How to compute eigenvalues for 3x3 matrices

Ex: $A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 2 & 0 & -2 \end{pmatrix}$

(This is Problem 4.3 b) from the workbook, the methods below are different from the solution in the workbook).

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & -1 \\ 0 & 1-\lambda & 1 \\ 2 & 0 & -2-\lambda \end{vmatrix} = 0$$

Develope the determinant along 1st column:

$$(2-\lambda) \cdot \begin{vmatrix} 1-\lambda & 1 \\ 0 & -2-\lambda \end{vmatrix} + 2 \cdot \begin{vmatrix} 1 & -1 \\ 1-\lambda & 1 \end{vmatrix} = 0$$

$$(2-\lambda) \cdot [(1-\lambda)(-2-\lambda) - 0] + 2(1 + 1-\lambda) = 0$$

$$\boxed{\underline{(2-\lambda)}(1-\lambda)(-2-\lambda) + 2 \cdot \underline{(2-\lambda)} = 0}$$

Notice that we need a factorization of L.H.S. to solve the equ.

Alt A: $(2-\lambda) \cdot [(1-\lambda)(-2-\lambda) + 2] = 0$

(2-λ) factor of both terms!

$$(2-\lambda) \cdot (\lambda^2 + \lambda - 2 + 2) = 0$$

$$(2-\lambda) \cdot (\lambda^2 + \lambda) = 0$$

$\underline{\lambda = 2}$ or $\lambda^2 + \lambda = 0$

$\underline{\lambda = 0}$, $\underline{\lambda = -1}$

Alt B: $(2-\lambda)(1-\lambda)(-2-\lambda) + 2(2-\lambda) = 0$

In case you didn't notice factorization in Alt. A.

$$(2-\lambda)(\lambda^2 + \lambda - 2) + 4 - 2\lambda = 0$$

$$-\lambda^3 + \lambda^2 + 4\lambda \cancel{-4} \cancel{+4} - 2\lambda = 0$$

$$-\lambda^3 + \lambda^2 + 2\lambda = 0$$

$$-\lambda^3+\lambda^2+2\lambda=0$$

$$\lambda\cdot(-\lambda^2+\lambda+2)=0$$

The constant term is zero, hence λ is a factor.

$\underline{\lambda=0}$ or $-\lambda^2+\lambda+2=0$

$$\lambda=\frac{-1\pm\sqrt{1^2-4\cdot(-1)\cdot 2}}{-2}$$

$$=\frac{-1\pm 3}{-2}=-1,2$$

$\underline{\lambda=-1}$, $\underline{\lambda=2}$

What to do if the constant term is non-zero?

<u>Useful fact:</u>

If $\lambda^3+a\lambda^2+b\lambda+c=0$ has an integer solution λ^*, then λ^* must <u>divide c</u>.

<u>Ex</u>: You get $\lambda^3-\lambda^2+3\lambda-10=0$ for the char. eqn. of some matrix.

If there is an integer solution, it must divide -10.

That is, the possibilities are $\pm1, \pm2, \pm5, \pm10$. We check if any of these are solutions. ±1 <u>not</u> soln.

$\lambda=2$ <u>is</u> solution.

This means that the char. eqn. can be written

$$\lambda^3-\lambda^2+3\lambda-10=(\lambda-2)\cdot Q(\lambda)=0$$

$\lambda=2$ is soln. means $(\lambda-2)$ is factor

Find $Q(\lambda)$ by polynomial division:

$$\lambda^3-\lambda^2+3\lambda-10 : \lambda-2 = \lambda^2+\lambda+5$$

$$\begin{array}{l} -(\lambda^3-2\lambda^2) \\ \hline \quad \lambda^2+3\lambda-10 \\ \quad -(\lambda^2-2\lambda) \\ \hline \qquad 5\lambda-10 \\ \qquad 5\lambda-10 \,/\, 0 \end{array}$$

This gives $(\lambda-2)\cdot(\lambda^2+\lambda+5)=0$

$\underline{\lambda=2}$ or $\lambda=\dfrac{-1\pm\sqrt{1^2-4\cdot1\cdot5}}{2}$ (no solution)

So $\lambda=2$ is ~~the~~ only eigenvalue