Exam	Final exam in GRA 6035 Mathematics
Date	January 5th, 2018 at 0900 - 1200

This exam consists of 12+1 problems (one additional problem is for extra credits, and can be skipped). Each problem has a maximal score of 6p, and 72p (12 solved problems) is marked as 100% score.

You must give reasons for your answers. Precision and clarity will be emphasized when evaluating your answers.

Question 1.

We consider the quadratic form f given by $f(x, y, z, w) = x^2 + y^2 + z^2 + w^2 - 2xw$.

(a) **(6p)** Find the symmetric matrix A of the quadratic form f, and compute the rank of A.
(b) **(6p)** Determine the definiteness of the quadratic form f.
(c) **(6p)** Find two vectors $\mathbf{v}_1, \mathbf{v}_2$ such that $\text{span}(\mathbf{v}_1) = E_0$ and $\text{span}(\mathbf{v}_2) = E_2$, where E_λ is the set of eigenvectors of A with eigenvalue λ.

Question 2.

Find the general solutions of the following differential equations:

(a) **(6p)** $y'' + 2y' - 15y = 18e^{4t}$
(b) **(6p)** $e^t y' = 2e^{y-t}$

Find all equilibrium states of the following differential equation, and determine their stability. Are any of the equilibrium states globally asymptotically stable?

(c) **(6p)** $y' = (y^2 - 3) \cdot \ln(y)$

Question 3.

Let u be the function given by $u(x, y, z) = 2 + x^2 + 2y^2 + 8z^2 + 2xy - 4xz$, and consider the composite function $f(x, y, z) = u/\ln(u)$ with $u = u(x, y, z)$.

(a) **(6p)** Find the minimal value of $u = u(x, y, z)$, if it exists.
(b) **(6p)** Compute the first order partial derivatives of $f = f(x, y, z)$.
(c) **(6p)** Determine the maximum and minimum values of f, it they exist.

Question 4.

We consider the following Kuhn-Tucker problem:

$$\max\ f(x, y) = xy(x + y) \text{ subject to } x^2 + y^2 + (x + y)^2 \leq 6$$

(a) **(6p)** Write down all Kuhn-Tucker conditions for this problem.
(b) **(6p)** Find all points $(x, y; \lambda)$ with $x, y \neq 0$ that satisfy the Kuhn-Tucker conditions.
(c) **(6p)** Show that the Kuhn-Tucker problem has a maximum, and find the maximum value.
(d) **Extra credit (6p)** Consider the new Kuhn-Tucker problem where the constraint is replaced by $x^2 + y^2 + (x + y)^2 \leq 5.7$. State the relevant envelope theorem, and use it to estimate the new maximum value.