REVIEW: INTEGRATION

GRA 6035

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MATHEMATICS

Basic functions and their integrals

① $\int t^n \, dt = \frac{1}{n+1} t^{n+1} + C$ for all $n \neq -1$

② $\int \frac{1}{t} \, dt = \ln |t| + C$ (for $n = -1$)

③ $\int e^t \, dt = e^t + C$

Integration of sums, differences and scalar multiplication

④ $\int (u+v) \, dt = \int u \, dt + \int v \, dt$ u, v any functions of t

⑤ $\int (u-v) \, dt = \int u \, dt - \int v \, dt$ — " —

⑥ $\int (c \cdot u) \, dt = c \cdot \int u \, dt$ $u = u(t)$ any function of t, c a constant

Integration by parts (to integrate products)

⑦ $\int u' \cdot v \, dt = uv - \int u \cdot v' \, dt$ u, v any functions of t

Substitution: (to integrate w.r.t. a new variable $u = u(t)$)

⑧ $\int f(t) \, dt = \int \frac{f(t)}{u'} \, du = \int g(u) \, du$ (with $g(u) = \frac{f(t)}{u'}$)

$u = u(t)$
$du = u'(t) \cdot dt$

Ex:

i) $$\int\left(t-t^3+2\sqrt{t}\right)dt = \frac{1}{2}t^2-\frac{1}{4}t^4+2\cdot\frac{2}{3}t^{3/2}+C$$
$$=\frac{1}{2}t^2-\frac{1}{4}t^4+\frac{4}{3}t\sqrt{t}+C$$

ii) $$\int e^{2t}\,dt \underset{\boxed{\begin{array}{l}u=2t\\ u'=2\\ du=2\cdot dt\end{array}}}{=} \int e^u\cdot\frac{du}{2} = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u+C = \frac{1}{2}e^{2t}+C$$

iii) $$\int \ln t\,dt = \int \underset{u'}{1}\cdot\underset{v}{\ln t}\,dt = \underset{u}{t}\cdot\underset{v}{\ln t} - \int \underset{u}{t}\cdot\underset{v'}{\frac{1}{t}}\,dt$$
$$= t\ln t - \int 1\,dt = t\ln t - t + C$$

iv) $$\int 2t\sqrt{t^2+1}\,dt \underset{\boxed{\begin{array}{l}u=t^2+1\\ du=2t\cdot dt\end{array}}}{=} \int \cancel{2t}\cdot\sqrt{u}\cdot\frac{du}{\cancel{2t}} = \int\sqrt{u}\,du$$
$$=\frac{2}{3}u^{3/2}+C = \frac{2}{3}(t^2+1)^{3/2}+C = \frac{2}{3}(t^2+1)\sqrt{t^2+1}+C$$

v) $$\int \frac{t^2}{1-t^2}\,dt :$$ Step I: Perform polynomial divison to get a fraction where the degree of the numerator is less than the degree of the denumerator:

$$\frac{t^2}{1-t^2} = -1+\frac{1}{1-t^2}$$

$$\left\{\begin{array}{l} t^2 : 1-t^2 = t^2 : -t^2+1 = -1 \leftarrow \text{quotient} \\ \underline{-(t^2-1)} \\ \quad 1 \leftarrow \text{remainder}\end{array}\right.$$

Step 2: Simplify $\frac{1}{1-t^2}$, the fraction that remains, in the following way:

(a) $1-t^2 = (1-t)\cdot(1+t)$ $\longleftarrow$ factorize denumerator

(b) Write and find constants A, B that fits.

$$\frac{1}{1-t^2} = \frac{A}{1-t} + \frac{B}{1+t}$$

$$\frac{1}{1-t^2}\cdot(1-t^2) = \frac{A}{1-t}\cdot(1-t^2) + \frac{B}{1+t}\cdot(1-t^2)$$

$$1 = A\cdot(1+t) + B(1-t)$$

$t=-1$ gives: $1 = A\cdot 0 + B\cdot 2 \Rightarrow \underline{B = 1/2}$

$t = 1$ " $1 = A\cdot 2 + B\cdot 0$ $\quad A = 1/2$

Concl: $\underline{\frac{1}{1-t^2} = \frac{1/2}{1-t} + \frac{1/2}{1+t}}$

Step 3: Do the integral

$$\int \frac{t^2}{1-t^2}\,dt = \int -1 + \frac{1}{1-t^2}\,dt = \int -1\,dt + \int \frac{1}{2}\cdot\frac{1}{1-t} + \frac{1}{2}\cdot\frac{1}{1+t}\,dt$$

$$= -t + \frac{1}{2}\ln|1-t|\cdot(-1) + \frac{1}{2}\ln|1+t| + C$$

$$= \frac{1}{2}\ln|1+t| - \frac{1}{2}\ln|1-t| - t + C$$

$$= \underline{\frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| - t + C}$$