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LECTURE 7

GRA 6035 MATHEMATICS

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OKT 6, 2016

Plan:

1. Constrained optimization and admissible points
2. Lagrange problems
3. Kuhn-Tucker problems

Reading:

[ME] 18.1–18.7, (12.3–12.5), 21.1

Review: Unconstrained optimization

$$f(\underline{x}) = f(x_1, x_2, \dots, x_n) \rightsquigarrow \max_{\underline{x}}/\min f(\underline{x})$$

Method:

① $\left.\begin{array}{l} f'_{x_1} = 0 \\ f'_{x_2} = 0 \\ \vdots \\ f'_{x_n} = 0 \end{array}\right\}$ FOC $\rightsquigarrow$ Solutions = Stationary points candidates for max/min

② Classify stationary points

$\underline{x}^* = (x_1^*, x_2^*, \dots, x_n^*)$ stationary point

Second derivative test (SOC)

$\begin{pmatrix} f''_{x_1x_1} & f''_{x_1x_2} & \cdots \\ \vdots & & \end{pmatrix}$

$H(f)(\underline{x}^*)$ pos. definite $\Rightarrow \underline{x}^*$ is a local min

$H(f)(\underline{x}^*)$ neg. definite $\Rightarrow \underline{x}^*$ is a local max

$H(f)(\underline{x}^*)$ indefinite $\Rightarrow \underline{x}^*$ is a saddle point

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3 Convex/concave functions

f is convex $\iff$ $H(f)(\underline{x})$ is positive semidefinite for all $\underline{x}$ (in D_f)

f is concave $\iff$ $H(f)(\underline{x})$ is negative semidefinite for all $\underline{x}$ (in D_f)

If f is convex: Any stationary point is global min
— " — concave: — " — is global max

Ex: $f(x,y) = x^2y^3 + y^2 - 2y$

1. $$\left.\begin{aligned} f'_x &= 2xy^3 = 0 \\ f'_y &= 3x^2y^2 + 2y - 2 = 0 \end{aligned}\right\} \text{FOC}$$

$2xy^3 = 0 \iff x = 0$ or $y = 0$

$x=0$	$y=0$
$2y-2=0$	$-2=0$
$y=1$	impossible
$\downarrow$	$\downarrow$
$(0,1)$	no solution

Stationary pts: $(x,y) = (0,1)$

2. $$H(f)(x,y) = H(f) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{pmatrix} = \begin{pmatrix} 2y^3 & 6xy^2 \\ 6xy^2 & 6x^2y+2 \end{pmatrix}$$

$$H(f)(0,1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$D_1 = 2$
$D_2 = 4$

$H(f)(0,1)$ pos. definite
$(x,y) = (0,1)$ local min.

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③ f convex / concave ?

$$H(f)(x,y) = \begin{pmatrix} \boxed{2y^3} & 6xy^2 \\ 6xy^2 & 6x^2y+2 \end{pmatrix}$$

$D_1 = 2y^3$ ← What can we say about the sign of D_1?

$D_2 =$

$2y^3 > 0$ or $2y^3 < 0$

$(y>0)$ $(y<0)$

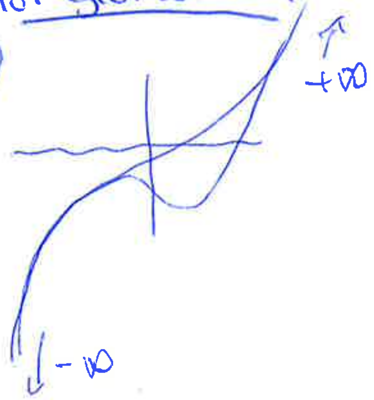
$\Downarrow$

f is not convex, not concave

$\cancel{(D_1 \geq 0)}$ $\cancel{(D_1 \leq 0)}$

$f(x,y) = x^2y^3 + y^2 - 2y$

$(0,1)$ is local min $f(0,1) = -1$ ← not global min



$x=a,\ y=a$: $f(a,a) = \underline{\underline{a^5}} + a^2 - 2a$

$\lim_{a \to -\infty} f(a,a) = -\infty$

$\underline{a=-2}$: $f(-2,-2) = (-2)^5 + (-2)^2 - 2(-2)$

$= 8 - 32 = -24$

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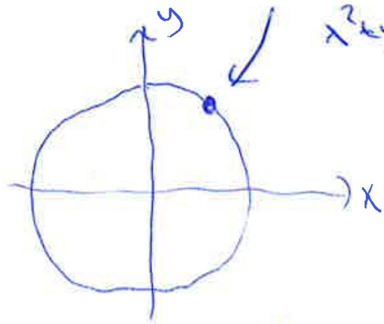
1 Constrained optimization

Ex: max $\boxed{f(x,y) = x+3y}$ when $\boxed{x^2+y^2=1}$

What is the maximal value of $x+3y$ among the points on the circle

(x,y) s.t. $x^2+y^2=1$



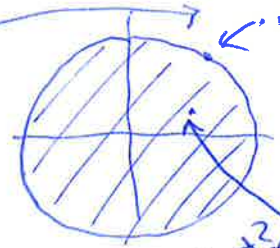
circle = admissible points

Constraints: – Equality constraints (ex: $x^2+y^2=1$)

– Inequality constraints (ex: $x^2+y^2\leq 1$)

boundary points

$x^2+y^2=1$



interior points

$x^2+y^2<1$

a) Equality constraints: Lagrange problem

b) Closed inequality constraints: Kuhn-Tucker problem

(constraints given by $\leq$ or $\geq$)

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Problem: max/min $f(\underline{x})$ when $\underbrace{\underline{x} \text{ in } S}$

$f(\underline{x}) = f(x_1, x_2, \dots, x_n)$

Lagrange case: max/min $f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) = a_1 \\ g_2(\underline{x}) = a_2 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$

Kuhn-Tucker case: max/min $f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) \le a_1 \\ g_2(\underline{x}) \le a_2 \\ \vdots \\ g_m(\underline{x}) \le a_m \end{cases}$

Notation: $f(\underline{x})$ objective function

$S =$ set of all points that satisfy all constraints } set of admissible points

a) Lagrange case:

max/min $f(\underline{x}) = f(x_1, \dots, x_n)$ such that $\begin{cases} g_1(\underline{x}) = a_1 \\ g_2(\underline{x}) = a_2 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$

Ex: max/min $f(x,y) = x + 3y$ when $\underbrace{x^2+y^2}_{g(x,y)} = \underset{a}{10}$

$$\mathcal{L} = \mathcal{L}(x,y;\lambda) = f(x,y) - \lambda \cdot g(x,y)$$
$$= x + 3y - \lambda \cdot (x^2+y^2)$$

Lagrangian
λ: Lagrange multiplier

FOC: $\mathcal{L}'_x = f'_x - \lambda \cdot g'_x = 1 - \lambda \cdot 2x = 0$

$\mathcal{L}'_y = f'_y - \lambda \cdot g'_y = 3 - \lambda \cdot 2y = 0$

C: $x^2 + y^2 = 10$

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FOC+C: Lagrange conditions

We solve FOC+C → Candidates for max/min in the Lagrange problem

$1 - \lambda \cdot 2x = 0$ → $1 = \lambda \cdot 2x \to x = \frac{1}{2\lambda}$

$3 - \lambda \cdot 2y = 0$ → $3 = \lambda 2y \to y = 3/2\lambda$

$x^2 + y^2 = 10$

what if $\lambda = 0$? no solution

$$\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{3}{2\lambda}\right)^2 = 10$$

$$\frac{1}{4\lambda^2} + \frac{9}{4\lambda^2} = 10$$

$$\frac{10}{4\lambda^2} = 10 \qquad 4\lambda^2 = 1 \qquad \lambda^2 = 1/4$$

$$\lambda = \pm 1/2$$

$\lambda = 1/2$: $x = 1$, $y = 3$

$\lambda = -1/2$: $x = -1$, $y = -3$

Candidates for max/min:

$(x, y; \lambda) = (1, 3; 1/2)$, $(-1, -3; -1/2)$

$f(1,3) = 10$ — best candidate for max

$f(-1,-3) = -10$ — best candidate for min

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In general:

max/min $f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) = a_1 \\ g_2(\underline{x}) = a_2 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$

$\underline{x}$ = $f(x_1, x_2, \ldots, x_n)$

$$\mathcal{L} = \mathcal{L}(x_1, x_2, \ldots, x_n; \lambda_1, \lambda_2, \ldots, \lambda_m) \quad \text{Lagrangian}$$
$$= f(\underline{x}) - \lambda_1 \cdot g_1(\underline{x}) - \lambda_2 \cdot g_2(\underline{x}) - \ldots - \lambda_m \cdot g_m(\underline{x})$$

FOC: $\boxed{\begin{array}{l} \mathcal{L}'_{x_1} = 0 \\ \mathcal{L}'_{x_2} = 0 \\ \vdots \\ \mathcal{L}'_{x_n} = 0 \end{array}}$ C: $\boxed{\begin{array}{l} g_1(\underline{x}) = a_1 \\ g_2(\underline{x}) = a_2 \\ \vdots \\ g_m(\underline{x}) = a_m \end{array}}$

FOC+C: Lagrange conditions $\begin{cases} m+n \text{ equations} \\ m+n \text{ variables} \end{cases}$

$x_1, x_2, \ldots, x_n$

$\lambda_1, \lambda_2, \ldots, \lambda_m$

Lagrange multipliers

{ solve the system of equations

Candidates for max/min.

Extreme value theorem (EVT)

If $f(\underline{x})$ is a continuous function on a compact set S, then f has a (global) max/min.

Application in the Lagrange setting:

$$\max/\min f(\underline{x}) \quad \text{when} \quad \begin{cases} g_1(\underline{x}) = a_1 \\ g_2(\underline{x}) = a_2 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$$

* $f(\underline{x})$ is continuous : ok
* $S = \{\underline{x} : g_1(\underline{x}) = a_1, g_2(\underline{x}) = a_2, \ldots, g_m(\underline{x}) = a_m\}$
  set of admissible points, i.e. points that satisfy all constraints.

Question: Is S compact?

$$S \text{ compact} \iff \begin{cases} S \text{ is closed} \quad \text{ok} \\ + \\ S \text{ is bounded} \end{cases}$$

Defn: Compact = closed and bounded

closed: all boundary points of S are included in S

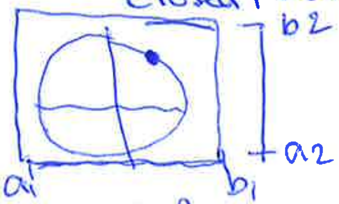
bounded: there are (finite) numbers $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ such that

$$\boxed{\begin{array}{l} a_1 \leq x_1 \leq b_1 \\ a_2 \leq x_2 \leq b_2 \\ \vdots \\ a_n \leq x_n \leq b_n \end{array}}$$

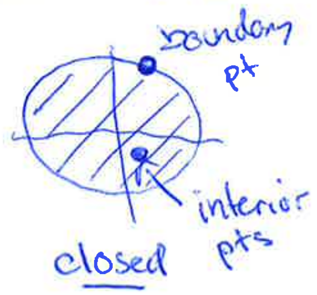
for all $(x_1, \ldots, x_n)$ in S

ex: $x^2 + y^2 = 10$

closed, bounded



ex: $x^2 + y^2 \leq 10$



In a Lagrange problem, if the set S of admissible points is bounded, then the problem has a solution (global max/min).

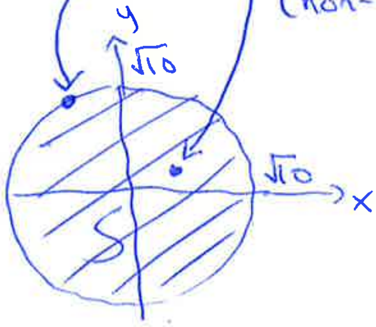
3 Kuhn-Tucker problems:

Closed inequality constraints: $\boxed{\leq \quad \geq}$

max $f(x,y) = x+3y$ when $x^2+y^2 \leq 10$

Divide: $x^2+y^2 = 10$ (binding) $\longrightarrow$ Lagrange problem

$x^2+y^2 < 10$ (non-binding) $\longrightarrow$ unconstrained problem



Kuhn-Tucker formulation:

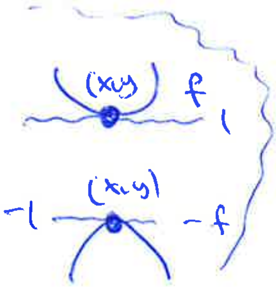
Std. form: max $f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) \leq a_1 \\ g_2(\underline{x}) \leq a_2 \\ \vdots \\ g_m(\underline{x}) \leq a_m \end{cases}$

How: Ex: $xy \geq 1 \quad | \cdot (-1)$

$-xy \leq -1$

Ex: min x^2+y^2 when $x+y \leq 10$

$= \max -(x^2+y^2)$ when $x+y \leq 10$



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Assume: (Std. form) max $f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) \leq a_1 \\ \vdots \\ g_m(\underline{x}) \leq a_m \end{cases}$

$$\mathcal{L} = f(\underline{x}) - \lambda_1 g_1(\underline{x}) - \lambda_2 g_2(\underline{x}) - \dots - \lambda_m \cdot g_m(\underline{x})$$

FOC: $\boxed{\begin{matrix} \mathcal{L}'_{x_1} = 0 \\ \mathcal{L}'_{x_2} = 0 \\ \vdots \\ \mathcal{L}'_{x_n} = 0 \end{matrix}}$ C: $\boxed{\begin{matrix} g_1(\underline{x}) \leq a_1 \\ g_2(\underline{x}) \leq a_2 \\ \vdots \\ g_m(\underline{x}) \leq a_m \end{matrix}}$ CSC:

CSC: Complementary slackness conditions

$\lambda_1 \geq 0$ and $\lambda_1 \cdot (g_1(\underline{x}) - a_1) = 0 \iff \begin{cases} \lambda_1 \geq 0 \text{ and} \\ \lambda_1 = 0 \text{ if constraint 1 is nonbinding.} \end{cases}$

$\lambda_2 \geq 0$ and $\lambda_2 \cdot (g_2(\underline{x}) - a_2) = 0$

$\vdots$

$\lambda_m \geq 0$ $\quad \lambda_m (g_m(\underline{x}) - a_m) = 0$

Kuhn-Tucker conditions: FOC + C + CSC

$\Big\}$ solve them

Candidates for max