Solutions	Midterm exam in GRA 6035 Mathematics
Date	October 13th, 2017 at 1500 - 1600

Correct answers: D-B-A-B-C-D-B-C

QUESTION 1.

Since $\text{rk}\, A = 4$, we also have $\text{rk}(A|\mathbf{b}) = 4$, and the linear system is consistent with $6 - 4 = 2$ degrees of freedom. The correct answer is alternative **D**.

QUESTION 2.

We form the matrix with the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ as columns, and compute its determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & t \end{vmatrix} = 1(2t - 12) - 1(t - 3) + 1(4 - 2) = t - 7$$

This shows that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent when $t = 7$, and linearly independent otherwise. The correct answer is alternative **B**.

QUESTION 3.

We compute the determinant of the matrix A using cofactor expansion along the first column:

$$|A| = \begin{vmatrix} t & 1 & 1 \\ 1 & t & 1 \\ 1 & 1 & t \end{vmatrix} = t(t^2 - 1) - 1(t - 1) + 1(1 - t) = t(t + 1)(t - 1) - 2(t - 1)$$
$$= (t - 1)(t(t + 1) - 2) = (t - 1)(t^2 + t - 2) = (t - 1)(t + 2)(t - 1)$$

This means that $\text{rk}(A) = 3$ for $t \neq 1, -2$. When $t = 1$, we clearly have that $\text{rk}(A) = 1$. When $t = -2$, we have $\text{rk}(A) = 2$ since there are non-zero 2-minors, for example

$$\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 4 - 1 = 3 \neq 0$$

Therefore, $\text{rk}(A) = 2$ if and only if $t = -2$. The correct answer is alternative **A**.

QUESTION 4.

We compute the eigenvalues of A by solving the characteristic equations $\det(A - \lambda I) = 0$, which gives

$$\begin{vmatrix} 3 - \lambda & 5 & 2 \\ 0 & 2 - \lambda & 0 \\ 1 & 3 & 4 - \lambda \end{vmatrix} = 0$$

We compute the determinant by cofactor expansion along the middle row, which gives

$$(2 - \lambda) \cdot \begin{vmatrix} 3 - \lambda & 2 \\ 1 & 4 - \lambda \end{vmatrix} = (2 - \lambda)(\lambda^2 - 7\lambda + 10) = (2 - \lambda)(\lambda - 2)(\lambda - 5) = 0$$

Therefore, $\lambda = 2$ is an eigenvalue of multiplicity two, and $\lambda = 5$ is an eigenvalue of multiplicity one. The correct answer is alternative **B**.

Question 5.

The eigenvalues are $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = -1$. Since $\lambda = 1$ is the only eigenvalue of multiplicity $m = 2 > 1$, we consider the eigenspace E_1 of solutions of the linear system $(A - \lambda I) \cdot \mathbf{x} = \mathbf{0}$ for $\lambda = 1$. In matrix form, it can be written

$$\begin{pmatrix} 0 & s & 1 \\ 0 & 0 & s \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

If $s = 0$, then both x and y are free. If $s \neq 0$, then only x is free. Therefore, $\dim E_1 = 2$ if and only if $s = 0$, and A is therefore diagonalizable exactly when $s = 0$. The correct answer is alternative **C**.

Question 6.

The symmetric matrix of the quadratic form $f(x, y, z) = x^2 + 4xy + 2xz + 4y^2 + 4yz$ is given by

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

The leading principal minors are $D_1 = 1$, $D_2 = 4 - 4 = 0$ and $D_3 = |A| = 1(0) - 2(0) = 0$ (we use cofactor expansion along the last column). Since $D_2 = D_3 = 0$, we compute all principal minors, and find that $\Delta_1 = 1, 4, 0 \geq 0$, that $\Delta_2 = 0, -4, -1$ and $\Delta_3 = |A| = 0$. Since there are second order principal minors Δ_2 that are negative, f is indefinite. The correct answer is alternative **D**.

Question 7.

The function $f(x, y, z) = 1 - x^4 - 2x^2 + 4xz - y^2 - z^4 - 2z^2$ has a stationary point in $(x, y, z) = (0, 0, 0)$ since the first order partial derivatives

$$f'_x = -4x^3 - 4x + 4z, \quad f'_y = -2y, \quad f'_z = 4x - 4z^3 - 4z$$

are zero at this point. The Hessian matrix of f is given by

$$H(f) = \begin{pmatrix} -12x^2 - 4 & 0 & 4 \\ 0 & -2 & 0 \\ 4 & 0 & -12z^2 - 4 \end{pmatrix}$$

The leading principal minors are $D_1 = -12x^2 - 4$, $D_2 = -2D_1$, and $D_3 = -2(144x^2z^2 + 48x^2 + 48z^2)$. We see that $D_1 < 0$, $D_2 > 0$ and $D_3 \leq 0$ for all points (x, y, z). At points (x, y, z) with $D_3 < 0$, it is clear that $H(f)$ is negative definite. At points (x, y, z) with $D_3 = 0$, the matrix $H(f)$ has rank two, and therefore $H(f)$ is negative semidefinite. We conclude that f is concave, and $(x, y, z) = (0, 0, 0)$ is a global maximum for f. The correct answer is alternative **B**.

Question 8.

Eigenvectors of A for $\lambda = 1$ are given by the linear system $(A - I)\mathbf{x} = \mathbf{0}$, where

$$A - I = \begin{pmatrix} -0.20 & 0.20 \\ 0.20 & -0.20 \end{pmatrix}$$

Therefore, we see that $-0.20x + 0.20y = 0$ and $x = y$. The eigenvectors are therefore given by

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The condition $x + y = 1$ for a state vector gives $x = y = 1/2$, and the market share of Firm A in the long run is $x = 0.50 = 50\%$. The correct answer is alternative **C**.