

LECTURE 10

ERVIND ERIKSEN

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6KA 6035

MATHEMATICS

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Plan:

Review Lecture 9

- ① First order differential equations
- ② Separable equations
- ③ Linear equations
- ④ Exact equation

Reading:

[MET] 24.1-24.2,
(24.4-24.6)

[ODE] 1.1-1.7, A1-A5

Review:

- Bordered Hessians: $\left\{ \begin{array}{l} \text{Not core curriculum this year.} \\ \text{Method to find out if a candidate pt.} \\ \text{is local max/min in L/KT problem.} \end{array} \right.$
- Envelope Theorems: $\left\{ \begin{array}{l} \text{Lagrange / KT - problems} \\ \text{unconstrained problems} \end{array} \right. \leftarrow \text{Important!}$

Ex: $\max x+3y$ when $x^2+y^2 \leq 10$

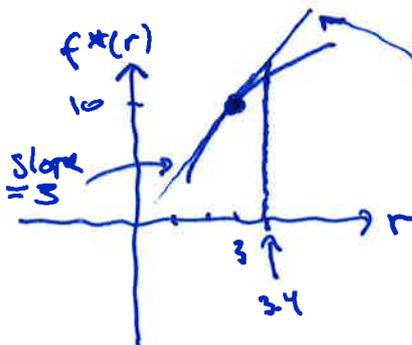
① Solve KT-problem: $(x,y;\lambda) = (1,3; 1/2)$ is max, $f(1,3) = 10$

② What happens if we want to solve: $\max x+ry$ when $x^2+y^2 \leq 10$ for $r \neq 3$
 $L = x+ry - \lambda(x^2+y^2-10)$

Reformulation using the parameter $r \rightarrow (x^*(r), y^*(r); \lambda^*(r))$ max
 $f^*(r)$ max value

Special case $r=3$:

$$\begin{cases} x^*(3) = 1 & y^*(3) = 3 \\ \lambda^*(3) = 1/2 & f^*(3) = 10 \end{cases}$$



③ Envelope Thm:

$$\frac{df^*(r)}{dr} = \frac{\partial L}{\partial r} (x^*(r), y^*(r); \lambda^*(r)) = y^*(r)$$

$$\text{at } \underline{r=3}: \frac{df^*(r)}{dr} (3) = y^*(3) = 3$$

$$\textcircled{4} f^*(3.4) \approx f^*(3) + \underbrace{0.4}_{\Delta r} \cdot 3 = 10 + 1.2 = \underline{\underline{11.2}}$$

① First order differential equations

Defn:

A differential equation in the unknown function $y = y(t)$ is an equation involving t , $y = y(t)$ and the derivation of $y = y(t)$; i.e. the first order derivative $y' = y'(t)$, or higher order derivatives such as $y''(t)$, or both.

Ex: 1) $y'(t) = y(t) + t$
 II
 $y' = y + t$

first order diff. equ.
 (involves y' , but no higher derivatives)

2) $y' = 2t^3$

first order
 — II —

3) $y' = y^2 - 1$

4) $y'' - 2y' + y = t$

second order diff. equ.

Order of a differential equ is the highest order of a derivative that appears in the equ.

First order diff. equ:

$$y' = F(t, y)$$

Ex:

$$y' = 2t$$

$$y = \int 2t \, dt$$

$$y = \underline{\underline{t^2 + C}}$$

Solutions: All functions $y(t)$ such that $y'(t) = 2t$.

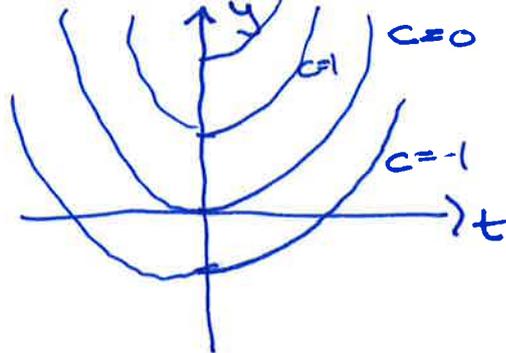
General solution = contains all solutions for different values of C .

$$y' = 2t$$

$$y = t^2 + C$$

$$y(t) = \underline{t^2 + C}$$

general solution

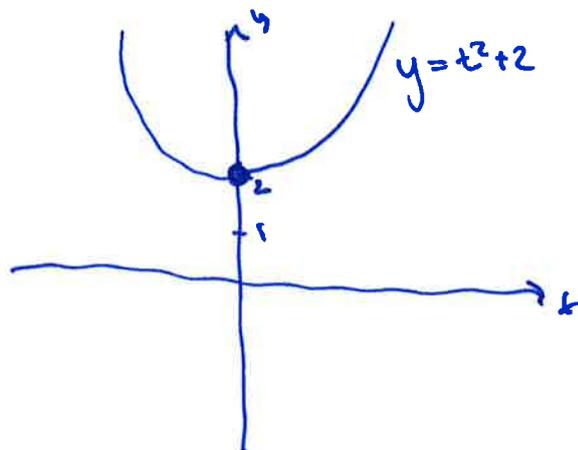


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$$\underline{y' = 2t}, \quad \underline{y(0) = 2} : \quad \text{initial value problem}$$

differential equation

initial value



$$y' = 2t \Rightarrow y = \underline{t^2 + C}$$

general solution

$$\underline{y(0) = 2} : \quad 2 = 0^2 + C$$
$$\underline{C = 2}$$

$$y = \underline{t^2 + 2}$$

particular solution

Motivation:

$$y' = F(t, y) \quad \leftarrow \quad \text{Interpretation:}$$



assumption
of the rate
of growth
of $y = y(t)$

t : time
 y' : change (rate of change)

Conclusion:

The general solution of $y' = F(t, y)$
will depend on one undetermined parameter.

One initial condition \rightarrow unique solution.

To solve differential equation, we must use integration.

$$\boxed{y' = g(t)} \Rightarrow y = \int g(t) dt$$

Ex: $y' = 2y = 2 \cdot y \quad | : y$

$$\frac{1}{y} y' = 2$$

$$\int \frac{1}{y} y' dt = \int 2 dt$$

$$\int \frac{1}{y} dy = 2t + C_2$$

$$\ln |y| + C_1 = 2t + C_2$$

$$\ln |y| = 2t + C_2 - C_1$$

$$e^{\ln |y|} = e^{2t + C_2 - C_1}$$

$$|y| = e^{2t} \cdot e^{C_2 - C_1}$$

$$y = \pm e^{2t} \cdot e^{C_2 - C_1}$$

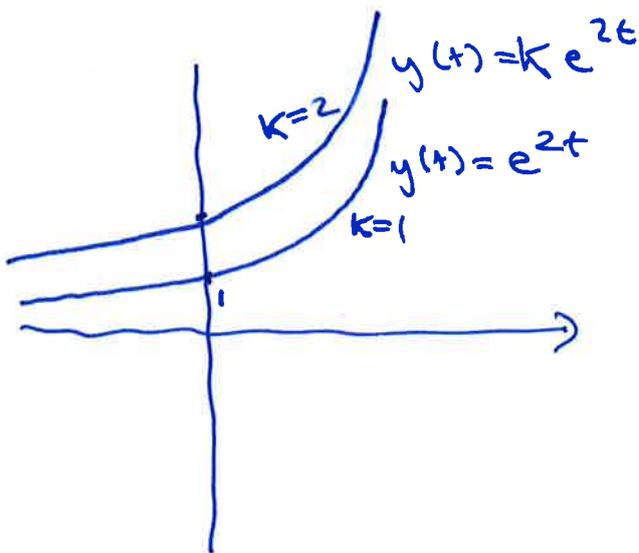
$$y = \underline{\underline{K \cdot e^{2t}}}$$

$$(K = \pm e^{C_2 - C_1})$$

equation, but
not a diff. eqn.

↓
Solution in
implicit form

↑ general solution
(in explicit form)



③ Linear first order equations diff.

Defn: A first order diff. eqn. $y' = F(t, y)$ is linear if it can be written

$$\boxed{y' + a(t) \cdot y = b(t)}$$

$$\Updownarrow$$

$$y' = \underbrace{b(t) - a(t)y}_{\text{linear in } y}$$

Ex: $y' = y + t = t + y$
 \Updownarrow
 $b(t) \quad a(t) = -1$

$$\Downarrow$$

$$\underline{y' - y = t} \quad \underline{\text{linear}} \quad (\text{but not separable})$$

Ex: * $y' = 1 - y^2$ not linear (but separable)

* $y' = 2y$ linear (and separable)

$$\Downarrow$$

$$y' - 2y = 0$$

* $y' = t^2 - y$ linear (not separable)

$$\Downarrow$$

$$y' + y = t^2$$

Solution method:

Integrating factor

Ex: $y' + y = t \cdot u(t)$ $\left\{ \begin{array}{l} a(t) = -1 \\ b(t) = t \end{array} \right.$
linear

$uy' + u'y$
=?

$$u \cdot y' - uy = u \cdot t$$

$$(uy)' dt = \int u(t) \cdot t dt$$

Yes, if
 $u = e^{-t}$!

$$(e^{-t} \cdot y)' dt = \int e^{-t} \cdot t dt$$

$$e^{-t} \cdot y = \int t \cdot e^{-t} dt$$

$$y = e^t \cdot \int t \cdot e^{-t} dt$$

$$= e^t \left(\underbrace{-e^{-t}}_u \cdot \underbrace{t}_v - \int \underbrace{-e^{-t}}_u \cdot \underbrace{1}_{v'} dt \right)$$

$$= e^t (-te^{-t} + (-e^{-t})) + C$$

$$= e^t (-te^{-t} - e^{-t} + C)$$

$$y = \underline{\underline{-t - 1 + C \cdot e^t}}$$

need:

$$u' = -u$$

$$u = \int -1 dt = e^{-t+C} = e^{-t} \cdot e^C = e^{-t}$$

$$u = e^{-t} \quad (C=0)$$

integrating factor

Method of integrating factor:

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$$y' + a(t) \cdot y = b(t) \quad | \cdot u \quad \leftarrow$$

$$u \cdot y' + a(t) u y = b(t) \cdot u$$

$$(u \cdot y)' = b(t) \cdot u$$

$$u \cdot y = \int b(t) \cdot u(t) dt$$

$$y = \frac{1}{u(t)} \int u(t) \cdot b(t) dt$$

Integrating factor:

$$e^{\int a(t) dt} = u$$

||

$$u' = u \cdot a(t)$$

$$(u \cdot y)' = u \cdot y' + u' \cdot y$$

$$= u y' + u \cdot a(t) \cdot y$$

Ex:

$$y' - 3y = 1 \quad | \cdot e^{-3t} \quad \leftarrow$$

$$(e^{-3t} \cdot y)' = e^{-3t}$$

$$e^{-3t} y = \int e^{-3t} dt = -\frac{1}{3} e^{-3t} + C$$

$$e^{-3t} y = -\frac{1}{3} e^{-3t} + C \quad | \cdot e^{3t}$$

$$y = \underline{\underline{-\frac{1}{3} + C \cdot e^{3t}}}$$

Integrating factor:

$$\int -3 dt = -3t + C$$

$$u = e^{-3t+C} = \underline{e^{-3t}}$$

④ Exact diff. equations

Defn. A first order differential eqn. $y' = F(t, y)$ is exact if it can be written

$$p(t, y) + q(t, y) \cdot y' = 0$$

and there exist a function $h = h(t, y)$ such that:

$$\begin{aligned} p(t, y) &= \frac{\partial h}{\partial t} \\ q(t, y) &= \frac{\partial h}{\partial y} \end{aligned}$$



the differential equation can be written

$$\frac{dh}{\partial t} + \frac{\partial h}{\partial y} \cdot y' = 0$$

Solution Method:

$$h(t, y) = C$$

Why? When $y = y(t)$, the

total derivative $\rightarrow \frac{dh(t, y)}{dt} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial y} \cdot y' = 0$

$$\frac{dh(t, y)}{dt} = 0 \Rightarrow h(t, y) = C$$

Ex: $1 + ty^2 + t^2y \cdot y' = 0$

\Downarrow
 $\frac{t^2y \cdot y'}{t^2y} = \frac{-1 - ty^2}{t^2y}$

$y' = \frac{-1 - ty^2}{t^2y}$ not linear
not separable

Exact? $(1 + ty^2) + (t^2y) \cdot y' = 0$

$\frac{dh}{dt} = \frac{dh}{dt} + \frac{\partial h}{\partial y} \cdot \frac{dy}{dt} = 0$

Try to find h s.t: $\textcircled{1} h'_t = 1 + ty^2$
 $\textcircled{2} h'_y = t^2y$ ← partial derivation

$\textcircled{1} h'_t = 1 + ty^2$
 $h = t + \frac{1}{2}t^2 \cdot y^2 + C(y)$
 $h = \int (1 + ty^2) dt = t + y^2 \cdot \int t dt = t + y^2 \cdot \frac{1}{2}t^2 + C(y)$

$\textcircled{2} h'_y = 0 + \frac{1}{2}t^2 \cdot 2y + C'(y) = t^2y$
 $t^2y + C'(y) = t^2y$ ok if $C'(y) = 0$

\Downarrow
 $h = t + \frac{1}{2}t^2y^2 = C$ implicit solution ← choose $C(y) = 0$
eqn. is exact

$\frac{\frac{1}{2}t^2y^2}{\frac{1}{2}t^2} = \frac{C-t}{\frac{1}{2}t^2}$
 $y^2 = \frac{C-t}{\frac{1}{2}t^2} \cdot 2 = \frac{2(C-t)}{t^2}$

$y = \pm \sqrt{\frac{2(C-t)}{t^2}}$ ← $y = \pm \sqrt{\frac{2(C-t)}{t^2}}$ general solution in explicit form.

Explanation: Partial derivatives and total derivative

$h(t, y)$ function of two variables

→ partial derivatives $\left\{ \begin{array}{l} \frac{\partial h}{\partial t} = h'_t \\ \frac{\partial h}{\partial y} = h'_y \end{array} \right.$ ← compute by thinking of y/t as constants

When $y = y(t)$, then we have:

t changes from t to $t + \Delta t$

⇓

y changes to $y(t + \Delta t) \approx y(t) + \Delta t \cdot y'(t)$

⇓

$h(t, y)$ changes to $h(t + \Delta t, y(t + \Delta t))$
 $\approx h(t + \Delta t, y(t) + \Delta t \cdot y'(t))$

$\approx h(t, y) + \underbrace{\frac{\partial h}{\partial t} \cdot \Delta t}_{\text{direct change caused by } t} + \underbrace{\frac{\partial h}{\partial y} \cdot \Delta t \cdot y'}_{\text{indirect change caused by change in } y, \text{ coming from change in } t}$

direct change caused by t

indirect change caused by change in y , coming from change in t

$= h(t, y) + \left(\frac{\partial h}{\partial t} + \frac{\partial h}{\partial y} \cdot y' \right) \Delta t$
 $= \frac{dh}{dt} \cdot \Delta t$ total derivative

$\frac{dh}{dt} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial y} \cdot y'$

total change in $h(t, y)$