Solutions	Final exam in GRA 6035 Mathematics
Date	June 20th, 2018 at 0900 - 1200

Question 1.

(a) The matrix A has determinant

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & t & t^2 \\ 1 & -t & t^2 \end{vmatrix} = 1(t^3 + t^3) - 1(t^2 - t^2) + 1(-t - t) = 2t^3 - 2t = 2t(t-1)(t+1)$$

(b) Since $\det(A) = 2t(t-1)(t+1)$, we have that $\det(A) = 0$ for $t = 0$ and $t = \pm 1$. Therefore, $\text{rk}(A) = 3$ for $t \neq 0, \pm 1$. Moreover, none of the values $t = 0$, $t = 1$ or $t = -1$ make both the minors

$$\begin{vmatrix} 1 & 1 \\ 1 & t \end{vmatrix} = t - 1, \quad \begin{vmatrix} 1 & 1 \\ 1 & -t \end{vmatrix} = -t - 1$$

vanish, so $\text{rk}(A) = 2$ in these three cases. This gives

$$\text{rk}(A) = \begin{cases} 2, & t = 0, 1, -1 \\ 3, & t \neq 0, 1, -1 \end{cases}$$

(c) If $t \neq 0, 1, -1$, then $\det(A) \neq 0$, and $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution of the linear system. If $t = 0, 1, -1$, then $t^3 = t$, and therefore $(x, y, z) = (0, 1, 0)$ is one of the solutions of the linear system. We could alternatively use Gaussian elimination in all three cases to solve the linear system (which would have infinitely many solutions). Therefore, the linear system is consistent for all values of t.

Question 2.

(a) The differential equation $y'' - 29y' + 100y = 100t - 29$ is second order linear and we can solve it using the superposition principle, with $y = y_h + y_p$. Since the characteristic equation $r^2 - 29r + 100 = 0$ has solutions $r = 4$ and $r = 25$, the homogeneous solution is given by $y_h = C_1 e^{4t} + C_2 e^{25t}$. For the particular solution, we guess that there is a solution of the form $y = At + B$. This gives $y' = A$ and $y'' = 0$, and therefore

$$0 - 29 \cdot A + 100 \cdot (At + B) = 100t - 29$$

This gives $100A = 100$, or $-29A + 100B = -29$. The solution is $A = 1$ and $B = 0$, and therefore $y_p = t$, and the general solution is

$$y = y_h + y_p = C_1 e^{4t} + C_2 e^{25t} + t$$

(b) The differential equation $y' + 2ty = 4e^{-t^2}$ is linear, with $a(t) = 2t$ and $b(t) = 4e^{-t^2}$. The integrating factor is e^{t^2} since $\int 2t \, \mathrm{d}t = t^2 + C$. Hence, the differential equation can be re-written as

$$(y \cdot e^{t^2})' = 4 \quad \Rightarrow \quad ye^{t^2} = 4t + C$$

We solve this exaction for y, and obtain

$$y = (4t + C)e^{-t^2}$$

(c) The differential equation $ty' = y\ln(y)$ is separable, since it can be written

$$y' = \frac{y \ln y}{t} \quad \Rightarrow \quad \frac{1}{y \ln y} y' = \frac{1}{t}$$

Integrating on both sides gives

$$\int \frac{1}{y \ln y} \, \mathrm{d}y = \int \frac{1}{t} \, \mathrm{d}t$$

Using the substitution $u = \ln y$ and $\mathrm{d}u = (1/y)\mathrm{d}y$, we obtain

$$\int \frac{1}{u} \mathrm{d}u = \ln |u| + C = \ln |\ln y| + C$$

on the left-hand side, and therefore

$$\ln|\ln y| = \ln|t| + C \quad \Rightarrow \quad |\ln y| = e^C|t|$$

This can be written $\ln y = Kt$ with $K = \pm e^C$, and the general solution is given by

$$y = e^{Kt}$$

Question 3.

(a) The stationary points of u are given by

$$u'_x = 4x - 2y - 4z = 0, \quad u'_y = -2x + 2y = 0, \quad -4x + 8z = 0$$

This gives $x = 2z$ from the last equation, $y = x = 2z$ from the second equation, and therefore $4(2z) - 2(2z) - 4z = 0$ from the first equation, which gives $0 = 0$. This implies that z is free, and $(x, y, z) = (2z, 2z, z)$ are therefore the stationary points of u.

(b) The Hessian of u is given by

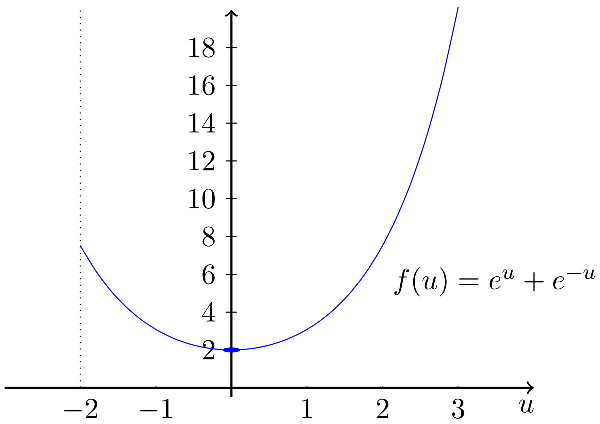
$$H(u) = \begin{pmatrix} 4 & -2 & -4 \\ -2 & 2 & 0 \\ -4 & 0 & 8 \end{pmatrix}$$

and $D_1 = 4$, $D_2 = 8 - 4 = 4$ and $D_3 = -4(8) + 8(4) = 0$ when the use cofactor expansion along the last column to compute D_3. Since $H(u)$ has rank two, it follows from the reduced rank condition that it is positive semidefinite, and therefore u is convex, and the stationary points are global minimum points. The global minimum value is $u(0, 0, 0) = -2$ since $(0, 0, 0)$ is one of the stationary points found in (a), with $z = 0$, and all the others, with $z \neq 0$, would give the same value of u.

(c) The outer function $f(u) = e^u + e^{-u}$ has derivative

$$f'(u) = e^u + e^{-u} \cdot (-1) = e^u - \frac{1}{e^u} = \frac{e^{2u} - 1}{e^u} = \frac{(e^u - 1)(e^u + 1)}{e^u}$$

Since $e^u, e^u + 1 > 0$ for all u, and $f'(u) = 0$ for $e^u = 1$, or $u = 0$, we see that $f(u)$ decreasing for $u \leq 0$ and increasing for $u \geq 0$. When $u = u(x, y, z)$, we have seen in (b) that the possible values for u are $V_u = [-2, \infty)$ since $u = -2$ is the minimum value of u. Hence $u = 0$ gives the minimum value $f_{\text{min}} = e^0 + e^{-0} = 2$ of f. There is no maximum value of f since $f(u) \to \infty$ when $u \to \infty$.



Question 4.

(a) The Lagrangian is $\mathcal{L} = xy(x - y) - \lambda(x^2 + y^2 + (x - y)^2) = x^2y - xy^2 - \lambda(2x^2 - 2xy + 2y^2)$ since the Kuhn-Tucker problem is in standard form. The first order conditions (FOC) are

$$\mathcal{L}'_x = 2xy - y^2 - \lambda(4x - 2y) = 0$$

$$\mathcal{L}'_y = x^2 - 2xy - \lambda(-2x + 4y) = 0$$

the constraint (C) is given by $2x^2 - 2xy + 2y^2 \leq 96$, and the complentary slackness conditions (CSC) are given by

$$\lambda \geq 0 \quad \text{and} \quad \lambda(2x^2 - 2xy + 2y^2 - 96) = 0$$

The Kuhn-Tucker conditions are FOC+C+CSC.

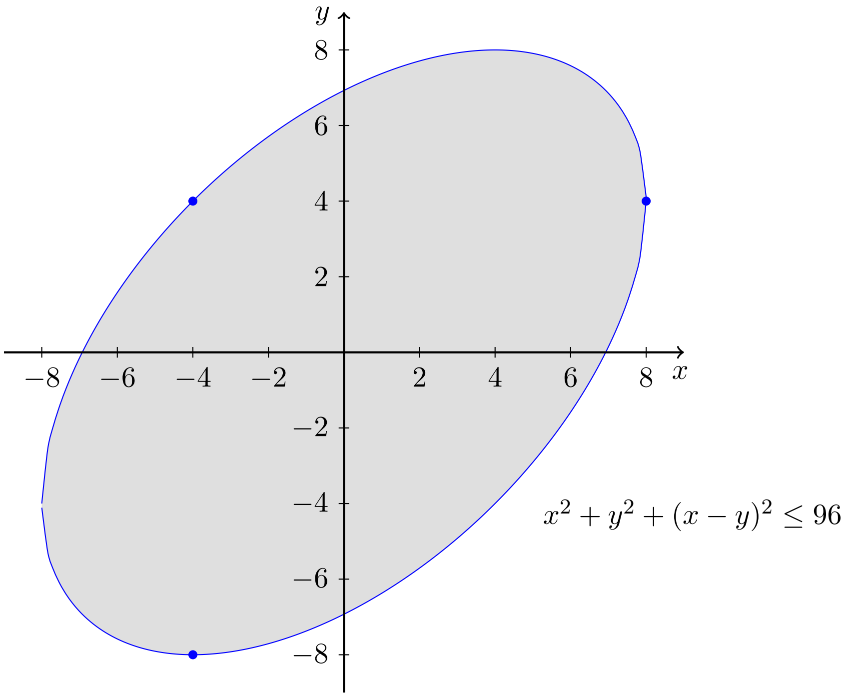
(b) We look at the cases when (i) $2x^2 - 2xy + 2y^2 < 96$ and (ii) $2x^2 - 2xy + 2y^2 = 96$ separately. In each case, we find all points $(x, y; \lambda)$ with $x, y \neq 0$ that satisfies FOC+C+CSC. We start with case (i): With $\lambda = 0$, we get $2xy - y^2 = x^2 - 2xy = 0$, or $y(2x - y) = x(x - 2y) = 0$. With $x, y \neq 0$, this gives $2x - y = x - 2y = 0$, and this implies that $x = y = 0$. Hence there are no candidates with $x, y \neq 0$ in this case. We consider case (ii), and write the FOCs as

$$2xy - y^2 - \lambda(4x - 2y) = y(2x - y) - 2\lambda(2x - y) = (y - 2\lambda)(2x - y) = 0$$
$$x^2 - 2xy - \lambda(-2x + 4y) = x(x - 2y) - 2\lambda(-x + 2y) = (x + 2\lambda)(x - 2y) = 0$$

There are four sub-cases to consider:
If $y = 2\lambda$ and $x = -2\lambda$, then the constrain gives $2x^2 - 2xy + 2y^2 = 24\lambda^2 = 96$. This gives $\lambda^2 = 4$ and $\lambda = 2$ (since $\lambda \geq 0$). We get the candidate point $(x, y; \lambda) = (-4, 4; 2)$ with $f = 128$.
If $y = 2\lambda$ and $x - 2y = 0$, then $x = 4\lambda$. The constrain gives $2x^2 - 2xy + 2y^2 = 24\lambda^2 = 96$. This gives $\lambda^2 = 4$ and $\lambda = 2$ (since $\lambda \geq 0$). We get the candidate point $(x, y; \lambda) = (8, 4; 2)$ with $f = 128$.
If $x = -2\lambda$ and $2x - y = 0$, then $y = -4\lambda$. The constrain gives $2x^2 - 2xy + 2y^2 = 24\lambda^2 = 96$. This gives $\lambda^2 = 4$ and $\lambda = 2$ (since $\lambda \geq 0$). We get the candidate point $(x, y; \lambda) = (-4, -8; 2)$ with $f = 128$.
Finally, if $2x - y = -x + 2y = 0$, we get $x = y = 0$. There are therefore three candidate points $(x, y; \lambda)$ with $x, y \neq 0$ that satisfy FOC+ C+CSC, given by

$$(x, y; \lambda) = (-4, 4; 2),\ (8, 4; 2),\ (-4, -8; 2)$$

shown in the figure below.



(c) The set of points (x, y) such that $x^2 + y^2 + (x - y)^2 \leq 96$ is bounded. In fact, $x^2, y^2, (x - y)^2 \geq 0$ and therefore $x^2, y^2, (x - y)^2 \leq 96$, which means that $-\sqrt{96} \leq x, y \leq \sqrt{96}$. We can also see that the set if bounded by noticing that it consist of an ellipse and the inside of an ellipse. By the EVT, the Kuhn-Tucker problem therefore has a maximum. The possible maximum points are the candidate points with $x, y \neq 0$ found in (b), candidate points with $x = 0$ or $y = 0$ that satisfy FOC+C+CSC, and admissible points where NDCQ fails. The candidate points found in (b) have $f = 128$. Therefore, possible candidate points with $x = 0$ or $y = 0$, where $f = 0$, cannot be maximum points. The NDCQ in the case $x^2 + y^2 + (x-y)^2 = 96$ is given by

$$\operatorname{rk}\begin{pmatrix} 2x + 2(x-y) & 2y - 2(x-y) \end{pmatrix} = 1$$

and it fails if $4x - 2y = -2x + 4y = 0$, which gives $x = y = 0$. Since this point does not satisfy $x^2 + y^2 + (x-y)^2 = 96$, NDCQ holds for all points on the ellipse. Since there is no NDCQ condition in case $x^2 + y^2 + (x-y)^2 < 96$, there are no admissible points where NDCQ fails inside the ellipse either. We conclude that $f = 128$ is the maximum value, and that $(x, y) = (-4, 4), (-4, -8), (8, 4)$ are the maximum points. The SOC gives no conclusion in this case.

Question 5.

We solve the differential equation $y' = ry(1 - y/K)$ by separation of variables, given by

$$Ky' = ry(K-y) \quad \Rightarrow \quad \frac{K}{y(K-y)}y' = r \quad \Rightarrow \quad \int \frac{K}{y(K-y)}\,\mathrm{d}y = \int r\,\mathrm{d}t$$

The integral on the left-hand side can be computed by decomposing the fraction as

$$\frac{K}{y(K-y)} = \frac{A}{y} + \frac{B}{K-y}$$

where multiplication by $y(K-y)$ gives $K = A(K-y) + By = (B-A)y + AK$. This gives $A = 1$ and $B = 1$, and therefore

$$\int \frac{1}{y} + \frac{1}{K-y}\mathrm{d}y = \ln|y| - \ln|K-y| + C_1 \quad \Rightarrow \quad \ln\left|\frac{y}{K-y}\right| = rt + C_2$$

Using the exponential function on both sides, we get

$$\frac{y}{K-y} = \pm e^{rt}e^{C_2} = Ce^{rt} \quad \Rightarrow \quad y = (K-y)Ce^{rt}$$

with $C = \pm e^{C_2}$. Solving for y, we get

$$y(1 + Ce^{rt}) = CKe^{rt} \quad \Rightarrow \quad y = \frac{KCe^{rt}}{1 + Ce^{rt}} = K\frac{Ce^{rt}}{1 + Ce^{rt}}$$

The differential equation has equilibrium states $y = 0$ and $y = K$. Since the equation can be written as $y' = F(y) = ry - ry^2/K$, we have that

$$F' = r - 2ry/K \quad \Rightarrow \quad \begin{cases} F'(0) & = r > 0 \\ F'(K) & = -r < 0 \end{cases}$$

Therefore, $y = 0$ in unstable and $y = K$ is stable by the Stability Theorem. We show the graph of $y = y(t)$ below for typical values of $r > 0$ and $K > 0$.