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LECTURE 4

GRA6035

MATHEMATICS

EIVIND ERIKSEN

SEP 15, 2016

Plan:

1. Review: Vectors and linear dependence
2. Eigenvalues and eigenvectors
3. Diagonalization

1 Review: Vectors and linear dependence

$\{\underline{v}_1, \ldots, \underline{v}_n\}$: Linearly dependent if one of the vector (at least) can be written as a linear comb. of the others

$\Updownarrow$

The linear system $x_1 \cdot \underline{v}_1 + x_2 \cdot \underline{v}_2 + \ldots + x_n \underline{v}_n = \underline{0}$ has non-trivial solutions $(x_1, \ldots, x_n) \neq (0, 0, \ldots, 0)$.

$\Updownarrow$

The linear system has free variables (infinitely many solutions).

Method: $A = (\underline{v}_1 | \underline{v}_2 | \ldots | \underline{v}_n) \rightsquigarrow$ Find pivot positions.

If there are pivot positions in col's $i_1, i_2, \ldots, i_k$, then $\{\underline{v}_{i_1}, \underline{v}_{i_2}, \ldots, \underline{v}_{i_k}\}$ are a maximal subset of linear independent vectors from $\{\underline{v}_1, \ldots, \underline{v}_n\}$.

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$\mathbb{R}^n$: Euclidean n-space, the set of all n-vectors with entries in $\mathbb{R}$, the real numbers

$$\mathbb{R}^n = \left\{ \underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} : v_1, v_2, \dots, v_n \in \mathbb{R} \right\}$$

":" such that; "$\in$" "is an element of"

<u>Linear subspace:</u>

A subset V of $\mathbb{R}^n$ that can be written as a span

$$V = \text{span}\{\underline{v}_1, \dots, \underline{v}_k\}$$

is called <u>linear subspace</u> of $\mathbb{R}^n$.

A <u>base</u> of a linear subspace V is a set of vectors $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k\}$ such that

i) $\text{span}(\underline{v}_1, \dots, \underline{v}_k) = V$

ii) $\{\underline{v}_1, \dots, \underline{v}_k\}$ are linearly independent

The <u>dimension</u> of V is the number of vectors in a base of V.

<u>Ex:</u> $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ $\underline{v}_3 = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix}$ $\rightsquigarrow A = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 2 & 4 \\ 1 & 0 & -2 \end{pmatrix}$

$V = \text{span}(\underline{v}_1, \underline{v}_2, \underline{v}_3)$ linear subspace of $\mathbb{R}^3$.

<u>not</u> linearly independent : $\underline{v}_3 = -2\underline{v}_1 + 3\underline{v}_2$

$\{\underline{v}_1, \underline{v}_2\}$ is base of V and $\underline{\dim V = 2}$.

<u>Fact:</u> The set of all solutions of a <u>homogeneous</u> $\left\{ \begin{matrix} m \times n\text{-} \\ \text{linear} \end{matrix} \right.$ system $A \cdot \underline{x} = \underline{0}$ is a linear subspace of $\mathbb{R}^n$.

2 Eigenvalues and eigenvectors

A (n×n-matrix) $\rightsquigarrow$ "Function" (Linear transformation) $\underline{v}$ (n-vect) $\rightsquigarrow$ $A\cdot\underline{v}$ (n-vect.)

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \longrightarrow T(\underline{v}) = A\cdot\underline{v}$

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Definition:

A number λ is an eigenvalue (characteristic value) of A if the equation

$$A\cdot\underline{v} = \lambda\cdot\underline{v}$$

homogeneous linear system with parameter λ

has non-trivial solutions, $\underline{v} \neq \underline{0}$. In that case,

$E_\lambda = \{\underline{v} : A\underline{v} = \lambda\underline{v}\}$ (all solutions of this equation)

is called the set of eigenvectors of A with eigenvalue λ. E_λ is called the eigenspace.

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad \underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad \underline{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$A\underline{v}_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3\cdot\underline{v}_1$$

$\lambda = 3$ is an eigenvalue, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector

$$A\underline{v}_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1\cdot\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1\cdot\underline{v}_2$$

$\lambda = 1$ is an eigenval. $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenv.

$$A\cdot\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\cdot\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \end{pmatrix} \neq \lambda\cdot\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad \underline{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad \lambda$

$A \cdot \underline{v} = \lambda \underline{v}$: $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \cdot \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{pmatrix} 2x + y \\ x + 2y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}$$

$$\begin{aligned} 2x + y &= \lambda x \\ x + 2y &= \lambda y \end{aligned}$$

$$\begin{aligned} (2-\lambda)x + y &= 0 \\ x + (2-\lambda)y &= 0 \end{aligned}$$

$$\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff A\underline{v} = \lambda \underline{v}$$

Fact: λ is an eigenvalue for A $\iff$ $\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$ characteristic equation

$$\begin{aligned} (2-\lambda)\cdot(2-\lambda) - 1^2 &= 0 \\ \lambda^2 - 4\lambda + 3 &= 0 \end{aligned}$$

$\underline{\lambda = 3}, \underline{\lambda = 1}$

Eigenvalues for A: $\lambda_1 = \underline{3}$, $\lambda_2 = \underline{1}$

Eigenvectors:

$\underline{\lambda = 3}$: $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad -x + y = 0$, $\cancel{x - y = 0}$ $\quad$ y free, $x = y$

Solutions: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \text{span}(\underline{v_1})$, $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$E_3 = \text{span}(\underline{v_1})$

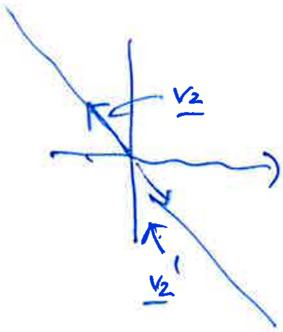
$\underline{\lambda = 1}$: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad x + y = 0$, $\cancel{x + y = 0}$ $\quad y$ free, $x = -y$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \rightarrow \text{span}(\underline{v_2}), \ v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$E_1 = \text{span}(\underline{v_2}), \ \underline{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$\underline{\text{Alt:}}$ x free, $y = -x$ $\quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -x \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$E_1 = \text{span}(\underline{v_2}'), \ v_2' = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



General method:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$A\underline{u} = \lambda \underline{u}$$

$$A\underline{u} - \lambda \underline{u} = \underline{0}$$

$$A\underline{u} - \lambda I \underline{u} = 0$$

$$(A - \lambda I)\underline{u} = \underline{0}$$ ← homogeneous lin. sys. with par. λ.

$$\begin{pmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Characteristic equation:

$$\det(A - \lambda I) = 0$$ $\longleftrightarrow$ $\underline{\text{Eigenvalues}}$ of A is the solutions of the char. eqn.

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$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

$(-\lambda)^n$ + lower degree term $= 0$

polynomial eqn. of degree n.

Facts: (1) If we can factorize $\det(A-\lambda I)$ in linear factors:

$(-1)^n \cdot (\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n) = 0$

then there are n eigenvalues

$\lambda_1, \lambda_2, \ldots, \lambda_n$

If $\lambda_1 = \lambda_2 = \ldots = \lambda_m$ is repeated m times, we say that ~~the~~ the eigenvalue λ_1 has multiplicity m.

(2) If there are n eigenvalues $\lambda_1, \ldots, \lambda_n$, then

$\det(A) = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n$

$a_{11} + a_{22} + \ldots + a_{nn} = \text{tr}(A) = \lambda_1 + \lambda_2 + \ldots + \lambda_n$

(trace of A)

(3) If A ~~is~~ symmetric $(A^t = A)$, then A has n eigenvalues.

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$

$(a-\lambda)(d-\lambda) - bc = 0$

$\lambda^2 - (a+d)\lambda + (ad-bc) = 0$

$$\boxed{\lambda^2 - tr(A)\lambda + det(A) = 0}$$

Ex: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\lambda^2 - 0 \cdot \lambda + 1 = 0$

$\lambda^2 + 1 = 0$

no real solutions $\longrightarrow$ no eigenvalues.

Eigenvectors:

If $\lambda = \lambda_i$ is an eigenvalue, then solve

$$\begin{pmatrix} a_{11}-\lambda_i & a_{12} & - \\ a_{21} & a_{22}-\lambda_i & - \\ \vdots & \vdots & - \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Ex: $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

$\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 3-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$

Char. eqn.

$(3-\lambda) \cdot \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$(3-\lambda) \cdot (\lambda^2 - 4\lambda + 3) = 0$

$\lambda = 3$ or $\lambda^2 - 4\lambda + 3 = 0$

$\lambda = 3, \lambda = 1$

$\underline{\lambda_1 = \lambda_2 = 3}$, $\underline{\lambda_3 = 1}$

Eigenvectors:

$\underline{\lambda = 3}$: $\begin{pmatrix} -1 & 0 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{1} & \cancel{0} & \cancel{-1} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

y, z free
$x = z$

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y, z : free
$x = z$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y \\ z \end{pmatrix} = y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$E_3 = \text{span}(\underline{v_1}, \underline{v_2}), \quad \underline{v_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\underline{v_2} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

If free variables are chosen correctly, then $\{\underline{v_1}, \underline{v_2}\}$ is a base.

Conclusion: $\dim E_\lambda = \#$ free var's in $(A - \lambda I) \cdot \underline{x} = \underline{0}$.

Fact: If λ is an eigenvalue for A of multiplicity m, then $1 \leq \dim E_\lambda \leq m$.

3 Diagonalization.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix} \quad \cdots \quad A^{100} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^{100} & 0 \\ 0 & 0 & 3^{100} \end{pmatrix}$$

Defn: An $n \times n$-matrix A is diagonalizable if there is a diagonal matrix D and an invertible matrix P such that

$$P^{-1} A P = D$$

Ex: If $P^{-1} A P = D$, then $A = \underline{P D P^{-1}}$ This means

$$A^n = (PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}}) \cdots (\cancel{P}DP^{-1}) = PD \cdot D \cdots D P^{-1}$$
$$= P \cdot D^n P^{-1}$$

How to diagonalize a matrix A:

$\underset{n\times n}{A} \rightsquigarrow$

i) Compute eigenvalues of A:

$\lambda_1, \lambda_2, \ldots, \lambda_k \qquad (k \leq n)$

(repeated according to multiplicity)

If $k=n$, then $D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$

ii) Compute eigenvectors for A:

$\lambda \rightsquigarrow E_\lambda = \text{span}(\underline{v}_{\lambda,1}, \underline{v}_{\lambda,2}, \ldots, \underline{v}_{\lambda,r})$

$r \leq m$, the multiplicity of λ

If $r=m$ for all eigenvalues λ, then

$$P = \left(\underline{v}_1 \middle| \underline{v}_2 \middle| \cdots \middle| \underline{v}_n \right)$$

Then P is invertible since $\{\underline{v}_1, \ldots, \underline{v}_n\}$ are linearly independent

A diagonalizable $\iff$ $\begin{cases} \text{i) A has enough eigenvalues } (k=n) \\ \text{ii) A has enough eigenvectors } (r=m \text{ for each } \lambda) \end{cases}$

Explanation: $P^{-1}AP = D \iff AP = PD$

$$AP = A \cdot \left(\underline{v}_1 \middle| \underline{v}_2 \middle| \cdots \middle| \underline{v}_n \right) = \left(A\underline{v}_1 \middle| A\underline{v}_2 \middle| \cdots \middle| A\underline{v}_n \right) = \left(\lambda_1 \cdot \underline{v}_1 \middle| \lambda_2 \underline{v}_2 \middle| \cdots \middle| \lambda_n \underline{v}_n \right)$$

$$PD = \left(\underline{v}_1 \middle| \underline{v}_2 \middle| \cdots \middle| \underline{v}_n \right) \cdot \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} = \left(\lambda_1 \cdot \underline{v}_1 \middle| \lambda_2 \underline{v}_2 \middle| \cdots \middle| \lambda_n \underline{v}_n \right)$$

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Fact: If A is symmetric, then it is diagonalizable.

Note: If A is not symmetric, it may or may not be diagonalizable. One should check eigenvalues / eigenvectors.

Ex: $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ A symmetric, so it is diagonalizable.

Eigenvalues:

$$\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 3-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = (3-\lambda)(\lambda^2 - 4\lambda + 3) = 0$$

$\lambda_1 = 3$ or $\lambda^2 - 4\lambda + 3 = 0$, $\lambda_2 = 3$, $\lambda_3 = 1$

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Eigenvectors:

$\lambda = 3$: $\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ y, z free, $x = z$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y \\ z \end{pmatrix} = y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$E_3 = \text{span}(\underline{v_1}, \underline{v_2})$, $\underline{v_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 1$: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ z free, $y = 0$, $x = -z$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$E_1 = \text{span}(\underline{v_3})$, $\underline{v_3} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$$P = \left(\underline{v_1} \,\middle|\, \underline{v_2} \,\middle|\, \underline{v_3} \right) = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

We know that $P^{-1}AP = D$

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Worked examples:

① Compute the rank of $A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & t \\ 4 & 7-t & -6 \end{pmatrix}$:

Note: $\operatorname{rk} A = 3 \Leftrightarrow |A| \neq 0$.

$$\begin{vmatrix} 1 & 3 & 2 \\ 2 & 5 & t \\ 4 & 7-t & -6 \end{vmatrix} = 1 \cdot (-30 - t(7-t)) - 3(-12-4t) + 2(2(7-t)-20)$$

$$= t^2 - 7t - 30 + 36 + 12t + 28 - 4t - 40$$

$$= t^2 + t - 6$$

$|A| = 0$: $t^2 + t - 6 = 0$

$$t = \frac{-1 \pm \sqrt{1+24}}{2} = -\frac{1}{2} \pm \frac{5}{2} = 2, -3$$

$t \neq 2, -3$: $\operatorname{rk} A = 3$

$t = 2$: $\operatorname{rk} A < 3$, $M_{12,12} = \begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = 5 - 6 = -1 \neq 0 \Rightarrow \operatorname{rk} A = 2$

$t = -3$: $\operatorname{rk} A < 3$, — 〃 — $= -1 \neq 0 \Rightarrow \operatorname{rk} A = 2$

Conclusion:

$$\operatorname{rk} A = \begin{cases} 3, & t \neq 2, -3 \\ 2, & t = 2, -3 \end{cases}$$

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2 Is $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ diagonalizable?

Not symmetric, so must compute eigenvalues/eigenvectors:

Eigenvalues: $\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = 0$

$\lambda^2 - 2\lambda + 1 = 0$

$(\lambda-1)^2 = 0$ or $\lambda = \frac{+2 \pm \sqrt{4-4}}{2} = 1 \pm 0$

$\lambda_1 = \lambda_2 = 1$ (multiplicity 2)

$D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ we have enough eigenvalues (2)

Eigenvectors: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ x free, $y = 0$ $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$E_1 = \text{span}(\underline{v_1})$, $\underline{v_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\dim E_1 = 1 < 2 = \text{mult.}(\lambda = 1)$

Not enough eigenvectors

$P = \left(\underline{v_1} \,\middle|\, ?\right) = \left(\begin{matrix} 1 \\ 0 \end{matrix} \,\middle|\, ?\right)$

Conclusion:
$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.