Solutions	Final exam in GRA 6035 Mathematics
Date	January 5th, 2018 at 0900 - 1200

Question 1.

(a) The matrix A, and an echelon form of A, is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \quad \to \quad \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We have that $\text{rk}(A) = 3$ since the echelon form has three pivots.

(b) The leading principal minors are $D_1 = 1$, $D_2 = 1$, and $D_3 = 1$ by direct computation, and $D_4 = 0$ since A has rank 3. This implies that A is positive semidefinite by the RRC (reduced rank criterion)

(c) To find $\mathbf{v}_1$, we solve $A\mathbf{x} = \mathbf{0}$ using Gaussian elimination, which gives an echelon form of the augmented matrix of the form

$$(A|\mathbf{0}) = \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \end{array}\right) \quad \to \quad \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Hence w is a free variable, $y = z = 0$, and $x = w$ from the first three equations. This gives solutions of the form

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} w \\ 0 \\ 0 \\ w \end{pmatrix} = w \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = w \cdot \mathbf{v}_1$$

To find $\mathbf{v}_2$, we solve $(A - 2I)\mathbf{x} = \mathbf{0}$ using Gaussian elimination, which gives an echelon form of the augmented matrix of the form

$$(A - 2I|\mathbf{0}) = \left(\begin{array}{cccc|c} -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 \end{array}\right) \quad \to \quad \left(\begin{array}{cccc|c} -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Hence w is a free variable, $y = z = 0$, and $x = -w$ from the first three equations. This gives solutions of the form

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -w \\ 0 \\ 0 \\ w \end{pmatrix} = w \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = w \cdot \mathbf{v}_2$$

Hence, we may choose the vectors as

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Note that these vectors are not unique; any non-zero scalar multiple of each vector would have the required properties.

Question 2.

(a) The differential equation $y'' + 2y' - 15y = 18e^{4t}$ is second order linear and we can solve it using the superposition principle, with $y = y_h + y_p$. Since the characteristic equation $r^2 + 2r - 15 = 0$ has solutions $r = 3$ and $r = -5$, the homogeneous solution is $y_h = C_1 e^{3t} + C_2 e^{-5t}$. For the particular solution, we guess that there is a solution of the form $y = Ae^{4t}$. This gives $y' = 4Ae^{4t}$ and $y'' = 16Ae^{4t}$, and therefore

$$16Ae^{4t} + 2 \cdot 4Ae^{4t} - 15 \cdot Ae^{4t} = 18e^{4t}$$

This gives $16A + 8A - 15A = 18$, or $9A = 18$. It has solution $A = 2$, and therefore $y_p = 2e^{4t}$, and the general solution is

$$y = y_h + y_p = C_1 e^{3t} + C_2 e^{-5t} + 2e^{4t}$$

(b) The differential equation $e^t y' = 2e^{y-t}$ is separable since it can be re-written as

$$y' = e^{-t} \cdot 2e^{y-t} = e^y \cdot 2e^{-2t} \quad \Rightarrow \quad e^{-y} y' = 2e^{-2t}$$

Integration on both sides with respect to t gives

$$\int e^{-y}\, \mathrm{d}y = \int 2e^{-2t}\, \mathrm{d}t \quad \Rightarrow \quad -e^{-y} = -e^{-2t} + C$$

We solve this exaction for y, and obtain $e^{-y} = e^{-2t} - C$, and hence

$$-y = \ln(e^{-2t} - C) \quad \Rightarrow \quad y = -\ln(e^{-2t} - C)$$

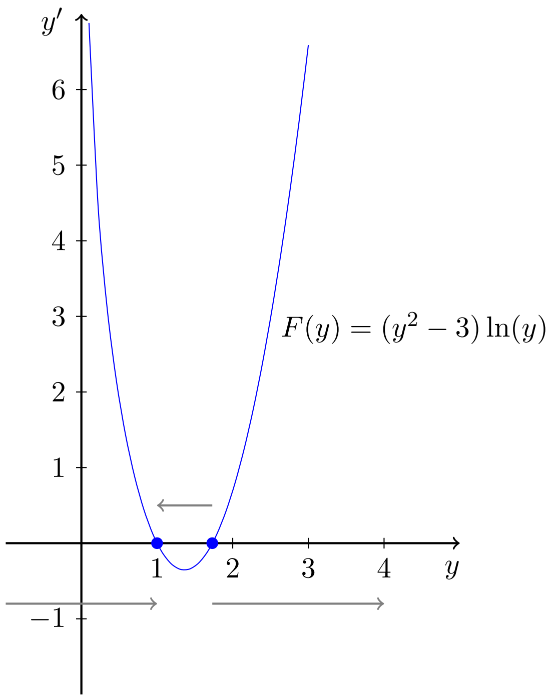
(c) The differential equation $y' = (y^2 - 3) \cdot \ln(y)$ is autonomous, with $F(y) = (y^2 - 3) \cdot \ln(y)$, which is defined for $y > 0$. The equilibrium states are therefore given by

$$F(y) = (y^2 - 3) \cdot \ln(y) = 0 \quad \Rightarrow \quad y = \sqrt{3} \quad \text{or} \quad y = 1$$

since $\ln(y) = 0$ gives $y = 1$, and since F is not defined at $y = -\sqrt{3}$. Hence $y_e = 1$ and $y_e = \sqrt{3}$ are the equilibrium states. To determine their stability, we compute $F'(y_e)$. Since $F'(y) = 2y \ln(y) + (y^2 - 3) \cdot 1/y$, we get

$$F'(1) = -2 < 0, \quad F'(\sqrt{3}) = 2\sqrt{3} \ln(\sqrt{3}) > 0$$

Therefore, $y_e = 1$ is stable and $y_e = \sqrt{3}$ is unstable by the Stability Theorem. We can also see this from the phase diagram below, where the arrows show the time development of $y = y(t)$ as time passes. None of the equilibrium states are globally asymptotically stable, since an



initial value $y_0 > \sqrt{3}$ will give a solution curve that moves away from both equilibrium states as time passes. In fact, since $y' = F(y) = (y^2 - 3)\ln(y) > 0$ for $y > \sqrt{3}$, the solution curve will be increasing.

Question 3.

(a) The stationary points of u are given by

$$u'_x = 2x + 2y - 4z = 0, \quad u'_y = 2x + 4y = 0, \quad -4x + 16z = 0$$

This gives $x = 4z$ from the last equation, $y = -x/2 = -2z$ from the second equation, and therefore $2(4z) + 2(-2z) - 4z = 0$ from the first equation, which gives $0 = 0$. Thies implies that z is free, and $(x, y, z) = (4z, -2z, z)$ are therefore the stationary points of u. The Hessian of u is given by

$$H(u) = \begin{pmatrix} 2 & 2 & -4 \\ 2 & 4 & 0 \\ -4 & 0 & 16 \end{pmatrix}$$

and $D_1 = 2$, $D_2 = 4$ and $D_3 = 16(4) - 4(16) = 0$ (by cofactor expansion along the last row). It follows that $\operatorname{rk} H(u) = 2$, and from the RRC (reduced rank condition) we see that $H(u)$ is positive semidefinite. This means that u is a convex function. Therefore, $u(0, 0, 0) = 2$ is the minimum value of u, since $(0, 0, 0)$ is one of the statinary points of u (with $z = 0$) that we know is a global minimum point.

(b) The outer function $f(u) = u/\ln(u)$ has derivative

$$f'(u) = \frac{1 \cdot \ln(u) - u \cdot (1/u)}{\ln(u)^2} = \frac{\ln(u) - 1}{\ln(u)^2}$$

Therefore, the partial derivatives of f are given by

$$f'_x = \frac{\ln(u) - 1}{\ln(u)^2} \cdot u'_x = \frac{\ln(u) - 1}{\ln(u)^2} \cdot (2x + 2y - 4z)$$

$$f'_y = \frac{\ln(u) - 1}{\ln(u)^2} \cdot u'_y = \frac{\ln(u) - 1}{\ln(u)^2} \cdot (2x + 4y)$$

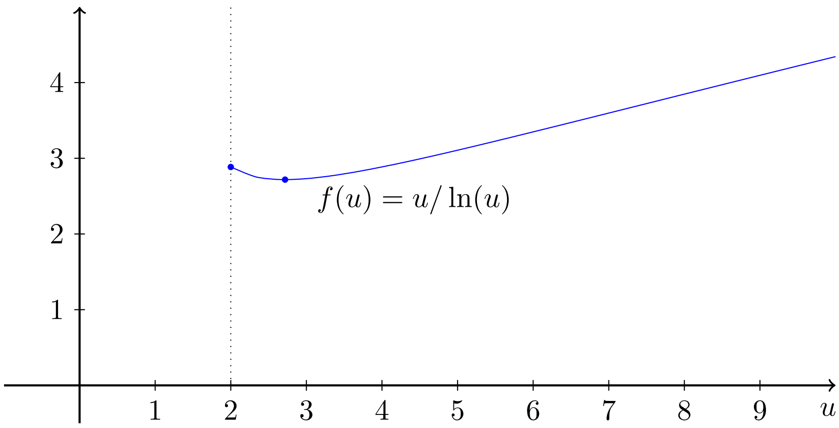
$$f'_z = \frac{\ln(u) - 1}{\ln(u)^2} \cdot u'_z = \frac{\ln(u) - 1}{\ln(u)^2} \cdot (-4x + 16z)$$

with $u = 2 + x^2 + 2y^2 + 8z^2 + 2xy - 4xz$.

(c) From (a) we know that the values of the inner function are $u \geq 2$, and from (b) we know that $f'(u) = (\ln u - 1)/\ln(u)^2$ is the derivative of the outer function. This means that $f'(u) = 0$ for $\ln(u) = 1$, or $u = e$, that $f(u)$ is decreasing for u in $[2, e]$, and that $f(u)$ is increasing for u in $[e, \infty)$. When $u \to \infty$, we have that $f(u) = u/\ln(u) \to \infty$. This means that the minimum value of f is

$$f_{\min} = f(e) = e/1 = e$$

and that f has no maximum value.



Question 4.

(a) The Lagrangian is $\mathcal{L} = xy(x+y) - \lambda(x^2+y^2+(x+y)^2) = x^2y + xy^2 - \lambda(2x^2+2xy+2y^2)$ since the Kuhn-Tucker problem is in standard form. The first order conditions (FOC) are

$$\mathcal{L}'_x = 2xy + y^2 - \lambda(4x+2y) = 0$$
$$\mathcal{L}'_y = x^2 + 2xy - \lambda(2x+4y) = 0$$

the constraint (C) is given by $2x^2+2xy+2y^2 \leq 6$, and the complentary slackness conditions (CSC) are given by

$$\lambda \geq 0 \quad \text{and} \quad \lambda(2x^2+2xy+2y^2-6) = 0$$

The Kuhn-Tucker conditions are FOC+C+CSC.

(b) We look at the cases when (i) $2x^2+2xy+2y^2 < 6$ and (ii) $2x^2+2xy+2y^2 = 6$ separately. In each case, we find all points $(x,y;\lambda)$ with $x,y \neq 0$ that satisfies FOC+C+CSC. We start with case (i): With $\lambda = 0$, we get $2xy+y^2 = x^2+2xy = 0$, or $y(2x+y) = x(x+2y) = 0$. With $x,y \neq 0$, this gives $2x+y = x+2y = 0$, and this implies that $x = y = 0$. Hence there are no candidates with $x,y \neq 0$ in this case. We consider case (ii), and write the FOCs as

$$2xy + y^2 - \lambda(4x+2y) = y(2x+y) - 2\lambda(2x+y) = (y-2\lambda)(2x+y) = 0$$
$$x^2 + 2xy - \lambda(2x+4y) = x(x+2y) - 2\lambda(x+2y) = (x-2\lambda)(x+2y) = 0$$

There are four sub-cases to consider. If $x = y = 2\lambda$, then the constrain gives $2x^2+2xy+2y^2 = 24\lambda^2 = 6$. This gives $\lambda^2 = 1/4$ and $\lambda = 1/2$ (since $\lambda \geq 0$). We get the candidate point $(x,y;\lambda) = (1,1;1/2)$ with $f = 2$.

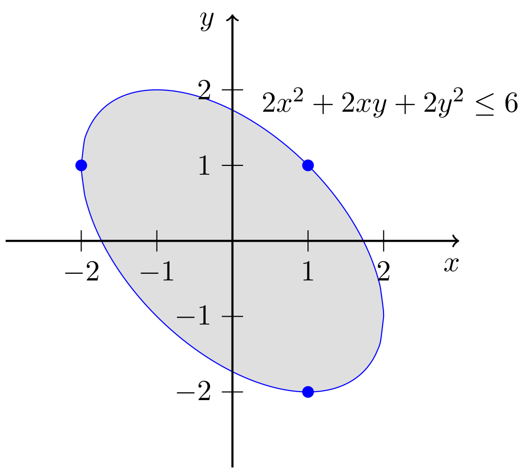
If $y = 2\lambda$ and $x+2y = 0$, then $x = -4\lambda$. The constrain gives $2x^2+2xy+2y^2 = 24\lambda^2 = 6$. This gives $\lambda^2 = 1/4$ and $\lambda = 1/2$ (since $\lambda \geq 0$). We get the candidate point $(x,y;\lambda) = (-2,1;1/2)$ with $f = 2$.

If $x = 2\lambda$ and $2x+y = 0$, then $y = -4\lambda$. The constrain gives $2x^2+2xy+2y^2 = 24\lambda^2 = 6$. This gives $\lambda^2 = 1/4$ and $\lambda = 1/2$ (since $\lambda \geq 0$). We get the candidate point $(x,y;\lambda) = (1,-2;1/2)$ with $f = 2$.

Finally, if $2x+y = x+2y = 0$, we get $x = y = 0$. There are therefore three candidate points $(x,y;\lambda)$ with $x,y \neq 0$ that satisfy FOC+ C+CSC, given by

$$(x,y;\lambda) = (1,1;1/2),\ (-2,1;1/2),\ (1,-2;1/2)$$

shown in the figure below.



(c) The set of points (x,y) such that $2x^2+2xy+2y^2 \leq 6$ is bounded. In fact, $x^2, y^2, (x+y)^2 \geq 0$ and therefore $x^2, y^2, (x+y)^2 \leq 6$, which means that $-\sqrt{6} \leq x, y \leq \sqrt{6}$. We can also see that the set if bounded by noticing that it consist of an ellipse and the inside of an ellipse. By the EVT, the Kuhn-Tucker problem therefore has a maximum. The possible maximum points are the candidate points with $x,y \neq 0$ found in (b), candidate points with $x = 0$ or $y = 0$ that satisfy FOC+C+CSC, and admissible points where NDCQ fails. The candidate points found in (b) have $f = 2$. Therefore, possible candidate points with $x = 0$ or $y = 0$, where $f = 0$, cannot be maximum points. The NDCQ in the case $2x^2 + 2xy + 2y^2 = 6$ is given by

$$\text{rk}\begin{pmatrix} 4x + 2y & 2x + 4y \end{pmatrix} = 1$$

and it fails if $4x + 2y = 2x + 4y = 0$, which gives $x = y = 0$. Since this point does not satisfy $2x^2+2xy+2y^2 = 6$, NDCQ holds for all points on the ellipse. Since there is no NDCQ condition in case $2x^2+2xy+2y^2 < 6$, there are no admissible points where NDCQ fails inside the ellipse either. We conclude that $f = 2$ is the maximum value, and that $(x, y) = (1, 1), (1, -2), (-2, 1)$ are the maximum points. The SOC gives no conclusion in this case.

(d) We formulate a Kuhn-Tucker problem with a parameter a, given by

$$\max\ f(x, y) = xy(x + y) \text{ subject to } x^2 + y^2 + (x + y)^2 - a \le 0$$

Its Lagrangian is $\mathcal{L} = xy(x+y) - \lambda(x^2+y^2+(x+y)^2 - a)$ and $\mathcal{L}'_a = \lambda$. The Envelope Theorem for this situation is that

$$\frac{\mathrm{d}f^*(a)}{\mathrm{d}a} = \mathcal{L}'_a(x^*(a), y^*(a), z^*(a); \lambda^*(a)) = \lambda^*(a)$$

For $a = 6$, we have maximum value $f^*(6) = 2$ and $\lambda^*(2) = 1/2$ from the computations in (b) and (c). This means that an estimate for the new maximum value is

$$f^*(5.7) \approx f^*(6) + \Delta a \cdot \lambda^*(6) = 2 + (-0.3) \cdot 0.5 = 1.85$$

For completeness, one may note that there is a maximum value $f^*(a)$ for all values of a close to 6 since $2x^2 + 2xy + 2y^2 \le a$ is bounded for any $a \ge 0$.