Solutions:	**GRA 60353** **Mathematics**	
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Answer sheets:	Squares	
	Counts 80% of GRA 6035	The subquestions are weighted equally
Extraordinary re-sit exam		Responsible department: Economics

Question 1.

(a) We compute the partial derivatives $f'_x = 2xe^u$, $f'_y = -e^u + 1$ and $f'_z = 2z$, where we write $u = x^2 - y$. The stationary points are given by the equations

$$2xe^u = 0, \quad 1 - e^u = 0, \quad 2z = 0$$

The first equation gives $x = 0$ and the third gives $z = 0$. From the second equation, we get that $e^u = 1$, or that $u = x^2 - y = 0$, and this gives $y = 0$ (since $x = 0$). The stationary points are therefore given by $(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$.

(b) We compute the second order partial derivatives of f and form the Hessian matrix

$$f'' = \begin{pmatrix} (2+4x^2)e^u & -2xe^u & 0 \\ -2xe^u & e^u & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

We see that the matrix has leading principal minors $D_1 = (2+4x^2)e^u > 0$, $D_2 = 2e^{2u} > 0$ and $D_3 = 4e^{2u} > 0$. Since all leading principal minors are positive, f is **convex but not concave**.

Question 2.

(a) To compute the determinant of A, we develop it along the third column:

$$\det(A) = \begin{vmatrix} 1 & 3s+1 & -2 \\ 3 & 7s-2 & 0 \\ 2 & 7s & -4 \end{vmatrix} = -2(21s - 2(7s-2)) - 4(1(7s-2) - 3(3s+1))$$

This gives

$$\det(A) = -2(7s+4) - 4(-2s-5) = \mathbf{-6s + 12 = -6(s-2)}$$

This means that A is has rank 3 if $s \neq 2$, since $\det(A) \neq 0$. For $s = 2$, we see that A has rank 2 since $\det(A) = 0$ and there is a minor of order two that is non-zero:

$$\begin{vmatrix} 3 & 0 \\ 2 & -4 \end{vmatrix} = -12 \neq 0$$

Therefore it follows that

$$\mathrm{rk}(A) = \begin{cases} 2 & s = 2 \\ 3 & s \neq 2 \end{cases}$$

(b) To check if $\mathbf{v}$ is an eigenvector of A, we compute

$$A\mathbf{v} = \begin{pmatrix} 1 & 3s+1 & -2 \\ 3 & 7s-2 & 0 \\ 2 & 7s & -4 \end{pmatrix} \cdot \begin{pmatrix} -8 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6s-12 \\ 14s-28 \\ 14s-28 \end{pmatrix}$$

We know that $\mathbf{v}$ is an eigenvector with eigenvalue λ if and only if

$$A\mathbf{v} = \lambda\mathbf{v} \quad \Leftrightarrow \quad \begin{pmatrix} 6s-12 \\ 14s-28 \\ 14s-28 \end{pmatrix} = \lambda \cdot \begin{pmatrix} -8 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -8\lambda \\ 2\lambda \\ 3\lambda \end{pmatrix}$$

From the last two equations, we see that $2\lambda = 3\lambda$, which means that $\lambda = 0$. When we substitute $\lambda = 0$ in all three equations, we see that $s = 2$ is a solution. This means that $\mathbf{v}$ is an eigenvector if and only if $\mathbf{s} = \mathbf{2}$, and the corresponding eigenvalues is $\lambda = \mathbf{0}$.

(c) We substitute $s = 2$ in A, and find that

$$A = \begin{pmatrix} 1 & 7 & -2 \\ 3 & 12 & 0 \\ 2 & 14 & -4 \end{pmatrix}$$

The we write down the characteristic equation $A - \lambda I = 0$, which gives

$$\begin{vmatrix} 1-\lambda & 7 & -2 \\ 3 & 12-\lambda & 0 \\ 2 & 14 & -4-\lambda \end{vmatrix} = -3(7(-4-\lambda)+28) + (12-\lambda)((1-\lambda)(-4-\lambda)+4) = 0$$

After we simplify this equation, we get

$$-3(-7\lambda) + (12-\lambda)(\lambda^2+3\lambda) = \lambda(-\lambda^2+9\lambda+57) = 0$$

The eigenvalues of A for $s = 2$ are therefore $\lambda = \mathbf{0}$ and $\lambda = \frac{\mathbf{-9\pm\sqrt{309}}}{\mathbf{-2}}$.

Question 3.

(a) The homogeneous equation $y'' + 3y' - 10y = 0$ has characteristic equation $r^2 + 3r - 10 = 0$, and therefore roots $r = 2, -5$. Hence the homogeneous solution is $y_h(t) = C_1e^{2t} + C_2e^{-5t}$. To find a particular solution of $y'' + 3y' - 10y = 2t$, we try $y = At + B$. This gives $y' = A$ and $y'' = 0$, and substitution in the equation gives $3A - 10(At + B) = 2t$. Hence $A = -1/5$ and $B = -3/50$ is a solution, and $y_p(t) = -\frac{1}{5}t - \frac{3}{50}$ is a particular solution. This gives general solution

$$y(t) = \mathbf{C_1e^{2t} + C_2e^{-5t} - \frac{1}{5}t - \frac{3}{50}}$$

(b) We re-write the differential equation as

$$3y^2y' = 2te^{t^2} - 2t$$

This differential equation is separable, and we integrate on both sides to solve it:

$$\int 3y^2\,\mathrm{d}y = \int (2te^{t^2} - 2t)\,\mathrm{d}t \quad \Rightarrow \quad y^3 = e^{t^2} - t^2 + \mathcal{C} \quad \Rightarrow \quad y = \sqrt[3]{e^{t^2} - t^2 + \mathcal{C}}$$

(c) We rewrite the differential equation as $y^2-1+2ty\cdot y'=0$, and try to find a function $u=u(y,t)$ such that $u'_t=y^2-1$ and $u'_y=2ty$ to find out if the equation is exact. We see that $u=y^2t-t$ is a solution, so the differential equation is exact, with solution $y^2t-t=\mathcal{C}$. The initial condition $y(1)=3$ gives $9-1=\mathcal{C}$, or $\mathcal{C}=8$. The solution is therefore

$$t(y^2-1)=8 \quad \Rightarrow \quad y=\sqrt{\mathbf{\frac{8}{t}+1}}$$

Question 4.

We rewrite the optimization problem in standard form as

$$\max\ -(x^2+y^2+z^2) \text{ subject to } -2x^2-6y^2-3z^2\le -36$$

(a) The Lagrangian for this problem is given by $\mathcal{L}=-(x^2+y^2+z^2)-\lambda(-2x^2-6y^2-3z^2)$, and the first order conditions are

$$\begin{aligned}\mathcal{L}'_x &= -2x+4x\lambda=0\\ \mathcal{L}'_y &= -2y+12y\lambda=0\\ \mathcal{L}'_z &= -2z+6z\lambda=0\end{aligned}$$

We solve the first order conditions, and get $x=0$ or $\lambda=\frac{1}{2}$ from the first equation, $y=0$ or $\lambda=\frac{1}{6}$ from the second, and $z=0$ or $\lambda=\frac{1}{3}$ from the third. The constraint is $-2x^2-6y^2-3z^2\le -36$, and the complementary slackness conditions are that $\lambda\ge 0$, and moreover that $\lambda=0$ if the constraint is not binding (that is, if $-2x^2-6y^2-3z^2<-36$). We shall find all admissible points that satisfy the first order condition and the complementary slackness condition. In the case where the constraint is not binding (that is, $-2x^2-6y^2-3z^2<-36$), we have $\lambda=0$ and therefore $x=y=z=0$ from the first order conditions. This point does not satisfy $-2x^2-6y^2-3z^2<-36$, and it is therefore not a solution. In the case where the constraint is binding ($-2x^2-6y^2-3z^2=-36$), we have that either $x=y=z=0$, or that at least one of these variables are non-zero. In the first case, $x=y=z=0$ does not satisfy $-2x^2-6y^2-3z^2=-36$, so this is not a solution. In the second case, we see that exactly one of the variables is non-zero (otherwise λ would take two different values), so we have the following possibilities:

$$\begin{cases} x=\pm\sqrt{18}, y=z=0, \lambda=\frac{1}{2}\\ x=0, y=\pm\sqrt{6}, z=0, \lambda=\frac{1}{6}\\ x=y=0, z=\pm\sqrt{12}, \lambda=\frac{1}{3}\end{cases}$$

These six points are the admissible points that satisfies the first order conditions and the complementary slackness conditions.

(b) We compute the value of the function $f(x,y,z)=x^2+y^2+z^2$ in the six points we found above, and get

$$f(\pm\sqrt{18},0,0)=18,\ f(0,\pm\sqrt{6},0)=6,\ f(0,0,\pm\sqrt{12})=12$$

Hence the best candidates for minimum are the points $(x,y,z;\lambda)=(0,\pm\sqrt{6},0;\frac{1}{6})$. We compute the Hessian of the Lagrangian function $\mathcal{L}(x,y,z,1/6)$ and find that

$$\mathcal{L}''(x,y,z,1/6)=\begin{pmatrix}-2+4/6 & 0 & 0\\ 0 & -2+12/6 & 0\\ 0 & 0 & -2+6/6\end{pmatrix}=\begin{pmatrix}-4/3 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$$

We see that this Lagrangian is concave, hence the given points are max for $-f$, and therefore min for f. It therefore follows that $f=6$ **at** $(x,y,z)=(0,\pm\sqrt{6},0)$ **is the minimum**.