BI NORWEGIAN SCHOOL OF MANAGEMENT

Department of Economics

Solutions:	**GRA 60353 Mathematics**
Examination date:	30.05.2011, 09:00 – 12:00
Permitted examination aids:	Bilingual dictionary BI-approved exam calculator: TEXAS INSTRUMENTS BA II Plus™
Answer sheets:	Squares
Total number of pages:	3

QUESTION 1.

(a) We compute the partial derivatives $f'_x = 5x^4 + y^2$, $f'_y = 2xy$, $f'_z = -w$ and $f'_w = -z$. The stationary points are given by

$$5x^4 + y^2 = 0, \quad 2xy = 0, \quad -w = 0, \quad -z = 0$$

and this gives $z = w = 0$ from the last two equations, and $x = y = 0$ from the first two. The stationary points are therefore $(x, y, z, w) = \boxed{(0, 0, 0, 0)}$.

(b) We compute the second order partial derivatives of f and form the Hessian matrix

$$f'' = \begin{pmatrix} 20x^3 & 2y & 0 & 0 \\ 2y & 2x & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

We see that the second order principal minor obtained from the last two rows and columns is

$$\begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} = -1 < 0$$

hence the Hessian is indefinite. Therefore, the stationary point is a $\boxed{\text{saddle point}}$.

QUESTION 2.

(a) The determinant of A is given by

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & s & s^2 \\ 1 & -1 & 1 \end{vmatrix} = 2s^2 - 2 = \boxed{2(s-1)(s+1)}$$

It follows that the rank of A is 3 if $s \neq \pm 1$ (since $\det(A) \neq 0$). When $s = \pm 1$, A has rank 2 since $\det(A) = 0$ but there is a non-zero minor of order two in each case. Therefore, we get

$$\boxed{\text{rk}(A) = \begin{cases} 3, & s \neq \pm 1 \\ 2, & s = \pm 1 \end{cases}}$$

(b) We compute that

$$A\mathbf{v} = \begin{pmatrix} 1 \\ 1+s-s^2 \\ -1 \end{pmatrix}, \quad \lambda\mathbf{v} = \begin{pmatrix} \lambda \\ \lambda \\ -\lambda \end{pmatrix}$$

and see that $\mathbf{v}$ is an eigenvector for A if and only if $\lambda = 1$ and $1 + s - s^2 = 1$, or $s = s^2$. This gives $\boxed{s = 0, 1}$.

(c) We compute the characteristic equation of A when $s = -1$, and find that

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & -1-\lambda & 1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = \lambda(2 + \lambda - \lambda^2) = -\lambda(\lambda - 2)(\lambda + 1)$$

Therefore, the eigenvalues of A are $\boxed{\lambda = 0, 2, -1}$ when $s = -1$. Since A has three distinct eigenvalues when $s = -1$, it follows that $\boxed{A \text{ is diagonalizable}}$.

Question 3.

(a) We have $x_{t+1} - 3x_t = 4$, and the homogenous solution is $x_t^h = C \cdot 3^t$. We try to find a particular solution of the form $x_t = A$, and substitution in the difference equation gives $A = 3A + 4$, so $A = -2$ is a particular solution. Hence the solution of the difference equation is

$$x_t = x_t^h + x_t^p = C \cdot 3^t - 2$$

The initial value condition is $2 = C - 2$, hence we obtain the solution

$$x_t = \boxed{4 \cdot 3^t - 2}$$

This gives $x_5 = \boxed{970}$.

(b) The homogeneous equation $y'' + 2y' - 35y = 0$ has characteristic equation $r^2 + 2r - 35 = 0$ and roots $r = 5$ and $r = -7$, so $y_h = C_1 e^{5t} + C_2 e^{-7t}$. We try to find a particular solution of the form $y = Ae^t + B$, which gives

$$y' = y'' = Ae^t$$

Substitution in the differential equation gives

$$Ae^t + 2Ae^t - 35(Ae^t + B) = 11e^t - 5 \Leftrightarrow -32A = 11 \text{ and } -35B = -5$$

This gives $A = -11/32$ and $B = 1/7$. Hence the general solution of the differential equation is $y = y_h + y_p = \boxed{C_1 e^{5t} + C_2 e^{-7t} - \frac{11}{32}e^t + \frac{1}{7}}$

(c) The differential equation can be written in the form

$$(2t + y) + (t - 4y)y' = 0$$

and we see that it is exact. Hence its solution can be written in the form $u(y, t) = C$, where $u(y, t)$ is a function that satisfies

$$\frac{\partial u}{\partial t} = 2t + y \quad \text{and} \quad \frac{\partial u}{\partial y} = t - 4y$$

One solution is $u(y, t) = t^2 + ty - 2y^2$, and the initial condition $y(0) = 0$ gives $C = 0$. Hence

$$t^2 + ty - 2y^2 = 0 \quad \Leftrightarrow \quad y = \frac{-t \pm 3t}{-4}$$

The solution to the initial value problem is therefore

$$\boxed{y = -\frac{1}{2}t \text{ or } y = t}$$

Question 4.

(a) We compute the Hessian of f, and find

$$f'' = e^{x+y}\begin{pmatrix} (x+2)y & (x+1)(y+1) \\ (x+1)(y+1) & x(y+2) \end{pmatrix}$$

The principal minors are

$$\Delta_1 = e^{x+y}(x+2)y,\ \Delta_1 = e^{x+y}x(y+2), \quad D_2 = (e^{x+y})^2(1-(x+1)^2-(y+1)^2)$$

Since $(x+1)^2+(y+1)^2 \leq 1$, D_f is a ball with center in $(-1,-1)$ and radius $r=1$, and it follows that $x,y<0$ and $x+2, y+2 \geq 0$, and therefore $\Delta_1 \leq 0$ and $D_2 \geq 0$. This means that f is $\boxed{\text{concave, but not convex}}$.

(b) Since D_f is closed and bounded, f has maximum and minimum values. We compute the stationary points of f: We have

$$f'_x = (x+1)ye^{x+y} = 0, \quad f'_y = x(y+1)e^{x+y} = 0$$

and $(x,y)=(0,0)$ and $(x,y)=(-1,-1)$ are the solutions. Hence there is only one stationary point $(x,y)=(-1,-1)$ in D_f, and the $f(-1,-1) = \boxed{e^{-2}}$ is the maximum value of f since f is concave. The minimum value most occur for (x,y) on the boundary of D_f. We see that $f(x,y) \geq 0$ for all $(x,y) \in D_f$ while $f(-1,0)=f(0,-1)=0$. Hence $\boxed{f(-1,0)=f(0,-1)=0}$ is the minimum value of f.