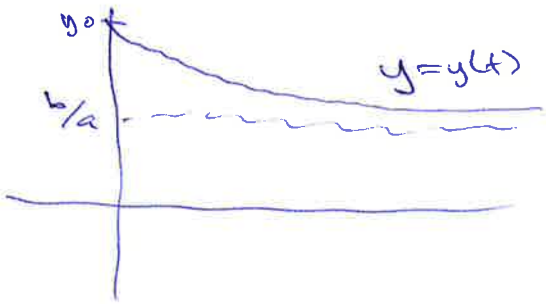
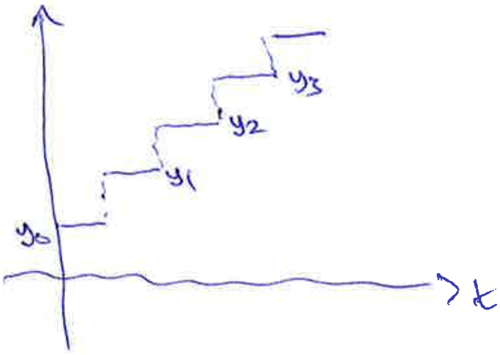
2 Linear difference equations

i) differential eqn. } models for change: i) continuous time

ii) difference eqn. } models for change: ii) discrete time

differential eqn's	difference eqn's
$y' + ay = b$	$y_{t+1} - y_t = 2 \Leftrightarrow y_{t+1} = y_t + 2$
$\Leftrightarrow y' = b - ay$	$\underbrace{y_{t+1} - y_t}$ change in y in discrete time
$y = \frac{b}{a} + Ce^{-at}$	





$y_0, y_1, y_2, y_3, \ldots$

$(y(0), y(1), y(2), y(3), \ldots)$

Ex: $y_{t+1} = y_t + 2$ ⟶ this hold for all t:

$\Updownarrow$

$y_{t+1} - y_t = 2$

$t=0$: $y_1 = y_0 + 2$

$t=1$: $y_2 = y_1 + 2 = y_0 + 4$

$t=2$: $y_3 = y_2 + 2 = y_0 + 6$

$\vdots$

$\boxed{y_t = y_0 + 2t}$ closed formula

$y_{t+1} = y_t + 2, \quad y_0 = 1$

$\Downarrow$

$y_t = 1 + 2t$

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First order linear difference equations:

$$\boxed{y_{t+1} + a y_t = f_t}$$

a constant
f_t expression in t

$$y_{t+1} = f_t - a y_t$$

$$\underbrace{y_{t+1} - y_t}_{\text{change}} = f_t - a y_t - y_t = F(y_t, t)$$

Remember:

$y_{t+1} + a y_t = f_t$ means that this equation should hold for all t

$t = 0, 1, 2, \ldots$

$y_1 + a y_0 = f_0$
$y_2 + a y_1 = f_1$
$\vdots$

Ex: $y_{t+1} - 2y_t = t$

$y_1 - 2y_0 = 0 \rightarrow y_1 = 2y_0$
$y_2 - 2y_1 = 1 \qquad y_2 = 2y_1 + 1$
$y_3 - 2y_2 = 2 \qquad y_3 = 2y_2 + 2$
$\vdots \qquad \vdots$

$y_0 = 1$: $y_1 = 2$
$y_2 = 5$
$y_3 = 12$
$\vdots$
$y_t = \cdots$?
closed form

Method: $y_t = y_t^h + y_t^p =$
(superposition)

y_t^h:

$y_{t+1} - 2y_t = 0$
$y_{t+1} = 2y_t$

$y_t^h = y_0 \cdot 2^t$

$y_1 = 2y_0$
$y_2 = 2y_1 = 4y_0$
$y_3 = 2y_2 = 8y_0$
$\vdots$
$y_t = y_0 \cdot 2^t$

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y_t^h:

$$y_{t+1} - 2y_t = 0$$

Char. eqn: $r - 2 = 0$

$r = 2$

$$y_t^h = C \cdot 2^t$$

r char root $\Leftrightarrow$ r^t sol.

y_t^p:

$$y_{t+1} - 2y_t = t$$

$f_t = t$

$f_{t+1} = t+1$

Guess: $y_t = At + B$

$$y_t = At + B$$

$$y_{t+1} = A(t+1) + B$$

$$\| $$

$$At + A$$

$$A(t+1) + B - 2 \cdot (At + B) = t$$

$$-At + (A + B - 2B) = t$$

$$-A = 1 \qquad A = -1$$

$$A - B = 0 \qquad B = -1$$

$$y_t^p = At + B = -t - 1$$

$$y_t = y_t^h + y_t^p = C \cdot 2^t - t - 1$$

Second order linear difference equations

$$\boxed{y_{t+2} + a y_{t+1} + b y_t = f_t}$$

a, b: constants
f_t: expr. in t

Ex: $y_{t+2} - 7 y_{t+1} + 10 y_t = t^2$

The first example $y_{t+2} - 4y_{t+1} + 3y_t = t^2$ was changed to give easier computations

$$y_t = y_t^h + y_t^p = \,?$$
(superposition)

r char root
$\Uparrow$
r^t solution of diff. eqn.

y_t^h: $y_{t+2} - 7y_{t+1} + 10 y_t = 0$

Char. eqn: $r^2 - 7r + 10 = 0$

$r = 2,\ r = 5 \Rightarrow y_t^h = C_1 \cdot 2^t + C_2 \cdot 5^t$

y_t^p: $y_{t+2} - 7y_{t+1} + 10y_t = t^2$

$f_t = t^2$
$f_{t+1} = (t+1)^2$
$f_{t+2} = (t+2)^2$
$= t^2 + 4t + 4$

Guess: $y_t = At^2 + Bt + C$

$y_{t+1} = A(t+1)^2 + B(t+1) + C$
$= A(t^2 + 2t + 1) + Bt + B + C$
$= At^2 + A \cdot 2t + A + Bt + B + C$

$y_{t+2} = A(t+2)^2 + B(t+2) + C$
$= At^2 + 4At + 4A + Bt + 2B + C$

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$$\left(At^2 + 4At + 4A + Bt + 2B + C\right) - 7\left(At^2 + 2At + A + Bt + B + C\right)$$
$$+ 10\left(At^2 + Bt + C\right) = t^2$$

$\underbrace{(A - 4A + 3A)}_{=0} t^2 + \ldots = t^2$

this will not work.

$\Downarrow$

$y_t = t \cdot (At^2 + Bt + C) = At^3 + Bt^2 + Ct$

For $a = -4$, $b = 3$: must multiply with t

$$(A - 7A + 10A)t^2 + (4A + B - 14A - 7B + 10B)t = t^2$$
$$+ (4A + 2B + C - 7A - 7B - 7C + 10C) \cdot 1$$

$$(4A)t^2 + (-10A + 4B)t + (-3A - 5B + 4C) = t^2$$

$4A = 1$

$-10A + 4B = 0$

$-3A - 5B + 4C = 0$

$A = 1/4$

$4B = 10A = 10/4 \Rightarrow B = \frac{10}{4 \cdot 4} = \frac{10}{16} = \frac{5}{8}$

$4C = 3A + 5B = \frac{3}{4} + \frac{25}{8} = \frac{31}{8}$

$\Rightarrow C = \frac{31}{32}$

$y_t^P = At^2 + Bt + C = \frac{1}{4}t^2 + \frac{5}{8}t + \frac{31}{32}$

$$y_t = C_1 \cdot 2^t + C_2 \cdot 5^t + \frac{1}{4}t^2 + \frac{5}{8}t + \frac{31}{32}$$

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Explanations:

$y' \rightsquigarrow y_{t+1} - y_t$ change

$y'' \rightsquigarrow (y_{t+2} - y_{t+1}) - (y_{t+1} - y_t)$

$= y_{t+2} - 2y_{t+1} + y_t$ Second order change

$y_{t+2} + a y_{t+1} + b y_t = 0$:

$\boxed{\text{Check if } r^t \text{ is a solution.}}$

$\downarrow$

$y_t = r^t$

$r^{t+2} + a r^{t+1} + b r^t = 0 \longleftarrow$

$r^t \cdot (r^2 + ar + b) = 0$

$r^2 + ar + b = 0$
char. eqn.

$\boxed{\begin{array}{c} y_t = r^t \text{ is a solution of the diff. eqn.} \\ \Updownarrow \\ r^2 + ar + b = 0 \\ r \text{ is char. root} \end{array}}$

Bank account: B_t balance at time t

cont. time:

$B'(t) = r \cdot B(t)$

$B(t) = \underline{B_0 \cdot e^{rt}}$

discrete time

$B_{t+1} - B_t = r \cdot B_t$

$B_{t+1} = B_t \cdot (1+r)$

$B_t = \underline{B_0 \cdot (1+r)^t}$

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$$y_{t+2} + a y_{t+1} + b y_t = 0$$

Char. eqn. $r^2 + ar + b = 0$

$$r = \frac{-a \pm \sqrt{a^2-4b}}{2}$$

i) $a^2-4b>0$: $r = r_1$, $r = r_2$ $\quad r_1 \neq r_2$

$$y_t = C_1 \cdot r_1^t + C_2 \cdot r_2^t$$

ii) $a^2-4b=0$: $r = r_1 = r_2 = -a/2$ $\quad$ double root

$$y_t = (C_1 + C_2 t) r_1^t$$

iii) $a^2-4b<0$: no real roots

$$y_t = (\sqrt{b})^t \left(C_1 \cdot \cos(\theta t) + C_2 \cdot \sin(\theta t) \right)$$

where $\theta = \cos^{-1}\left(a/(2\sqrt{b}) \right)$

You should know case i) – ii), and know that there is a solution in case iii).