Solutions	Midterm exam in GRA 6035 Mathematics
Date	May 31st, 2018 at 1400 - 1500

Correct answers: A-D-D-D-A-D-B-C

Question 1.

Since A is invertible, the system has a unique solution $\mathbf{x} = A^{-1}\mathbf{b}$, and the number of solutions does not depend on $\mathbf{b}$. The correct answer is alternative **A**.

Question 2.

We form the matrix with the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ as columns, and compute its determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & t \end{vmatrix} = 1(2t-12) - 1(t-3) + 1(4-2) = t - 7$$

This shows that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent when $t = 7$, and linearly independent otherwise. The correct answer is alternative **D**.

Question 3.

We compute the three 2-minors (maximal minors) of the matrix A:

$$M_{12,12} = 0, \quad M_{12,13} = t^2 - 1, \quad M_{12,23} = t^2 - 1$$

This means that $\mathrm{rk}(A) < 2$ when $t^2 = 1$, or when all maximal minors vanish, and that $\mathrm{rk}(A) = 2$ otherwise. Therefore, $\mathrm{rk}(A) = 2$ for $t \neq 1, -1$. The correct answer is alternative **D**.

Question 4.

We compute the eigenvalues of A by solving the characteristic equations $\det(A - \lambda I) = 0$, which gives

$$\begin{vmatrix} -1-\lambda & 0 & 0 \\ 0 & 2-\lambda & -1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

We compute the determinant by cofactor expansion along the first row, which gives

$$(-1-\lambda) \cdot \begin{vmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = (-1-\lambda)(\lambda^2 - 4\lambda + 5) = 0$$

Since $\lambda^2 - 4\lambda + 5 = 0$ has no real solutions, $\lambda = -1$ is the unique eigenvalue of A, of multiplicity one. The correct answer is alternative **D**.

Question 5.

The eigenvalues are $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 0$. Since $\lambda = 1$ is the only eigenvalue of multiplicity $m = 2 > 1$, we consider the eigenspace E_1 of solutions of the linear system $(A - \lambda I) \cdot \mathbf{x} = \mathbf{0}$ for $\lambda = 1$. In matrix form, it can be written

$$\begin{pmatrix} 0 & s & -1 \\ 0 & -1 & s \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We use Gaussian elimination, and obtain an echelon form (of the coefficient matrix) given by

$$\begin{pmatrix} 0 & s & -1 \\ 0 & -1 & s \\ 0 & 0 & 0 \end{pmatrix} \quad \to \quad \begin{pmatrix} 0 & -1 & s \\ 0 & 0 & s^2 - 1 \\ 0 & 0 & 0 \end{pmatrix}$$

We see that for $s = \pm 1$, the only pivot position is in the second column, and both x and z are free. The matrix A is therefore diagonalizable for $s = \pm 1$. For all other values of s, there are two pivot positions, only x is free, and A is not diagonalizable. The correct answer is alternative **A**.

Question 6.

The symmetric matrix of the quadratic form $f(x, y, z) = x^2 - 2xy - 4xz - 4yz + 4z^2$ is given by

$$A = \begin{pmatrix} 1 & -1 & -2 \\ -1 & 0 & -2 \\ -2 & -2 & 4 \end{pmatrix}$$

The first leading principal minors are $D_1 = 1$ and $D_2 = -1$. Since $D_2 < 0$, we conclude that f is indefinite. The correct answer is alternative **D**.

Question 7.

The function $f(x, y, z) = x^4 + 2x^2 + 4xz + y^2 + z^4 + 2z^2$ has first order partial derivatives

$$f'_x = 4x^3 + 4x + 4z, \quad f'_y = 2y, \quad f'_z = 4x + 4z^3 + 4z$$

and Hessian matrix

$$H(f) = \begin{pmatrix} 12x^2 + 4 & 0 & 4 \\ 0 & 2 & 0 \\ 4 & 0 & 12z^2 + 4 \end{pmatrix}$$

The leading principal minors are $D_1 = 12x^2 + 4$, $D_2 = 2D_1$, and $D_3 = 2(144x^2z^2 + 48x^2 + 48z^2)$. We see that $D_1 > 0$, $D_2 > 0$ and $D_3 \geq 0$ for all points (x, y, z). At points (x, y, z) with $D_3 > 0$, it is clear that $H(f)$ is positive definite. At points (x, y, z) with $D_3 = 0$, the matrix $H(f)$ has rank two, and therefore $H(f)$ is positive semidefinite. We conclude that f is convex, and $(x, y, z) = (0, 0, 0)$ is a global minimum for f. The correct answer is alternative **B**.

Question 8.

Eigenvectors of A for $\lambda = 1$ are given by the linear system $(A - I)\mathbf{x} = \mathbf{0}$, where

$$A - I = \begin{pmatrix} -0.36 & 0.18 \\ 0.36 & -0.18 \end{pmatrix}$$

Therefore, we see that $-0.36x + 0.18y = 0$ and $x = y/2$. The eigenvectors are therefore given by

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = y/2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The condition $x + y = 1$ for a state vector gives $y/2 = 1/(1 + 2)$, or $y = 2/3$ and the market share of Firm A in the long run is $x = 1/3$. The correct answer is alternative **C**.