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LECTURE 13

GRA 6035 MATHEMATICS

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Plan:

1. Review of important topics from the course
2. Final exam 12/2015

} topic by topic

A) Matrix methods

Key words:

- determinants, rk
- linear systems and Gaussian elimination

$\boxed{A\cdot\underline{x}=\underline{0} \Rightarrow \underline{x}=\text{span}(\underline{v_1},\dots,v_n) \text{ solution set.}}$

- eigenvalues/-vectors, diagonalization
- pos./neg. definiteness and (leading) principal minors

Ex 1: $A=\begin{pmatrix}0&1&1&0\\1&0&0&1\\1&0&0&1\\0&1&1&0\end{pmatrix}$

a) rk(A): $\text{rk}(A)=\underline{\underline{2}}$

$$\begin{pmatrix}0&1&1&0\\1&0&0&1\\1&0&0&1\\0&1&1&0\end{pmatrix}\to\begin{pmatrix}1&0&0&1\\0&1&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

$A\underline{x}=\underline{0}$

$$\left(\begin{array}{cccc|c}0&1&1&0&0\\1&0&0&1&0\\1&0&0&1&0\\0&1&1&0&0\end{array}\right)\to\begin{array}{c}\begin{array}{cccc}x&y&z&w\end{array}\\\left(\begin{array}{cccc|c}1&0&0&1&0\\0&1&1&0&0\\0&0&0&0&0\\0&0&0&0&0\end{array}\right)\end{array}$$

z, w : free

$x+w=0 \Rightarrow x=-w$

$y+z=0 \Rightarrow y=-z$

$$\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}-w\\-z\\z\\w\end{pmatrix}=z\cdot\begin{pmatrix}0\\-1\\1\\0\end{pmatrix}+w\begin{pmatrix}-1\\0\\0\\1\end{pmatrix}$$

$\text{span}(\underline{v_1},\underline{v_2})$, $\underline{v_1}=\begin{pmatrix}0\\-1\\1\\0\end{pmatrix}$ $\underline{v_2}=\begin{pmatrix}-1\\0\\0\\1\end{pmatrix}$

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b) A is diag. since it is symmetric.

Eigenvalues:

$$\begin{vmatrix} 0-\lambda & 1 & 1 & 0 \\ 1 & 0-\lambda & 0 & 1 \\ 1 & 0 & 0-\lambda & 1 \\ 0 & 1 & 1 & 0-\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & 1 & 1 & 0 \\ 1 & -\lambda & 0 & 1 \\ 1 & 0 & -\lambda & 1 \\ 0 & 1 & 1 & -\lambda \end{vmatrix} = 0$$

$$= -\lambda \cdot \left[-\lambda \cdot (\lambda^2-1) + 1 \cdot -\lambda \right] - 1 \cdot \begin{vmatrix} 1 & 0 & 1 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & -\lambda & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -\lambda \end{vmatrix}$$

$$= -\lambda\left(\lambda \cdot (1-\lambda^2+1)\right) + 1\left(1\cdot(\lambda^2-1)+1\cdot 1\right) + 1\left(1(-1)-1(\lambda^2-1)\right)$$

$$= -\lambda^2(2-\lambda^2) - \lambda^2 - \lambda^2 = -\lambda^2(2-\lambda^2+2) = -\lambda^2(4-\lambda^2) = 0$$

$$\lambda_1 = \lambda_2 = 0 \quad \lambda_3 = 2 \quad \lambda_4 = -2$$

c) $B = A^2$: B is diag. since it is symm.

$$B = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix}$$

$$\begin{vmatrix} 2-\lambda & 0 & 0 & 2 \\ 0 & 2-\lambda & 2 & 0 \\ 0 & 2 & 2-\lambda & 0 \\ 2 & 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)\cdot(2-\lambda)\cdot\left[(2-\lambda)^2-4\right] - 2\cdot 2\left[(2-\lambda)^2-4\right]$$

$$= \left[(2-\lambda)^2-4\right]\cdot\left[(2-\lambda)^2-4\right] = 0$$

$$= (\lambda^2-4\lambda)\cdot(\lambda^2-4\lambda) = \lambda\cdot(\lambda-4)\cdot\lambda(\lambda-4) = 0$$

$$\lambda_1 = \lambda_2 = 0, \quad \lambda_3 = \lambda_4 = 4$$

Alt: λ eigenvalue of $A \Rightarrow \lambda^2$ eigenvalues of $B = A^2$

$0, 0, 2, -2$ $\qquad$ $0, 0, 4, 4$

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d) Markov chain: $T = \begin{pmatrix} 0.55 & 0.10 & 0.15 \\ 0.10 & 0.80 & 0.05 \\ 0.35 & 0.10 & 0.80 \end{pmatrix}$

Eq. state: T regular $\Rightarrow$ the unique eigenvector for $\lambda = 1$ of the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ w. $x+y+z=1$ is the eq. state.

$\lambda = 1$:

$(A - \lambda I)\underline{x} = \underline{0}$ $\rightarrow$ $\left(\begin{array}{ccc|c} -0.45 & 0.10 & 0.15 & 0 \\ 0.10 & -0.20 & 0.05 & 0 \\ 0.35 & 0.10 & -0.20 & 0 \end{array}\right)$ $\leftarrow$ mult. each row by 20.

$$\left(\begin{array}{ccc|c} -9 & 2 & 3 & 0 \\ 2 & -4 & 1 & 0 \\ 7 & 2 & -4 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1/2 & 0 \\ -9 & 2 & 3 & 0 \\ 7 & 2 & -4 & 0 \end{array}\right) \begin{array}{l}]9 \\ \\ \leftarrow -7 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1/2 & 0 \\ 0 & -16 & 15/2 & 0 \\ \cancel{0} & \cancel{16} & \cancel{-15/2} & \cancel{0} \end{array}\right)$$

z free

$-16y = -15/2\, z$

$y = \dfrac{15}{32} z$

$x = 2y - \frac{1}{2} z$

$= 2\left(\frac{15}{32} z\right) - \frac{1}{2} z$

$= \frac{15}{16} z - \frac{8}{16} z$

$x = \dfrac{7}{16} z$

$x+y+z=1$:

$\frac{7}{16} z + \frac{15}{32} z + z = 1 \quad | \cdot 32$

$14z + 15z + 32z = 32$

$61z = 32$

$z = 32/61$

$x = \frac{7}{16} \cdot \frac{32}{61} = \underline{\underline{\frac{14}{61}}}$ $\quad y = \frac{15}{32} \cdot \frac{32}{61} = \underline{\underline{\frac{15}{61}}}$ $\quad z = \underline{\underline{\frac{32}{61}}}$

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(B) Differential / difference equations

Differential equations

- seperable
- linear first order
- exact

- linear second order
  $(y = e^{rt})$

Difference equations:

- first and second order linear
  $(y_t = r^t)$

2. a) $y'' - 4y' - 12y = 15e^t$

$$y = y_h + y_p = \underline{\underline{C_1 e^{6t} + C_2 e^{-2t} - e^t}}$$

y_h: $r^2 - 4r - 12 = 0$

$$r = \frac{4 \pm \sqrt{16 - 4\cdot(-12)}}{2}$$

$$= \frac{4 \pm 8}{2} = \underline{6}, \underline{-2}$$

$y_h = C_1 \cdot e^{6t} + C_2 \cdot e^{-2t}$

y_p: $y'' - 4y' - 12y = 15e^t$

$f = 15e^t$
$f' = 15e^t$
$f'' = 15e^t$

$\Downarrow$

$\begin{cases} y = Ae^t \\ y' = Ae^t \\ y'' = Ae^t \end{cases}$

$$Ae^t - 4Ae^t - 12Ae^t = 15e^t$$

$$\underbrace{(A - 4A - 12A)}_{-15A} e^t = 15e^t$$

$$-15A = 15$$

$$A = -1 \qquad y_p = \underline{-e^t}$$

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b) $y' = 3\sqrt{t}\ e^{-2y} \quad | \cdot e^{2y}$ not linear $y' \neq b(t) - a(t)y$

$e^{2y} y' = 3\sqrt{t}$

$\int e^{2y}\, dy = \int 3\sqrt{t}\, dt$

$\frac{1}{2} \cdot e^{2y} = 3 \cdot \int t^{1/2} dt = \cancel{3} \cdot \frac{2}{\cancel{3}} t^{3/2} + C$

$\frac{1}{2} e^{2y} = 2\ t^{3/2} + C \quad | \cdot 2$

$e^{2y} = 4 \cdot t^{3/2} + 2C$

$2y = \ln(4t^{3/2} + 2C)$

$y = \frac{1}{2} \ln(4t^{3/2} + 2C) = \frac{1}{2} \ln(4t\sqrt{t} + 2C)$

c) $(4yt + 4t^3 + 2t) + (2y - 1 + 2t^2)\, y' = 0 \quad , \quad y(1) = 0$

$\hookrightarrow y' = -\dfrac{4yt + 4t^3 + 2}{2y - 1 + 2t^2}$ not lin. not sep.

$\dfrac{\partial h}{\partial t} + \dfrac{\partial h}{\partial y} \dfrac{dy}{dt} = 0$ ($\dfrac{dy}{dt} = y'$)

$(4yt + 4t^3 + 2t) + (2y - 1 + 2t^2) y' = 0$

① $= h'_t$ ② $= h'_y$

$h = 4y \cdot \frac{1}{2} t^2 + t^4 + t^2 + C(y) = 2yt^2 + t^4 + t^2 + C(y)$

② $h'_y = \cancel{2t^2 + 0 + 0} + C'(y) = 2y - 1 + \cancel{2t^2}$

$C'(y) = 2y - 1$

$C(y) = y^2 - y$

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Cond: $h = 2yt^2 + t^4 + t^2 + y^2 - y$ solves ① and ②

$$\Downarrow$$

$$h = 2yt^2 + t^4 + t^2 + y^2 - y = C$$

$$1 \cdot y^2 + (2t^2 - 1)y + (t^4 + t^2 - c) = 0$$

$$y = \frac{-(2t^2-1) \pm \sqrt{(2t^2-1)^2 - 4 \cdot 1 \cdot (t^4+t^2-c)}}{2 \cdot 1}$$

$$y = \frac{1-2t^2}{2} \pm \frac{\sqrt{(2t^2-1)^2 - 4t^4 - 4t^2 + 4c}}{2}$$

$y(1)=0$:

$$0 = -\frac{1}{2} \pm \frac{\sqrt{1-8+4c}}{2}$$

$$= \frac{-1 + \sqrt{1-8+4c}}{2}$$

$$\sqrt{1-8+4c} = 1$$

$$1-8+4c = 1$$

$$4c = 8$$

$$c = 2$$

$$y = \frac{1-2t^2}{2} + \frac{\sqrt{(2t-1)^2 - 4t^4 - 4t^2 + 8}}{2}$$

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Exam: 3h.

12 problems + 1 extra pb.
each counts 6p.
12 pb = 100%

Grading scale: (not final but starting pt)

A: 92% B: 77% C: 58% D: 46% E: 40%

max/min $f(\underline{x})$
- find stationary pts.
- classify stat. pts
Second derivative test:
$H(f)(\underline{x}^*)$ w defn.
- global max/min
- convex/concave fn's.
- envelope thm.

(C) Unconstrained optimization

3. a) $f(x,y,z) = \ln(u+1)$, where $u = 2x^2 + 2xy + 3y^2 - 2xz + z^2$

Definiteness of u:

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$D_1 = 2$
$D_2 = 5$
$D_3 = -1\cdot 3 + 1\cdot 5 = 2$

u pos. defn.

$$H(u) = \begin{pmatrix} 4 & 2 & -2 \\ 2 & 6 & 0 \\ -2 & 0 & 2 \end{pmatrix} = 2A$$

$u'_x = 4x + 2y - 2z$
$u'_y = 2x + 6y$
$u'_z = -2x + 2z$

$D_1 = 4$
$D_2 = 20$
$D_3 = -2\cdot 12 + 2\cdot 20 = 16$

u pos. defn.

D_f: all pts (x,y,z) = $\mathbb{R}^3$

$f = \ln(u+1)$
Defined if $u+1 > 0$
$u > -1$

u pos. defn:
$u(x,y,z) \geq 0$
$u+1 \geq 1$

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b) $f = \ln(u+1)$, $u = 2x^2 + 2xy + 3y^2 - 2xz + z^2$

$f'_x = \frac{1}{u+1} \cdot u'_x = \frac{1}{u+1} \cdot \underline{(4x + 2y - 2z)} = 0$

$f'_y = \frac{1}{u+1} \cdot u'_y = \frac{1}{u+1} \cdot \underline{(2x + 6y \quad)} = 0$

$f'_z = \frac{1}{u+1} \cdot u'_z = \frac{1}{u+1} \cdot \underline{(-2x + 2z \quad)} = 0$

linear sys.
⇓
can use
Gauss.

$4x + 2y - 2z = 0$
$2x + 6y \quad = 0 \qquad y = -2x/6 = -x/3$
$-2x \quad + 2z = 0 \qquad \Rightarrow x = z$

$\underline{z = x}, \ y = \underline{-x/3}$

$4x + 2 \cdot (-x/3) - 2 \cdot x = 0 \quad | \cdot 3$

$12x - 2x - 6x = 0$

$4x = 0$

$\underline{x = 0} \quad \underline{y = 0} \quad \underline{z = 0}$

<u>Stat. pt:</u> $(0,0,0)$

c) If f has <u>min</u>, it must be $\underline{(0,0,0)}$ $\quad f(0,0,0) = \ln 1 = \underline{0}$

<u>Is f convex?</u> $H(f) = (f'')$ Difficult

$f''_{xx} = \left[\left(\frac{1}{u+1}\right) \cdot (4x + 2y - 2z)\right]'_x = -\frac{1}{(u+1)^2} \cdot u'_x \cdot u'_x + \frac{1}{u+1} \cdot 4$

$f''_{xy} = \left[\left(\frac{1}{u+1}\right) \cdot (4x + 2y - 2z)\right]'_y = -\frac{1}{(u+1)^2} \cdot u'_y \cdot u'_x + \frac{1}{u+1} \cdot 2$

⋮

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$$H(f) = \frac{1}{u+1}\cdot\begin{pmatrix} 4 & 2 & -2 \\ 2 & 6 & 0 \\ -2 & 0 & 2 \end{pmatrix} - \frac{1}{(u+1)^2}\begin{pmatrix} u'_x \cdot u'_x & u'_x \cdot u'_y & \dots \\ \vdots & & \\ \vdots & & \end{pmatrix}$$

not convex

$$D_1 = \frac{1}{u+1}\cdot 4 - \frac{1}{(u+1)^2}\left(\left(4x+2y-2z\right)^2\right)$$

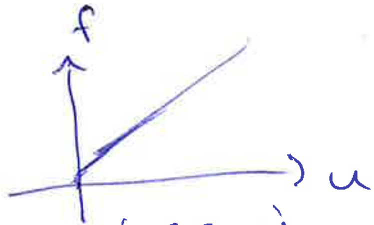
$$= \frac{4(u+1) - (4x+2y-2z)^2}{(u+1)^2}$$ both pos & neg. values

f is not convex

How to find min without the Hessian.

$f = \ln(u+1)$, $u = 2x^2 + \dots$

$f'_u = \frac{1}{u+1}\cdot 1 = \frac{1}{u+1} > 0$



increasing function.

Min for f is where u is minimal.

$u \geq 0 \rightsquigarrow u=0$ is the min. value of u

$f(u)=0$ is the minimal value of f.

$(x,y,z) = (0,0,0)$ gives $u=0$ and is global min

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D) Unconstrained optimization

Review:
- Lagrange : Cand pts: FOC + C
- K-T : Cand pts: FOC + C + CSC

$g(x_1, \dots, x_n) \leq a$:

CSC: $\lambda \geq 0$
$\lambda \cdot (g(\underline{x}) - a) = 0$

SOC:

$(x_1^*, \dots, x_n^*; \lambda_1^*, \dots, \lambda_m^*) \rightsquigarrow \mathcal{L}(x_1, \dots, x_n; \lambda_1^*, \dots, \lambda_m^*)$

fn. in $x_1, \dots, x_n$

concave $\to$ cand pt max
convex $\to$ " min

EVT: need $\{g_i(\underline{x}) = a_i\}$ is bounded
$\Downarrow$
there exists a max/min.

4. b) min $x^2 + y^2$ when $xy \geq 4$ KT.
max $-(x^2+y^2)$ $\quad -xy \leq -4$

$L = -x^2 - y^2 + \lambda(xy)$

FOC
$$\boxed{\begin{aligned} L'_x &= -2x + \lambda y = 0 \\ L'_y &= -2y + \lambda x = 0 \end{aligned}}$$

$xy = 4$	$xy > 4$
$\lambda \geq 0$	$\lambda = 0$
FOC	FOC
	$x = 0, y = 0$, $\lambda = 0$
	$xy \not> 4$
	no cand.

$x = \frac{\lambda}{2} y$

$-2y + \lambda \cdot (\frac{\lambda}{2} y) = 0$

$y(-2 + \lambda^2/2) = 0$

$y = 0$ or $\lambda^2 = 4 \Rightarrow \lambda = 2$

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b)

$\underline{y=0}$: $x=0$

$xy \neq 4$ not possible.

$\underline{\lambda=2}$: $-2x+2y=0$

$\underline{x=y}$

$xy=4$

$x^2=4$

$x=\pm 2$

$\underline{(2,2;2)}$, $\underline{(-2,-2;2)}$

$f=8$ $f=8$

$\underline{SOC}$:

$\mathcal{L}(x,y;2)=$

$-x^2-y^2+2xy$

$$H=\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \quad D_1=-2 \quad D_2=0 \quad \Delta_1=-2,-2 \quad \Delta_2=0$$

neg. semidefin.

$\Downarrow$

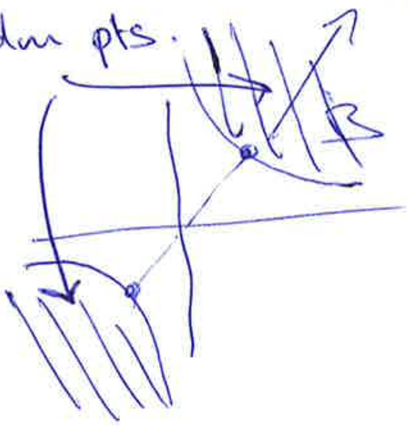
$\mathcal{L}(x,y;2)$ concave

$\Downarrow$

$\underline{\underline{(\pm 2,\pm 2) \text{ is min}}}$

(max for $-f$ = min f)

a) $\underline{xy \geq 4}$: Adm pts.



$xy=4$:

$y=4/x$

It is $\underline{\text{not bounded}}$

min $f=x^2+y^2$

$\shortparallel$

min $\sqrt{x^2+y^2}$

$\shortparallel$

dist. to $(0,0)$

$=\sqrt{u}$, $u=x^2+y^2$

there is a min since there is a pt in $\underline{R}$ with min. distance to the origin.