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Differential Equations

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Contents

Chapter 1
Differential Equations

1.1 Introduction to differential equations

Let $y(t)$ be a function in one variable. A *differential equation* in $y(t)$ is an equation that contains the derivative $y'(t)$, or higher order derivatives such as $y''(t)$ and $y'''(t)$, or both. The equations

$$y'(t) + 2y(t) = 1, \qquad y''(t) + y(t) = 1, \qquad y'(t) + y(t) \cdot y''(t) = t^2$$

are examples of differential equations in $y(t)$. We think of the function $y(t)$ as the unknown in these differential equations.

It is usual to write y, y', y'' for $y(t), y'(t), y''(t)$ to write differential equations in a more compact form. The differential equations above would be written as

$$y' + 2y = 1, \qquad y'' + y = 1, \qquad y' + yy'' = t^2$$

It is understood from the context that y is a function $y = y(t)$ in one variable, and that y' and y'' are the first and second order derivatives $y' = y'(t)$ and $y'' = y''(t)$ of this function. Sometimes, the notation $\dot{y}$ is used for y' and $\ddot{y}$ for y''.

The *order* of a differential equation is the highest order derivative the appears in the equation. A *first order* differential equation contains the derivative y' but no other derivatives. A *second order* differential equation contains y'', and possibly y', but no higher order derivatives. Hence $y' + 2y = 1$ is a first order differential equation, while $y'' + y = 1$ and $y' + yy'' = t^2$ are second order differential equations.

To *solve* a differential equation means to find all functions $y(t)$ that satisfy the equation. Let us consider the differential equation $y' + 2y = 1$ as an example. In this case, the function $y(t) = \mathrm{e}^{-2t} + 1/2$ is a solution of the differential equation. We can see this by substituting y with $y(t) = \mathrm{e}^{-2t} + 1/2$ and y' with its derivative $y'(t) = -2\mathrm{e}^{-2t}$. The left-hand side of the differential equation becomes

$$y'(t) + 2y(t) = \left(-2\mathrm{e}^{-2t} + 0\right) + 2\left(\mathrm{e}^{-2t} + 1/2\right) = -2\mathrm{e}^{-2t} + 2\mathrm{e}^{-2t} + 1 = 1$$

and equals the right-hand side of the equation. This shows that $y(t) = e^{-2t} + 1/2$ is a solution of $y' + 2y = 1$. It is not clear from this computation whether or not there are other solutions of this equation, and we need more systematic methods to investigate this.

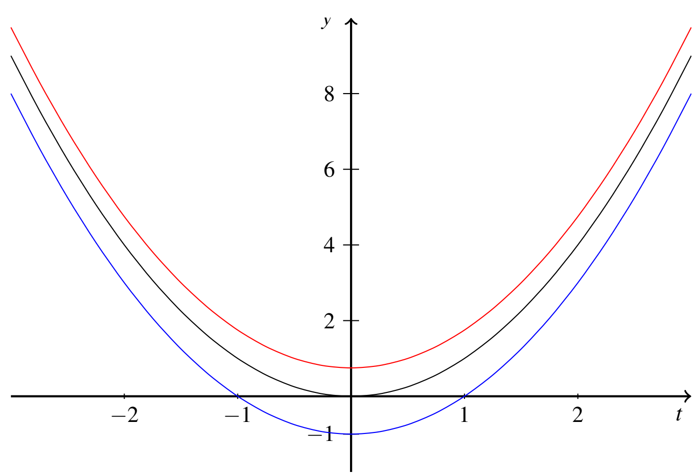
The equation $y' = 2t$ is an even simpler example of a differential equation. The solutions of this differential equation are all functions $y(t)$ with derivative $y'(t) = 2t$. That is, the solutions are all antiderivatives of $2t$, given by the indefinite integral

$$\int 2t \, dt = t^2 + C$$

The integration constant C is called an *undetermined coefficient*, since there is no information in the differential equation that can be used to determine the value of C. Hence, there is one solution $y(t) = t^2 + C$ for each value of C, and these solutions form an infinite family. We call $y(t) = t^2 + C$ the *general solution* of the differential equation; considered as a family, it contains all solutions of the differential equation. For any given value of C, we obtain a *particular solution* of the differential equation, such as

$$y(t) = t^2, \quad y(t) = t^2 - 1, \quad y(t) = t^2 + \frac{3}{4}$$

corresponding to the (randomly chosen) values $C = 0$, $C = -1$ and $C = 3/4$. We show the graphs of these particular solutions below. One may easily imagine the shape of the graph of other particular solutions, corresponding to other values of C.



It is a hopeless task to solve differential equations in general. In this chapter, we shall explain some techniques and methods that can be used to solve certain first and second order differential equations. Others are simply too difficult to solve in this way, and we must use approximation techniques (and often computers) to solve them.

Let us end this introduction by considering the first order differential equations that can be written in the form $y' = f(t)$. All equations that belong to this class, such as the example $y' = 2t$ considered above, can be solved by simple integration:

Differential equations solvable by simple integration
Let f be a continuous function. The first order differential equation $y' = f(t)$ has general solution

$$y(t) = \int f(t)\,\mathrm{d}t = F(t) + C$$

where $F(t)$ is an antiderivative of $f(t)$. The general solution depends on one undetermined coefficient C.

We remark that not all first order differential equations are of the form $y' = f(t)$. However, it is typical for first order differential equation that the general solution depends on *one* undetermined coefficient, and we often need to compute indefinite integrals to find the general solution. In Appendix A, we review indefinite integrals.

Problems

1.1. Show that $y = t - t^2$ is a solution of the differential equation $y' + y = 1 - t - t^2$.

1.2. Determine all values of the constant r such that $y = \mathrm{e}^{rt}$ is a solution of the differential equation $3y' + 6y = 0$.

1.3. Find the general solution of the differential equations:
a) $y' = 4t^3 + 1$ *b*) $ty' = 2\ln(t)$ *c*) $y' + t^3 = t^2$ *d*) $\mathrm{e}^t y' = t$

1.4. Find the general solution of the differential equations $y'' = 12t + 6$, and show that it depends on two undetermined coefficients.

1.2 Modelling change using differential equations

To model how a variable y changes, we use a differential equation in the function $y(t)$. We write t for the independent variable and think of this variable as the *time*. In this situation, we interpret the derivative

$$y' = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h}$$

as the *rate of change* in the variable y. A first order differential equation can often be written in the form $y' = F(t, y)$, and this equation specifies the rate of change in $y = y(t)$ using the expression $F(t, y)$. This is exactly what we mean by a model for change. Let us show some examples.

Example 1.1. Let $y(t)$ denote the UK population (in millions) t years after 1980. The UK population was 56.3 millons in 1980 and 58.9 millons in 2000. This gives us the data points $y(0) = 56.3$ and $y(20) = 58.9$. In order to model the population $y = y(t)$ as a function of t, we need to make some assumption about the growth of the population. Recall that the *rate of change* of the population $y = y(t)$ is its derivative

$$y' = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h}$$

where $\Delta y/\Delta t$ is the average rate of change in the period $[t, t+h]$. For example, in the period 1980-2000, the population increased by the average rate of change

$$\frac{y(20) - y(0)}{20} = \frac{58.9 - 56.3}{20} = 0.13$$

or 0.13 millions/year. An assumption about the rate of change leads to a differential equation. One choice of model is the *simple exponential growth model*. In this model, we assume that

$$y' = r \cdot y$$

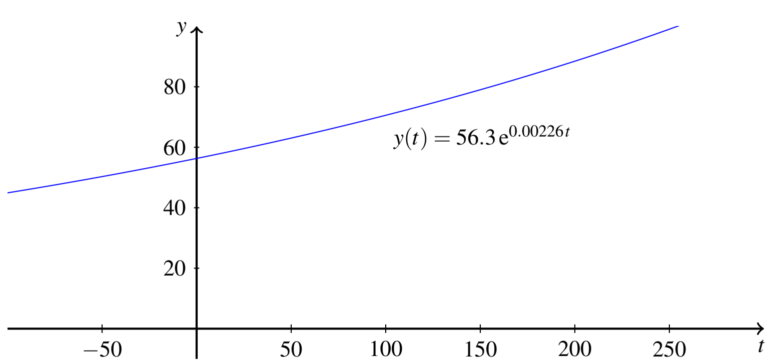
for a constant r. In other words, the rate of change $y' = y'(t)$ is proportional to the population $y = y(t)$. It turns out that the general solution of this differential equation is $y(t) = C \cdot \mathrm{e}^{rt}$, where C is an undetermined coefficient. You may verify that this is a solution of $y' = ry$, and we will show how to obtain the general solution of this equation in Section 1.4 - 1.5. When we fit the general solution $y(t) = C\mathrm{e}^{rt}$ to the given data points, we see that $y(0) = 56.3$ gives

$$56.3 = C \cdot \mathrm{e}^{r \cdot 0} = C \quad \Rightarrow \quad C = 56.3$$

and that $y(20) = 58.9$ gives

$$58.9 = 56.3\,\mathrm{e}^{r \cdot 20} \quad \Rightarrow \quad \mathrm{e}^{20r} = 58.9/56.3 \quad \Rightarrow \quad r = \frac{\ln(58.9/56.3)}{20} \approx 0.00226$$

This means that $y(t) = 56.3\,\mathrm{e}^{0.00226t}$. The graph of the solution is shown below.



According to this model, the UK population will reach 80 millions around the year 2135, since

$$56.3\,\mathrm{e}^{0.00226t} = 80 \quad \Rightarrow \quad \mathrm{e}^{0.00226t} = 80/56.3 \quad \Rightarrow \quad t = \frac{\ln(80/56.3)}{0.00226} \approx 155$$

and $t = 155$ corresponds to the year 2135. In the long term, the population will increase without bounds since

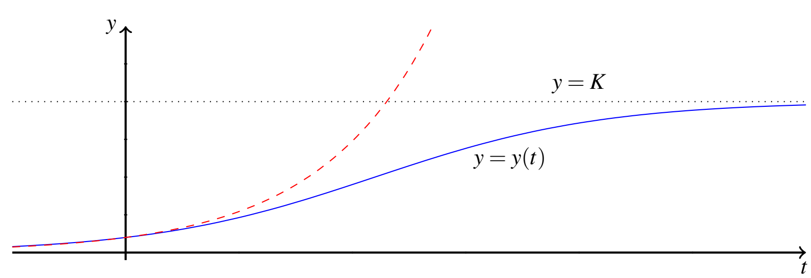
$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} 56.3\cdot \mathrm{e}^{0.00226t} = \infty$$

A differential equation in $y(t)$, where t is time, gives a model for the changes in the variable $y = y(t)$. The derivative y' is interpreted as the rate of change in y. We need additional information about an initial state to determine the solution $y = y(t)$ completely.

Example 1.2. The simple exponential growth model is seldom a realistic model. In most practical situation, the growth of $y = y(t)$ would be restricted by the size of y. For example, when y is a popluation, limited resources would keep the growth rate y' in the population from growing equally fast as y. Instead of the differential equation $y' = ry$, we could consider

$$y' = r\cdot y\left(1 - \frac{y}{K}\right)$$

This is called a *logistic growth model*. The positive constant K is called the *carrying capacity*. We see that when y is much smaller than K, the factor $1-y/K$ is close to 1, and $y(t)$ will have close to simple exponential growth. However, when y grows large and approaches K, the factor $1-y/K$ is close to zero, and $y(t)$ will have close to zero growth. The graph of the solution of the logistic growth model is shown below (blue curve). The carrying capacity K (dotted) and the corresponding solution of the simple exponential growth model (red curve) is shown for comparison.



It turns out that the general solution of the logistic differential equation is

$$y(t) = K \cdot \frac{C\,\mathrm{e}^{rt}}{1 + C\,\mathrm{e}^{rt}}$$

where C is an undetermined coefficient. It is possible to verify that this function is a solution of $y' = ry(1 - y/K)$, and we will show how to obtain the general solution of this equation in Section 1.4. Notice that $y(t) \to K$ when $t \to \infty$.

Problems

1.5. Show that $y = C\mathrm{e}^{rt}$ is a solution of the differential equation $y' = ry$.

1.6. Let $p = p(t)$ be the price of a product with demand function $d = d(t)$ and supply function $s = s(t)$. We consider the differential equation

$$p' = k(d - s)$$

for a positive constant $k > 0$. Explain the assumption on the rate of change in the price $p = p(t)$ that this differential equation expresses. What happens with the price when there is a demand surplus? What happens when there is a supply surplus?

1.7. Find a particular solution of the logistic differential equation $y' = ry(1 - y/K)$ such that $y(0) = 56.3$ and $y(20) = 58.9$ assuming that the carrying capacity $K = 80$. You may use that

$$y(t) = K \cdot \frac{C\,\mathrm{e}^{rt}}{1 + C\,\mathrm{e}^{rt}}$$

is the general solution of logistic differential equation. Compare the graphs by sketching them in the same coordinate system (for instance using Wolfram Alpha).

1.3 First order differential equations

A first order differential equation in the unknown function $y = y(t)$ is an equation that involves expressions in t, y and y'. Many first order differential equations can be written in the form

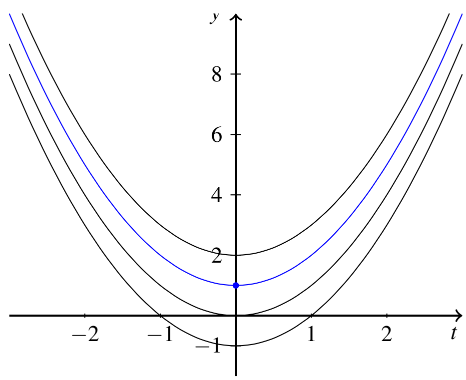
$$y' = F(t, y)$$

for some function F, and we shall only consider first order differential equations of this type. Examples of first order differential equations are

$$y' = -y^2\mathrm{e}^t, \quad y' = ty - 1, \quad y' = \frac{y^2 - 3t^2y}{t^3 - 2yt}$$

In Section 1.4 - 1.7, we shall explain methods for solving *certain*, but not all, first order differential equations that can be written as $y' = F(t, y)$.

For a general first order differential equation $y' = F(t, y)$, we expect that a general solution $y = y(t)$ exists and that it depends on one undetermined coefficient C. For example, the differential equation $y' = 2t$ has general solution $y(t) = t^2 + C$. If the value of $y = y(t)$ at an initial time $t = t_0$ is given, this is called an *initial value* for the differential equation. In this case, we expect that there is a unique solution of $y' = F(t,y)$ such that $y(t_0) = y_0$. For example, the differential equation $y' = 2t$ with initial value $y(0) = 1$ has a unique solution $y(t) = t^2 + 1$, since the condition $y(0) = 1$ in the general solution $y(t) = t^2 + C$ gives $1 = 0^2 + C$, which we can solve for C to determine that $C = 1$. This means that there is a unique solution with a graph that passes through the point (t_0, y_0). The unique particular solution $y(t) = t^2 + 1$ that passes through the point $(t_0, y_0) = (0, 1)$, corresponding to the initial condition $y(0) = 1$, is shown below (blue curve).



A first order differential equation $y' = F(t,y)$ with an initial condition $y(t_0) = y_0$ is called an *initial value problem*. It turns out that if $F(t,y)$ is a "nice" function, then the initial value problem $y' = F(t,y)$, $y(t_0) = y_0$ has a unique solution. To make the notion of a "nice" function precise, we define that $F(t,y)$ is a C^1 function if it is continuous and has continuous partial derivatives.

Existence and uniqueness of solutions
Let $y' = F(t,y)$, $y(t_0) = y_0$ be a first order initial value problem. If F is a C^1 function in a neighbourhood around the point (t_0, y_0), then the initial value problem has a unique solution $y = y(t)$.

Problems

1.8. Solve the initial value problem $y' = 3t^2 + 6$, $y(1) = 1$.

1.9. Solve the initial value problem $y' = 3\sqrt{t}$, $y(0) = 1$.

1.10. Write the differential equation $t^2 y' - ty = t + y$ in the form $y' = F(t,y)$, if possible. Is it solvable by simple integration?

1.11. Solve the initial value problem $ty' = 2\ln(t)$, $y(1) = 3$.

1.4 Separable differential equations

A first order differential equation is called *separable* if it can be written in the form

$$y' = f(t) \cdot g(y)$$

for functions $f(t)$, $g(y)$. The cases where $f(t)$ or $g(y)$ are constants, are special cases of separable differential equations. For example, the first order differential equation $y' = f(t)$ solvable by simple integration, considered in Section 1.1, is a special case of a separable differential equation with $g(y) = 1$.

In general, any separable differential equation can be solved by a technique called *separation of variables*. When the equation is written in the form $y' = f(t) \cdot g(y)$, the solution method is given by the following steps:

$$\begin{aligned} y' &= f(t) \cdot g(y) \\ \frac{1}{g(y)} \cdot y' &= f(t) \\ \int \frac{1}{g(y)} \cdot y' \,\mathrm{d}t &= \int f(t) \,\mathrm{d}t \\ \int \frac{1}{g(y)} \,\mathrm{d}y &= \int f(t) \,\mathrm{d}t \end{aligned}$$

We have divided both sides with $g(y)$ and integrated both sides with respect to t. To rewrite the integral on the left-hand side, we use the chain rule for the composite function $1/g(y)$ with $y = y(t)$, which can be written in the form $\mathrm{d}y = y' \,\mathrm{d}t$. We obtain two indefinite integrals that we can solve, at least in principle.

Separable differential equations
Let f, g be continuous functions. The first order separable differential equation $y' = f(t) \cdot g(y)$ has general solution given by

$$\int \frac{1}{g(y)} \,\mathrm{d}y = \int f(t) \,\mathrm{d}t$$

In particular, the general solution can be written as $G(y) + C_1 = F(t) + C_2$ in implicit form, and rewritten as $G(y) = F(t) + K$ with $K = C_2 - C_1$. It depends on one undetermined coefficient K.

When we use this method, we obtain the general solution as $G(y) = F(t) + K$. It is called an *implicit form* of the solution. We prefer to solve this equation for y, to obtain a solution $y = y(t)$ in *explicit form*, if possible.

To illustrate the method, consider the differential equation $y' = 2y$ as an example. Using the factorization $2y = 2 \cdot y$, with $f(t) = 2$ and $g(y) = y$, we obtain

$$
\begin{aligned}
y' &= 2 \cdot y \\
\frac{1}{y} \cdot y' &= 2 \\
\int \frac{1}{y} \cdot y' \, \mathrm{d}t &= \int 2 \, \mathrm{d}t \\
\int \frac{1}{y} \, \mathrm{d}y &= \int 2 \, \mathrm{d}t
\end{aligned}
$$

Then we have to compute these integrals. The integrals are in this case not difficult to compute, and we get

$$
\int \frac{1}{y} \, \mathrm{d}y = \ln|y| + C_1, \qquad \int 2 \, \mathrm{d}t = 2t + C_2
$$

and therefore $\ln|y| + C_1 = 2t + C_2$ is the general solution in implicit form. We solve for y to get it in explicit form $y = y(t)$. We get

$$
\begin{aligned}
\ln|y| + C_1 &= 2t + C_2 \\
\ln|y| &= 2t + C_2 - C_1 \\
|y| &= \mathrm{e}^{2t + C_2 - C_1} = \mathrm{e}^{2t} \cdot \mathrm{e}^{C_2 - C_1} \\
y &= \pm \mathrm{e}^{C_2 - C_1} \cdot \mathrm{e}^{2t} = K \mathrm{e}^{2t}
\end{aligned}
$$

with $K = \pm \mathrm{e}^{C_2 - C_1}$, which is an undetermined coefficient. The general solution of the differential equation in explicit form is therefore $y(t) = K \mathrm{e}^{2t}$.

Problems

1.12. Find the general solution of the differential equation $y' = ry$ when r is a given constant. This is the differential equation in the simple exponential growth model considered in Section 1.2.

1.13. Determine whether the differential equations are separable or not, and find the general solution of the separable equations.
a) $yy' = t$ *b)* $y' + y = \mathrm{e}^t$ *c)* $\mathrm{e}^y y' = t + 1$ *d)* $ty' + y^2 = 1$ *e)* $y' - \ln(t) = 1$

1.14. Find the general solution of the differential equation $y' = ry(1 - y/K)$ when r, K are given constants with $K > 0$. This is the differential equation in the logistic growth model considered in Section 1.2.

1.5 Linear first order differential equation

A first order differential equation $y' = F(t,y)$ is *linear* if $F(t,y)$ is linear in y. More precisely, the differential equation is linear if and only if it can be written in the form

$$y' + a(t) \cdot y = b(t) \quad \Leftrightarrow \quad y' = b(t) - a(t) \cdot y$$

for functions $a(t)$, $b(t)$. The form $y' + a(t) \cdot y = b(t)$ is called the *standard form* of linear first order differential equations, and it is the one we use when we solve these equations. The form $y' = b(t) - a(t) \cdot y$ is included to show that it has the form $y' = F(t,y)$ and that the function $F(t,y)$ is linear in y.

Let us postpone the general case for the moment, and start by considering the special case where $a(t)$ and $b(t)$ are constants. In this case, we write the equation as

$$y' + ay = b$$

where $a = a(t)$ and $b = b(t)$ are constants. If $a = 0$, then $y' = b$ is easy to solve by simple integration, and we get general solution $y = bt + C$. When $a \neq 0$, we notice that the equation is separable since it can be written

$$y' + ay = b \quad \Leftrightarrow \quad y' = b - ay = 1 \cdot (b - ay)$$

with factorization given by $f(t) = 1$ and $g(y) = b - ay$. By separation of variables, this gives

$$\int \frac{1}{b - ay}\, \mathrm{d}y = \int 1 \, \mathrm{d}t$$

The first integral is

$$\int \frac{1}{b - ay}\, \mathrm{d}y = \int \frac{1}{u} \cdot \frac{\mathrm{d}u}{(-a)} = \frac{1}{-a} \cdot \ln|u| + C_1 = -\frac{1}{a} \ln|b - ay| + C_1$$

using the substitution $u = b - ay$ with $\mathrm{d}u = -a \cdot \mathrm{d}y$ since $u' = -a$. This gives

$$\begin{aligned}
-\frac{1}{a} \ln|b - ay| + C_1 &= t + C_2 \\
\ln|b - ay| &= -a(t + C_2 - C_1) = -at - a(C_2 - C_1) \\
b - ay &= \pm \mathrm{e}^{-at - a(C_2 - C_1)} = K\mathrm{e}^{-at} \\
ay &= b - K\mathrm{e}^{-at} \\
y &= \frac{b}{a} - \frac{K}{a}\mathrm{e}^{-at} = \frac{b}{a} + C\,\mathrm{e}^{-at}
\end{aligned}$$

with $K = \pm \mathrm{e}^{-a(C_2 - C_1)}$ and $C = -K/a$. Therefore, $y = b/a + C\,\mathrm{e}^{-at}$ is the general solution of $y' + ay = b$ when a, b are constants with $a \neq 0$.

This a very useful special case. For instance, the differential equation $y' = ry$ of the simple exponential growth model can be solved using the formula above, since

$y' = ry$ can be written $y' - ry = 0$ with $a = -r$ and $b = 0$. This gives the solution $y = b/a + Ce^{-at} = Ce^{rt}$. It also illustrates the fact that some first order differential equations are both separable and linear.

In general, linear first order differential equations can be solved by a technique called *integrating factor*. We choose an integrating factor $u = u(t)$ and multiply the differential equation with this factor. This gives

$$\begin{aligned} y' + a(t) \cdot y &= b(t) \\ u\,y' + u\,a(t) \cdot y &= u\,b(t) \end{aligned}$$

We want to choose $u = u(t)$ such that the left-hand side of the above equation is $(uy)' = uy' + u'y$. For this to be the case, we see that we must choose $u = u(t)$ such that $u' = a(t)u$. This is a separable differential equation in $u = u(t)$ that we can solve by separation of variables

$$\begin{aligned} u' &= a(t) \cdot u \\ \frac{1}{u} \cdot u' &= a(t) \\ \int \frac{1}{u}\,\mathrm{d}u &= \int a(t)\,\mathrm{d}t \\ \ln|u| &= \int a(t)\,\mathrm{d}t \end{aligned}$$

and we therefore choose the integrating factor $u = u(t)$ to be given by

$$u(t) = e^{\int a(t)\,\mathrm{d}t}$$

We know that this choice for $u = u(t)$ makes $u\,y' + u\,a(t) \cdot y$ equal to $(uy)'$. We use this to solve the equation:

$$\begin{aligned} y' + a(t) \cdot y &= b(t) \\ uy' + u\,a(t) \cdot y &= u\,b(t) \\ (uy)' &= u(t) \cdot b(t) \\ uy &= \int u(t) \cdot b(t)\,\mathrm{d}t \\ y &= \frac{1}{u} \int u(t) \cdot b(t)\,\mathrm{d}t = \frac{1}{u(t)} \int u(t) \cdot b(t)\,\mathrm{d}t \end{aligned}$$

Notice that $u = u(t)$, and we switch between using u and $u(t)$ above. Also note that the formula for the integrating factor gives an undetermined coefficient. For instance, in the differential equation $y' + y = e^t$, the formula gives

$$\int a(t)\,\mathrm{d}t = \int 1\,\mathrm{d}t = t + C \quad \Rightarrow \quad u = e^{\int a(t)\,\mathrm{d}t} = e^{t+c}$$

We choose the integrating factor as simple as possible. In this case, we choose $u = e^t$.

Linear first order differential equations
Let f, g be continuous functions. The linear first order differential equation $y' + a(t)\,y = b(t)$ has general solution given by

$$y = \frac{1}{u(t)} \cdot \int u(t) \cdot b(t)\,\mathrm{d}t$$

where $u(t) = \mathrm{e}^{\int a(t)\,\mathrm{d}t}$ is an integrating factor.

To illustrate the method, consider the differential equation $y' - 2y = 4t$ as an example. In this case, the differential equation is first order linear in standard form, with $a(t) = -2$ (a constant) and $b(t) = 4t$. We first compute an integrating factor:

$$\int a(t)\,\mathrm{d}t = \int -2\,\mathrm{d}t = -2t + C \quad \Leftrightarrow \quad u = \mathrm{e}^{-2t+C} = \mathrm{e}^{-2t}$$

We have chosen the integrating factor $u = \mathrm{e}^{-2t}$ with $C = 0$, as any value of C would work. This gives

$$\begin{aligned} y' - 2y &= 4t \\ \mathrm{e}^{-2t}y' - 2\mathrm{e}^{-2t} \cdot y &= 4t\mathrm{e}^{-2t} \\ (\mathrm{e}^{-2t}y)' &= 4t\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t}y &= \int 4t\mathrm{e}^{-2t}\,\mathrm{d}t \\ y &= \mathrm{e}^{2t}\int 4t\mathrm{e}^{-2t}\,\mathrm{d}t \end{aligned}$$

Notice that the step where we simplify the left-hand side to $(\mathrm{e}^{-2t}y)'$ follows from the construction of $u(t)$; we have chosen u such that this would be the case. Finally, we have to solve the integral on the right-hand side using integration by parts

$$\begin{aligned} \int 4t\mathrm{e}^{-2t}\,\mathrm{d}t &= 4t \cdot \left(\frac{1}{(-2)}\mathrm{e}^{-2t}\right) - \int 4 \cdot \left(\frac{1}{(-2)}\mathrm{e}^{-2t}\right)\mathrm{d}t \\ &= -2t\mathrm{e}^{-2t} + \int 2\mathrm{e}^{-2t}\,dft \\ &= -2t\mathrm{e}^{-2t} - \mathrm{e}^{-2t} + C \end{aligned}$$

with $u' = \mathrm{e}^{-2t}$ and $v = 4t$, which gives $u = -\mathrm{e}^{-2t}/2$ and $v' = 4$. The general solution of the differential equation is therefore

$$y = \mathrm{e}^{2t} \cdot \left(-2t\mathrm{e}^{-2t} - \mathrm{e}^{-2t} + C\right) = -2t - 1 + C\mathrm{e}^{2t} = C\mathrm{e}^{2t} - 2t - 1$$

with undetermined coefficient C.

Problems

1.15. Show that the differential equation $y' = ry$ is both separable and linear when r is a given constant. Find the integrating factor, and use it to solve the differential equation.

1.16. Determine whether the differential equations are linear or not, and find the general solution of the linear equations.
a) $y' + y = \mathrm{e}^t$ *b)* $yy' = t$ *c)* $y' = 2t + y$ *d)* $t^2 y' + \ln(t) y = \ln(t)$ *e)* $y' - 2ty = 2t$

1.6 Superposition principle

In this section, we explain the *superposition principle* and how to use it to solve linear first order differential equations. The principle gives a decomposition of the general solution $y = y(t)$ into two components

$$y(t) = y_h(t) + y_p(t)$$

and a description of the two components that give an alternative method for solving linear first order differential equations. In many cases, this method is easier to use than integrating factors.

We shall introduce *operators* to explain the superposition principle. We think of an operator as a "function of functions", which takes a function as input and produces a new function as output. In concrete terms, the operators that we use in linear first order differential equations are the *first order differential operators* of the form $D = p(t)\,\mathrm{d}/\mathrm{d}t + q(t)$. It operates on an input function $y = y(t)$ by

$$D(y) = \left(p(t)\frac{\mathrm{d}}{\mathrm{d}t} + q(t) \right) y = p(t)\,y' + q(t)\,y$$

and is called a differential operator since it involves differentiating the input function $y = y(t)$.

Any linear first order differential equation $y' + a(t)\,y = b(t)$ can be written in the form $D(y) = b(t)$, where $y = y(t)$ is the unknown function, and $D = \mathrm{d}/\mathrm{d}t + a(t)$ is a differential operator, and we interpret a solution of the differential equation to be an input function $y = y(t)$ such that the output function $D(y)$ is $b(t)$. For example, we write the differential equation $y' + y = \mathrm{e}^t$ as $D(y) = \mathrm{e}^t$ with $D = \mathrm{d}/\mathrm{d}t + 1$. Using the (random) input functions $y(t) = t,\ t^2,\ \mathrm{e}^t$, we get

$$D(t) = 1 + t, \quad D(t^2) = 2t + t^2, \quad D(\mathrm{e}^t) = \mathrm{e}^t + \mathrm{e}^t = 2\mathrm{e}^t$$

and neither of these input functions are solutions of the linear differential equation since the requirement is that $D(y)$ is e^t.

In general, we say that an operator D is *linear* if $D(y_1+y_2)=D(y_1)+D(y_2)$ and $D(cy)=cD(y)$ for all functions $y_1=y_1(t)$, $y_2=y_2(t)$ and all constants c. It is not difficult to see that any first order differential operator $p(t)\,\mathrm{d}/\mathrm{d}t+q(t)$ is linear. To check the two requirement, we compute

$$\begin{aligned} D(y_1+y_2) &= p(t)\cdot(y_1+y_2)'+q(t)\cdot(y_1+y_2) \\ &= p(t)\cdot y_1'+p(t)\cdot y_2'+q(t)\,y_1+q(t)\,y_2 = D(y_1)+D(y_2) \\ D(cy) &= p(t)\cdot(cy)'+q(t)\cdot(cy) = cp(t)\,y'+cq(t)\,y = c\,D(y) \end{aligned}$$

The real reason why $y'+a(t)\,y=b(t)$ is called a *linear* differential equation is of course that it can be written as $D(y)=b(t)$ for a linear operator D.

First order differential operators
Any first order differential operator $D=p(t)\,\mathrm{d}/\mathrm{d}t+q(t)$ is linear.

A linear first order differential equation $y'+a(t)\,y=b(t)$ is called *homogeneous* if $b(t)=0$, and *inhomogeneous* otherwise. We define $y_h(t)$ to be the general solution of the homogeneous equation

$$y'+a(t)\,y=0$$

obtained by replacing $b(t)$ with zero. If $a(t)=a$ is constant, it is easy to find y_h. We can use the formula in the case $y'+ay=b$ from Section 1.5, which gives

$$y_h(t)=b/a+C\mathrm{e}^{-at}=C\mathrm{e}^{-at}$$

We say that the equation has *constant coefficients* when $a(t)=a$ is a constant. If $a(t)$ is non-constant, we must use an integrating factor to find $y_h(t)$.

We call $y_p=y_p(t)$ a *particular solution* of $y'+a(t)\,y=b(t)$ if it is a solution. We can often find y_p by considering special cases. For example, in the differential equation $t^2y'+\ln(t)\,y=\ln(t)$, we guess that a constant $y_p=A$ could be a solution. To verify this and find A, we substitute $y(t)=A$ and $y'(t)=0$ into the differential equation, which gives $\ln(t)\cdot A=\ln(t)$, or $A=1$. Therefore, $y_p(t)=1$ is a particular solution in this case.

The differential equation $y'+a(t)\,y=b(t)$ can be written as $D(y)=b(t)$. If we have found a particular solution y_p of $D(y)=b(t)$, and the general solution y_h of the homogeneous equation $D(y)=0$, then $y=y_h+y_p$ is clearly a solution of $D(y)=b(t)$ since

$$D(y)=D(y_h+y_p)=D(y_h)+D(y_p)=0+b(t)=b(t)$$

by definition. It also follows that any solution of the equation must be of the form $y=y_h+y_p$, since for any solution y we have

$$D(y-y_p)=D(y)-D(y_p)=b(t)-b(t)=0$$

and therefore that $y-y_p$ is a homogeneous solution, or $y-y_p=y_h$, which gives $y=y_h+y_p$. This proves the superposition principle:

Superposition principle
The general solution of the linear differential equation $y' + a(t)\,y = b(t)$ can be written as $y = y_h + y_p$, where the y_h is the general solution of the *homogeneous* equation $y' + a(t)\,y = 0$, and y_p is any particular solution of $y' + a(t)\,y = b(t)$.

To illustrate the usefulness of the superposition principle, let us use it to solve the differential equation $y' + y = \mathrm{e}^t$. By the superposition principle, the general solution is $y = y_h + y_p$. The homogeneous solution is easy to find, since $a(t) = 1$ is a constant:

$$y' + y = 0 \quad \Rightarrow \quad y = \frac{b}{a} + C\mathrm{e}^{-at} = C\mathrm{e}^{-t}$$

Therefore $y_h = C\mathrm{e}^{-t}$. To find y_p, we just need one particular solution of $y' + y = \mathrm{e}^t$. We try $y = \mathrm{e}^t$, which gives $D(y) = \mathrm{e}^t + \mathrm{e}^t = 2\mathrm{e}^t$ with $D = \mathrm{d}/\mathrm{d}t + 1$. The first attempt $y = \mathrm{e}^t$ is not a solution, but since $D(\mathrm{e}^t/2) = D(\mathrm{e}^t)/2 = 2\mathrm{e}^t/2 = \mathrm{e}^t$, it follows that $y_p = \mathrm{e}^t/2$ is a particular solution. The general solution is therefore

$$y = y_h + y_p = C\mathrm{e}^{-t} + \frac{1}{2}\mathrm{e}^t$$

by the superposition principle.

Problems

1.17. Use the superposition principle to find the general solution of the linear first order differential equations:
a) $y' + 3y = 4\mathrm{e}^t$ *b*) $y' - y = t$ *c*) $y' = 2t + y$ *d*) $6y' - 18y = 12t$

1.18. Use the superposition principle to find the general solution of the linear first order differential equation $t^2 y' + \ln(t)y = \ln(t)$.

1.19. Solve the initial value problem $ty' + 2y = t,\ y(1) = 1$.

1.7 Exact differential equations

A first order differential equation $y' = F(t,y)$ is *exact* if it can be written in the form

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial y} \cdot y' = 0$$

for a differentiable function $h = h(t,y)$. In this case, we call $h'_t + h'_y \cdot y' = 0$ an *exact form* of the differential equation. It is simple to write any first order differential equation $y' = F(t,y)$ as $p(t,y) + q(t,y) \cdot y' = 0$. However, the requirement that $p = h'_t$ and $q = h'_y$ for a common function $h = h(t,y)$ is restrictive and must be checked to see whether the differential equation is in exact form.

For example, the differential equation $y' = 2y$, which is both separable and linear, can be written in the form $p(t,y) + q(t,y) \cdot y' = 0$ in many different ways. Let us first write it as

$$-2y + 1 \cdot y' = 0$$

with $p(t,y) = -2y$ and $q(t,y) = 1$. There is no function $h(t,y)$ that satisfies the requirement that $h'_t = -2y$ and $h'_y = 1$. To see this, note that any function $h(t,y)$ satisfying $h'_y = 1$ must have the form $h(t,y) = y + \phi(t)$ for some function $\phi(t)$ that is constant in y. Using this, we see that $h'_t = 0 + \phi'(t) = \phi'(t)$. Since this derivative is constant in y, it cannot equal $2y$ for any choice of the function ϕ.

Let us instead take advantage of the fact that the differential equation $y' = 2y$ is separable, and re-write it as

$$\frac{1}{y} \cdot y' = 2 \quad \Longleftrightarrow \quad -2 + \frac{1}{y} \cdot y' = 0$$

with $p(t,y) = -2$ and $q(t,y) = 1/y$. It we choose $h(t,y) = -2t + \ln|y|$, we see that $h'_t = -2 = p$ and $h'_y = 1/y = q$. Hence the differential equation $y' = 2y$ is exact, and we obtained an exact form by multiplying the equation $-2y + y' = 0$ with $1/y$.

We could also take advantage of the fact that the differential equation $y' = 2y$ is linear. Writing it as $y' - 2y = 0$, we see that its integrating factor is $u = \mathrm{e}^{-2t}$. After multiplication with u, we can re-write the differential equation as

$$y' - 2y = 0 \quad \Longleftrightarrow \quad -2y\mathrm{e}^{-2t} + \mathrm{e}^{-2t} \cdot y' = 0$$

with $p(t,y) = -2y\mathrm{e}^{-2t}$ and $q(t,y) = \mathrm{e}^{-2t}$. It we choose $h(t,y) = y\mathrm{e}^{-2t}$, we see that $h'_t = -2y\mathrm{e}^{-2t} = p$ and $h'_y = \mathrm{e}^{-2t} = q$. Hence the differential equation is exact, and we obtained an exact form by multiplying it with the integrating factor e^{-2t}.

Separable and linear differential equations are exact
Any first order differential equation that is either separable or linear, is also exact.

In general, we use the following method for solving exact differential equations: First, we find a function $h = h(t,y)$ such that the differential equation can be written in the form

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial y} \cdot y' = 0$$

This is the hard part. Then, we recall that the solution of the differential equation is supposed to be a function $y = y(t)$. When we take this into consideration, and compute the derivative of $h = h(t,y) = h(t,y(t))$ with respect to t, we get

$$\frac{dh}{dt} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial y} \cdot \frac{dy}{dt} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial y} \cdot y'$$

This is sometimes called the *total derivative* of h. We can think of the two terms as the direct change in h as a result as a change in t, and the indirect change as a result of a change in y. The total derivative is equal to the left-hand side of the exact differential equation, which therefore simplifies to

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial y} \cdot y' = 0 \quad \Leftrightarrow \quad \frac{dh}{dt} = 0$$

This means that $h(t,y)$ is a constant. Therefore, the general solution of the exact differential equation is given by $h(t,y) = C$.

Exact differential equations
Let $h = h(t,y)$ be a differentiable function. The exact differential equation

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial y} \cdot y' = 0$$

has a general solution that can be written as $h(t,y) = C$ in implicit form.

Let us illustrate the method with an example, the first order differential equation $(2t+y)+(t-4y)y' = 0$. This equation is neither separable nor linear. We can see this by transforming it to the form $y' = F(t,y)$, which gives

$$(2t+y)+(t-4y)y' = 0 \quad \Leftrightarrow \quad y' = -\frac{2t+y}{t-4y}$$

We shall attempt to solve this differential equation as an exact equation, and we therefore try to write the differential equation

$$(2t+y)+(t-4y)y' = 0$$

in the form $h'_t + h'_y \cdot y' = 0$ for a function $h = h(t,y)$. The function h must have the properties that

$$h'_t = 2t+y \qquad \text{and} \qquad h'_y = t-4y$$

Any function h that satisfy the first condition, must have the form $h = t^2 + yt + \phi(y)$ for a function $\phi(y)$ that is constant in t. We see this by using the inverse operation of partial derivation with respect to t. We can think of this inverse operation as an integral

$$\int (2t+y)\,dt = t^2 + yt + C = t^2 + yt + \phi(y)$$

where we consider y to be constant. We interpret the integration constant C as any expression $\phi(y)$ that is constant in the integration variable t, since this gives $(\phi(y))'_t = 0$. After finding the functions $h(t,y) = t^2 + yt + \phi(y)$ that satisfies the first condition, we check if any of these functions also satisfy the second condition by computing

$$h'_y = \left(t^2 + yt + \phi(y)\right)'_y = 0 + t + \phi'(y) = t + \phi'(y)$$

This should equal $q(t,y) = t - 4y$, and this condition means that $\phi'(y) = -4y$, or that $\phi(y) = -2y^2 + K$. It follows that the function $h(t,y) = t^2 + yt - 2y^2$ satisfies both conditions. Therefore, the equation is exact with implicit solution

$$t^2 + yt - 2y^2 = C$$

Notice that it is enough to find one function $h(t,y)$ that satisfy the two conditions; we have used $h(t,y) = t^2 + ty - 2y^2 + K$ with $K = 0$. To find an explicit solution, we solve the implicit equation for y, and get

$$-2y^2 + ty + (t^2 - C) = 0 \quad \Rightarrow \quad y = \frac{-t \pm \sqrt{t^2 - 4(-2)(t^2 - C)}}{2(-2)} = \frac{t \pm \sqrt{9t^2 - 8C}}{4}$$

using the formula for quadratic equations. This is the general solution of the exact differential equation in explicit form.

Criterion for exactness
Let $p(t,y)$, $q(t,y)$ be C^1 functions, and consider the first order differential equation $p(t,y) + q(t,y) \cdot y' = 0$. There is a function $h = h(t,y)$ such that $h'_t = p$ and $h'_y = q$ if and only if the condition

$$\frac{\partial p}{\partial y} = \frac{\partial q}{\partial t}$$

holds. In this case, the differential equations is exact and $p(t,y) + q(t,y) \cdot y' = 0$ is an exact form.

Notice that if we use the exactness criterion and find that it is satisfied, we just know that the function $h = h(t,y)$ exists; we still have to find h in order to solve the exact differential equation. If the exactness criterion is not satisfied, it just means that $p(t,y) + q(t,y) \cdot y' = 0$ is not in exact form; it could still be possible to transform it to an exact form by multiplying it with a factor (such as an integrating factor).

It is not difficult to explain why an exact differential equation satisfy the criterion: If $p = h'_t$ and $q = h'_y$ for a common function $h = h(t,y)$, then we have that

$$p'_y = h''_{ty}, \qquad q'_t = h''_{yt}$$

and we know that the Hessian matrix of h is symmetric so that $h''_{ty} = h''_{yt}$. To prove the opposite implication, that any differential equation that satisfy the criterion must be exact, is much more difficult.

Problems

1.20. Solve the differential equation $1+2ty^2+2t^2y\cdot y'=0$, and find all solutions that satisfy the initial condition $y(1)=-1$.

1.21. Solve the following differential equations:
a) $2t-y+(2y-t)y'=0$ *b)* $y\mathrm{e}^t+\mathrm{e}^t\cdot y'=0$ *c)* $ty^2+y+(t^2y+t)y'=0$

1.22. Show that any separable differential equation is exact.

1.23. Show that any first order linear differential equation is exact.

1.8 Equilibrium states and stability

A first order differential equation $y'=F(t,y)$ is called *autonomous* if the right-hand side $F(t,y)$ is independent of the variable t. In other words, autonomous first order differential equations can be written $y'=F(y)$. An example of an autonomous first order differential equation is the linear equation $y'+ay=b$, where a,b are constants. This equation can be written as $y'=b-ay$, where the right-hand side $F(y)=b-ay$ is independent of t.

Equilibrium states. Let $y'=F(y)$ be an autonomous differential equation, and let y_e be a number. If $F(y_e)=0$, then we say that $y=y_e$ is an *equilibrium state* for $y'=F(y)$. If this is the case, then the constant function $y(t)=y_e$ is a solution of the differential equation $y'=F(y)$. This follows from the facts that $y'=0$ for the constant function $y(t)=y_e$, and that $F(y)=0$ at an equilibrium state $y=y_e$.

The particular solution of the differential equation $y'=F(y)$ with initial condition $y(0)=y_e$ for an equilibrium state y_e must be the constant solution $y(t)=y_e$. In other words, if we start at the equilibrium state, we will stay there for all time. This is the reason for the name equilibrium state.

Notice that it is often much easier to compute equilibrium states than to solve the differential equation. For example, the differential equation $y'+ay=b$ can be written $y'=b-ay$, and we can find equilibrium states by solving the equation

$$F(y)=b-ay=0 \quad \Rightarrow \quad y_e=\frac{b}{a}$$

Compare this with the general solution $y(t)=C\mathrm{e}^{-at}+b/a$ of $y'+ay=b$ found in Section 1.5. We notice that if $a>0$, then the limit

$$\bar{y}=\lim_{t\to\infty} y(t)=\lim_{t\to\infty}\left(C\mathrm{e}^{-at}+\frac{b}{a}\right)=\frac{b}{a}$$

is the equilibrium state of this equation. If $a<0$, then the limit above does not exist.

Another example is the logistic differential equation $y'=ry(1-y/K)$, which is autonomous with $F(y)=ry(1-y/K)$. The equilibrium states are given by

$$F(y) = ry\left(1 - \frac{y}{K}\right) = 0 \quad \Rightarrow \quad y = 0 \quad \text{or} \quad y = K$$

Therefore, this differential equation has two equilibrium states $y_e = 0$ and $y_e = K$. We may compare this with the general solution

$$y(t) = K \cdot \frac{C\,\mathrm{e}^{rt}}{1 + C\,\mathrm{e}^{rt}}$$

of $y' = ry(1 - y/K)$ found in Section 1.4. Notice that the limit

$$\bar{y} = \lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left(K \cdot \frac{C\,\mathrm{e}^{rt}}{1 + C\,\mathrm{e}^{rt}}\right) = \begin{cases} K, & r > 0 \\ 0, & r < 0 \end{cases}$$

is one of the two equilibrium states of this equation.

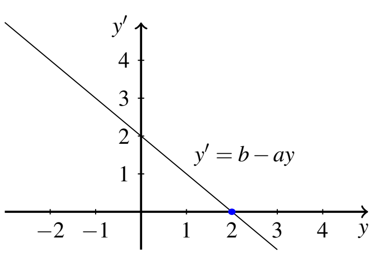
Convergence to equilibrium states
Let $y' = F(y)$ be an autonomous first order differential equation with $y = y(t)$ as a particular solution. If the limit

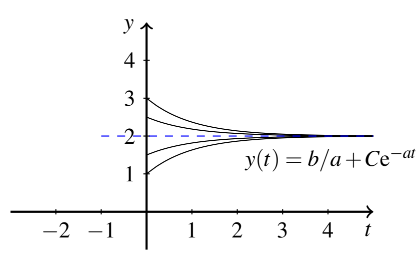
$$\bar{y} = \lim_{t\to\infty} y(t)$$

exists (and is finite), then $y = \bar{y}$ is a an equilibrium state for $y' = F(y)$.

Stability. Let $y' = F(y)$ be an autonomous first order differential equation with an equilibrium state $y = y_e$. We consider the initial value problem $y' = F(y)$, $y(0) = y_0$ when y_0 is close to the equilibrium state y_e but $y_0 \neq y_e$. If the solution $y(t)$ of the initial value problem moves away from y_e as t increases, then the equilibrium $y = y_e$ is called *unstable*. If it moves towards y_e, or at least doesn't move further away from it, then the equilibrium $y = y_e$ is called *stable*.

We consider the linear differential equation $y' + ay = b$ as an example. We have seen that it can be written $y' = b - ay$ with $F(y) = b - ay$, and if $a \neq 0$, then it has one equilibrium state $y_e = b/a$. Let us determine the stability of $y_e = b/a$.





We first consider the special case $a = 1$ and $b = 2$, and look at the diagrams shown above. The diagram on the left shows the plot of $y' = F(y)$ in the yy'-plane. This is called a *phase diagram*. The intersection with the horisontal axis is the equilibrium state $y_e = b/a = 2$. The diagram on the right is the solution curve $y(t) = b/a + C\mathrm{e}^{-at}$ drawn in the ty-plane for various initial values $y_0 \neq y_e = 2$. The equilibrium state is shown as the horisontal blue line $y = y_e = 2$.

We see that the equilibrium state $y = y_e = 2$ is stable, since the solution curves in the right-hand side diagram move towards $y = 2$ as t increases when the initial state y_0 is close to $y_e = 2$ but $y_0 \neq y_e$. We can also see this from the phase diagram on the left-hand side: At points to the right of the equilibrium state $y = 2$, the graph of $y' = F(y)$ lies under the horisontal axis, meaning that $y' < 0$ and that $y = y(t)$ will decrease. At points to the left of $y = 2$, the graph of $y' = F(y)$ lies over the horisontal axis, meaning that $y' > 0$ and that $y = y(t)$ will increase. In either case, y will move towards the equilibrium state $y = 2$ when t increases.

Using the same methods, we analyse the equilibrium state $y = b/a$ of $y' = b - ay$ when a, b are general constants with $a \neq 0$. We see that $y_e = b/a$ is stable when $a > 0$ and unstable when $a < 0$. In fact, it turns out that the slope of the tangent line of $y' = F(y)$ at the equilibrium state $y = y_e$ determines the stability:

Stability Theorem
Let $y' = F(y)$ be an autonomous first order differential equation with an equilibrium state $y = y_e$. Then we have the following:

1. If $F'(y_e) < 0$, then $y = y_e$ is a stable equilibrium state.
2. If $F'(y_e) > 0$, then $y = y_e$ is an unstable equilibrium state.

Stability of an equilibrium state $y = y_e$ is a local property, since it requires that $y(t)$ moves toward y_e when y_0 is close to y_e. We say that an equilibrium state $y = y_e$ is *globally asymptotically stable* if the particular solution $y = y(t)$ of the initial value problem $y' = F(y),\ y(0) = y_0$ moves towards y_e as t increases for all values of y_0.

For example, consider the linear differential equation $y' = b - ay$. We already know that $y_e = b/a$ is stable when $a > 0$. But since the general solution is given by $y(t) = b/a + C\mathrm{e}^{-at}$ with $C = y_0 - b/a$, it follows that

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \left(\frac{b}{a} + \left(y_0 - \frac{b}{a} \right) \mathrm{e}^{-at} \right) = \frac{b}{a}$$

for all values of y_0 when $a > 0$. This means that the equilibrium $y_e = b/a$ is globally asymptotically stable when $a > 0$.

Problems

1.24. Determine all equilibrium states of the differential equation $y' = 1 - y^2$, and determine their stability. If there are any stable equilibrium points, check if they are globally asymptotically stable.

1.25. Consider the differential equation $p' = k(d - s)$, where $p = p(t)$ is the price of a good with linear supply and demand functions

$$\begin{aligned} d &= a - bp \\ s &= c + dp \end{aligned}$$

We assume that $a, b, c, d, k > 0$ are positive constants. Find all equilibrium states for the price p, and determine their stability.

1.9 Second order differential equations

A second order differential equation contains the second derivative y'', and can often be written in the form $y'' = F(t, y, y')$ for some function F (we shall only consider second order differential equations of this type). A simple example is

$$y'' = 12t$$

We can solve this second order differential equation by simple integration in two steps. We get

$$y' = \int 12t \, \mathrm{d}t = 6t^2 + C \quad \Rightarrow \quad y = \int (6t^2 + C) \, \mathrm{d}t = 2t^3 + Ct + D$$

We notice that the general solution depends on two undetermined coefficients C, D. This is typical for second order differential equations, and implies that we need two initial conditions to determine a unique solution.

For example, the initial value problem $y'' = 12,\ y(0) = 1,\ y'(0) = 2$ has general solution $y(t) = 2t^3 + Ct + D$, with $y'(t) = 6t^2 + C$. The condition $y(0) = 1$ gives $1 = 2 \cdot 0^3 + C \cdot 0 + D$, or $D = 1$, and the condition $y'(0) = 2$ gives $2 = 6 \cdot 0^2 + C$, or $C = 2$. Therefore, the particular solution is $y(t) = 2t^3 + 2t + 1$, and it is unique.

Existence and uniqueness of solutions
Let $y'' = F(t, y, y'),\ y(t_0) = b, y'(t_0) = c$ be a second order initial value problem. If F is a C^1 function in a neighbourhood around the point (t_0, b, c), then the initial value problem has a unique solution $y = y(t)$.

Problems

1.26. Solve the differential equation $y'' = 0$.

1.27. Solve the differential equation $y'' = e^t - e^{-t}$, and find all solutions that satisfy the initial conditions $y(0) = -1$ and $y'(0) = 1$.

1.28. Solve the differential equation $y'' = 1 - y'$. Hint: Rewrite the equation as a differential equation in the variable $z = y'$.

1.10 Linear second order differential equations.

A second order differential equation $y'' = F(t, y, y')$ is *linear* if it can be written in the form

$$y'' + a(t)y' + b(t)y = h(t)$$

for functions $a(t),\ b(t),\ h(t)$. We may consider the left-hand side as $D(y)$, where D is the second order differential operator

$$D = \frac{\mathrm{d}^2}{\mathrm{d}t^2} + a(t)\,\frac{\mathrm{d}}{\mathrm{d}t} + b(t)$$

written using Leibniz' notation $\mathrm{d}y/\mathrm{d}t = y'$ and $\mathrm{d}^2y/\mathrm{d}t^2 = y''$. It operates on an input function $y = y(t)$ as

$$D(y) = \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + a(t)\,\frac{\mathrm{d}}{\mathrm{d}t} + b(t)\right) y = y'' + a(t)\,y' + b(t)\,y$$

One may show that any second order differential operator D is a linear, just as first order differential operators were shown to be linear in Section 1.6, and therefore the superposition principle applies to linear second order differential equations.

Superposition principle for linear second order differential equations
The general solution of the differential equation $y'' + a(t)\,y' + b(t)\,y = h(t)$ can be written as $y = y_h + y_p$, where y_h is the general solution of the *homogeneous* equation $y'' + a(t)\,y' + b(t)\,y = 0$, and y_p is a particular solution of the equation $y'' + a(t)\,y' + b(t)\,y = h(t)$.

Let us consider the linear second order differential equation $y'' + 3y' + 2y = 10$ as an example. By the superposition principle, its general solution is $y = y_h + y_p$. Therefore, it is enough to find the general homogeneous solution y_h and a particular solution y_p to solve this differential equation. It turns out that $y_h = C_1 \cdot e^{-t} + C_2 \cdot e^{-2t}$ and that $y_p = 5$ in this case, and we shall explain in detail how to find y_h and y_p below. It follows that

$$y = y_h + y_p = C_1 \cdot e^{-t} + C_2 \cdot e^{-2t} + 5$$

is the general solution of $y'' + 3y' + 2y = 10$.

The homogenous case. A homogeneous linear second order differential equation $y'' + a(t)\,y' + b(t)\,y = 0$ has *constant coefficients* if $a(t) = a$ and $b(t) = b$ are constants. In this case, we write the differential equation as

$$y'' + ay' + by = 0$$

We shall explain the solution method for this type of differential equation. It is based on the idea that $y = e^{rt}$ is a solution of $y'' + ay' + by = 0$ for certain values of r, and that we can determine those values of r by substituting $y = e^{rt}$ in the differential equation, using that $y' = re^{rt}$ and $y'' = r^2e^{rt}$. The left-hand side becomes

$$y'' + ay' + by = r^2e^{rt} + a(re^{rt}) + b(e^{rt}) = e^{rt}(r^2 + ar + b)$$

Therefore, $y = e^{rt}$ is a solution of the differential equation $y'' + ay' + by = 0$ if and only if $r^2 + ar + b = 0$. This is the *characteristic equation*, and it has solutions

$$r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

The number of solutions depends on the sign of the *discriminant* $\Delta = a^2 - 4b$.

Characteristic equation
Let $y'' + ay' + by = 0$ be a linear second order differential equation that is homogeneous with constant coefficients. The function $y(t) = e^{rt}$ is a solution if and only if r is a root in the characteristic equation $r^2 + ar + b = 0$.

When $\Delta > 0$, we have two distinct (real) roots $r \neq s$, and therefore $y_1 = e^{rt}$ and $y_2 = e^{st}$ are distinct solutions. Since the differential operator D is linear, it follows that any linear combination

$$y(t) = C_1 \cdot y_1 + C_2 \cdot y_2 = C_1 \cdot e^{rt} + C_2 \cdot e^{st}$$

is a solution. This is the general solution, with two undetermined coefficients.

When $\Delta = 0$, we have a double root $r = -a/2$, and therefore $y_1 = e^{rt}$ is a solution. One may show that also $y_2 = t \cdot e^{rt}$ is a solution. Since the differential operator D is linear, it follows that any linear combination

$$y(t) = C_1 \cdot y_1 + C_2 \cdot y_2 = C_1 \cdot e^{rt} + C_2 \cdot te^{rt} = (C_1 + C_2\,t)e^{rt}$$

is a solution. This is the general solution, with two undetermined coefficients.

When $\Delta < 0$, there are no real roots of the characteristic equation. But notice that formally, we can write the solutions

$$r = \frac{-a \pm \sqrt{a^2 - 4b}}{2} = \frac{-a}{2} \pm \frac{\sqrt{4b - a^2}\sqrt{-1}}{2} = \alpha + \beta \cdot \sqrt{-1}$$

with $\alpha = -a/2$ and $\beta = \sqrt{4b - a^2}/2$ since $4b - a^2 > 0$. The general solution of $y'' + ay' + by = 0$ is in this case given by

$$y(t) = e^{\alpha t} \cdot (C_1 \cos(\beta t) + C_2 \sin(\beta t))$$

with two undetermined coefficients.

General solution in the homogeneous case
Let $y'' + ay' + by = 0$ be a linear second order differential equation that is homogeneous with constant coefficients. The general solution is given by

$$\begin{aligned}
\Delta > 0: &\quad y(t) = C_1 e^{rt} + C_2 e^{st} \\
\Delta = 0: &\quad y(t) = (C_1 + C_2 t) e^{r \cdot t} \\
\Delta < 0: &\quad y(t) = e^{\alpha \cdot t} (C_1 \cos(\beta t) + C_2 \sin(\beta t))
\end{aligned}$$

As an example, let us consider the homogeneous equation $y'' + 3y' + 2y = 0$ with constant coefficients. Its characteristic equation is

$$r^2 + 3r + 2 = 0 \quad \Rightarrow \quad r = \frac{-3 \pm \sqrt{9-8}}{2} = \frac{-3 \pm 1}{2}$$

and there are two distinct characteristic roots $r = -1$ and $r = -2$. This corresponds to the case $\Delta > 0$, and the general solution of $y'' + 3y' + 2y = 0$ is therefore

$$y(t) = C_1 \cdot e^{-t} + C_2 \cdot e^{-2t}$$

The linear second order differential equation $y'' + 3y' + 2y = 10$ that we considered earlier in this section therefore has general homogeneous solution

$$y_h = C_1 \cdot e^{-t} + C_2 \cdot e^{-2t}$$

Another example is $y'' + 4y' + 4y = 0$. It has characteristic equation $r^2 + 4r + 4 = 0$, with double root $r = -2$. In this case, the general solution of the differential equation is $y(t) = (C_1 + C_2 t)e^{-2t}$.

We remark that if the homogeneous linear second order differential equation $y'' + a(t)\,y' + b(t)\,y = 0$ does not have constant coefficients, then the characteristic equation cannot be used to find the general solution.

Particular solutions. It is usually much simpler to find a particular solution of a differential equation than to find the general solution by solving the equation. A useful technique is to make an assumption about the *form* of the solution, or to “guess” a solution $y = y(t)$, and then substitute this form into the differential equation to check whether any function of the chosen form is a solution. This is called the method of *undetermined coefficients*.

For example, to find a particular solution of $y'' + 3y' + 2y = 10$, we may "guess" that it has a constant solution $y = A$. To check whether or not this satisfies the equation for any value of A, we assume that $y = A$, compute $y' = y'' = 0$, and substitute the values of y, y' and y'' into the differential equation $y'' + 3y' + 2y = 10$. This gives $2A = 10$, or $A = 5$. This means that $y = 5$ is a constant solution. We therefore say that $y_p = 5$ is a particular solution of $y'' + 3y' + 2y = 10$.

Notice that it is a good idea to "guess" a solution $y = y(t)$ that depends on one or more parameters, the undetermined coefficients. This means that we guess a form of the solution (for example a constant solution in the example above), rather than a specific solution. Also notice that it may happen that there are no solutions of the differential equation of the form we have guessed. If this happens, we must change our assumptions.

We consider the linear second order differential equation $y'' + 7y' + 12y = t$ as an example. If we try to guess a constant solution $y = A$, we get $12A = t$ when we substitute into the differential equation, and this has no solutions. Remember that A is assumed to be a constant. We notice that the problem is the non-constant function t on the right-hand side, and try to guess a linear form $y = At$ for a constant A instead. To substitute this into the diffential equation, we compute $y' = A$ and $y'' = 0$. This gives

$$0 + 7(A) + 12(At) = t \quad \Rightarrow \quad (12A)t + (7A) = t$$

We compare coefficients of the linear functions on the left and right side of the equation, and get $12A = 1$ and $7A = 0$. There is no solution for A, and this means that the differential equation does not have any particular solutions of the form $y_p = At$ either. We must change our assumption, and this time we notice that the problem is that the equation $(12A)t + (7A) = t$ has a constant term as well as the degree one term. This time, we therefore assume that the solution $y = At + B$ is a linear expression. We compute $y' = A$ and $y'' = 0$, and substitute into the differential equation:

$$0 + 7(A) + 12(At + B) = t \quad \Rightarrow \quad (12A)t + (7A + 12B) = t$$

Comparing coefficients, this gives $12A = 1$ and $7A + 12B = 0$. The first equation gives $A = 1/12$, and the second gives $12B = -7A = -7/12$, or $B = -7/144$. We find a solution for A and B, and this means that there is a particular solution of the chosen form,

$$y_p = At + B = \frac{7}{12} \cdot t - \frac{7}{144}$$

We know that $y'' + 7y' + 12y = t$ has general solution $y = y_h + y_p$ by the superposition principle. Since $y'' + 7y' + 12 = 0$ has characteristic equation $r^2 + 7r + 12 = 0$, with roots $r = -3$ and $r = -4$, the homogeneous solution is

$$y_h = C_1 e^{-3t} + C_2 e^{-4t}$$

Therefore, the general solution of $y'' + 7y' + 12y = t$ is given by

$$y = y_h + y_p = C_1 e^{-3t} + C_2 e^{-4t} + \frac{7}{12} \cdot t - \frac{7}{144}$$

From this example, we notice that it is a good idea to chose $y = y(t)$ of the same form as the right-hand side of the linear second order differential equation: Since the right-hand side is a linear expression t, we guess $y = y(t)$ is a linear expression $y = At + B$.

Method of undetermined coefficients
To find a particular solution of a linear second order differential equation $y'' + ay' + by = h(t)$, we guess a solution $y = y(t)$ and substitute it into the differential equation. We choose $y = y(t)$ such that

1. $y = y(t)$ depends on one or more undetermined coefficients
2. $y = y(t)$ has the same form as $h(t)$, $h'(t)$ and $h''(t)$

If the initial guess $y = y(t)$ does not work, we try to replace $y(t)$ with $t \cdot y(t)$, and repeat the process until we find a particular solution.

Let us reconsider the differential equation $y'' + 7y' + 12y = t$. We first look at the right-hand side $h(t) = t$ and its derivatives $h'(t) = 1$ and $h''(t) = 0$, and then choose $y = At + B$. This is an expression with undetermined coefficients that has the same form as h, h' and h'', in the sense that all these functions are special cases of a linear function $At + B$. This means that we find a guess $y = At + B$ that gives particular solutions when we substitute it into the differential equation, and we avoid trying $y = A$ and $y = At$.

Problems

1.29. Solve the initial value problem $y'' + y' - 2y = 4t, \quad y(0) = 1,\ y'(0) = 0.$

1.30. Solve the following linear second order differential equations:
a) $y'' - 4y = t + 1$ *b*) $y'' + 3y' = \mathrm{e}^{-t}$ *c*) $y'' + 5y' - 6y = t^2$

1.31. Solve the differential equation $y'' + 3y' - 4y = 2\mathrm{e}^t$.

1.32. Show that if $y'' + ay' + by = 0$ has a characteristic equation with double root r, then $y = te^{rt}$ is a solution of the differential equation.

1.33. Solve the linear first order differential equation $y' - 4y = te^t$, first using the superposition principle, the characteristic equation and the method of undetermined coefficients, and then using the integrating factor. Compare the methods.

1.34. Prove that the following second order differential operator is linear:

$$D = \frac{\mathrm{d}^2}{\mathrm{d}t^2} + a(t)\,\frac{\mathrm{d}}{\mathrm{d}t} + b(t)$$

Chapter 2
Systems of Differential Equations

2.1 Introduction to systems of differential equations

Let $y_1(t), y_2(t), \ldots, y_n(t)$ be functions in one variable t, where t represents time. A first order differential equation in $y_i(t)$ is called a *coupled* differential equation if it has the form

$$y_i' = F_i(t, y_1, y_2, \ldots, y_n)$$

for some function F_i in the $n+1$ variables $(t, y_1, y_2, \ldots, y_n)$. The growth rate of the variable $y_i = y_i(t)$ is affected by all of the variables $y_1(t), y_2(t), \ldots, y_n(t)$, in addition to the time t. For example, the differential equation

$$y_1' = y_1 + y_2$$

is coupled in the sense that y_2 will contribute to the growth rate of y_1, and therefore the solution $y_1(t)$ of the coupled differential equation will depend on $y_2 = y_2(t)$.

A *coupled system of first order differential equations* in $y_1(t), y_2(t), \ldots, y_n(t)$ is a system of differential equations in the form

$$\begin{aligned} y_1' &= F_1(t, y_1, \ldots, y_n) \\ y_2' &= F_2(t, y_1, \ldots, y_n) \\ &\vdots \\ y_n' &= F_n(t, y_1, \ldots, y_n) \end{aligned}$$

where $F_1, F_2, \ldots, F_n$ are functions in $(t, y_1, y_2, \ldots, y_n)$. We often abbreviate the name and call it a *system of differential equations*. The system is called *autonomous* if the functions $F_1, F_2, \ldots, F_n$ are independent of the time t, and only depend on the variables $(y_1, y_2, \ldots, y_n)$.

In contrast, the differential equations in Chapter 1 had the form $y_i' = F(t, y_i)$, which means that the growth rate y_i' of $y_i = y_i(t)$ only depends on the variable $y_i(t)$ and the time t. In the context of systems of differential equations, a differential equation of the form $y_i' = F(t, y_i)$ is called a *decoupled* differential equation. For decoupled differential equations, the solution will not depend on the other variables, and can often be found using the techniques of Chapter 1.

Given a system of differential equations, we may think of $(y_1, y_2, \dots, y_n)$ as an n-vector. We call it the *state vector* of the system, and write it

$$\mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix} \quad \text{or} \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

In other words, the state vector is a collection of n functions $y_1(t), y_2(t), \dots, y_n(t)$. A *solution* of the system of differential equations is a state vector $\mathbf{y} = \mathbf{y}(t)$ that satisfies all differential equations in the system. An *initial condition* for a coupled system is an initial state vector at an initial time $t = t_0$, given by $\mathbf{y}(t_0) = \mathbf{b}$ for a vector $\mathbf{b}$; in other words, it is given by the conditions

$$y_1(t_0) = b_1,\ y_2(t_0) = b_2,\ \dots,\ y_n(t_0) = b_n$$

where $b_1, b_2, \dots, b_n$ are given numbers.

Solutions of initial value problems for coupled systems
Let $y_1' = F_1(t, y_1, \dots, y_n), \dots, y_n' = F_n(t, y_1, \dots, y_n)$, $y_1(t_0) = b_1, \dots, y_n(t_0) = b_n$ be an initial value problem. If $F_1, \dots, F_n$ are C^1 functions in a neighbourhood around the point $(t_0, b_1, \dots, b_n)$, then the initial value problem has a unique solution $\mathbf{y} = \mathbf{y}(t)$.

Planar systems. Coupled systems of differential equations are called *planar* in the special case $n = 2$. In this case, there are two variables $y_1 = y_1(t)$ and $y_2 = y_2(t)$, and we can write y and z instead of y_1 and y_2. This is the simplest case of non-trivial coupled systems. Most of the examples we consider in these notes, are planar autonomous coupled systems. They can be written in the form

$$\begin{aligned} y' &= F(y, z) \\ z' &= G(y, z) \end{aligned}$$

where F, G are functions in two variables. A simple example of a planar autonomous system, is the *predator-prey system* given by

$$\begin{aligned} y' &= y(-a + bz) \\ z' &= z(c - dy) \end{aligned}$$

where a, b, c, d are positive constants. We may think of $y(t)$ and $z(t)$ as the population of two species (predator and prey) at time t, where the presence of predators has a negative impact on the growth of the prey population, and the presence of prey has a positive effect on the growth of the predator population.

Second order differential equations as coupled systems. A differential equation of second order can be rewritten as a planar coupled system of first order differential equations. Let us write the second order differential in the form $y'' = F(t,y,y')$, and let $u = y$ and $v = y'$. Then $u' = v$ and $v' = y'' = F(t,y,y') = F(t,u,v)$, and we obtain the coupled system

$$\begin{aligned} u' &= v \\ v' &= F(t,u,v) \end{aligned}$$

Let us consider the second order differential equation $y'' - 4y' + 3y = 6$ as a simple example. It can be written as $y'' = 6 - 3y + 4y'$. Using $u = y$ and $v = y'$, we can therefore rewrite it as the planar coupled system

$$\begin{aligned} u' &= v \\ v' &= 6 - 3u + 4v \end{aligned}$$

In a similar way, any n'th order differential equation can be rewritten as a coupled system of n first order differential equations in n variables.

Problems

2.1. Solve the planar system of differential equations given by

$$\begin{aligned} y' &= y + z \\ z' &= 2z \end{aligned}$$

and find the particular solutions with $y(0) = 1$ and $z(0) = 2$.

2.2. Rewrite the second order differential equation $y'' - 7y' + 12y = 4$ as a planar system of differential equations.

2.2 Linear systems of differential equations

An autonomous coupled system of first order differential equations is called *linear* if it can be written in the form

$$\begin{aligned} y_1' &= a_{11}\,y_1 + a_{12}\,y_2 + \cdots + a_{1n}\,y_n \\ y_2' &= a_{21}\,y_1 + a_{22}\,y_2 + \cdots + a_{2n}\,y_n \\ &\vdots \\ y_n' &= a_{n1}\,y_1 + a_{n2}\,y_2 + \cdots + a_{nn}\,y_n \end{aligned}$$

In other words, the coupled system is linear if and only if $F_1, \ldots, F_n$ are linear forms. Writing $\mathbf{y} = \mathbf{y}(t)$ for the state vector, and $\mathbf{y}' = \mathbf{y}'(t)$ for the vector of derivatives, we can write the system in matrix form as $\mathbf{y}' = A \cdot \mathbf{y}$, or

$$\mathbf{y}'(t) = \begin{pmatrix} y_1'(t) \\ y_2'(t) \\ \vdots \\ y_n'(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \vdots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} y_1(t) \\ y_2(t) \\ \ldots \\ y_n(t) \end{pmatrix} = A \cdot \mathbf{y}(t)$$

where A is an $n \times n$ matrix. It turns out that we can solve this system by decoupling it, and the decoupling uses the eigenvalues and eigenvectors of the matrix A.

Solution method using eigenvalues and eigenvectors. We consider the case when $\mathbf{y}' = A \cdot \mathbf{y}$ for an $n \times n$ diagonalizable matrix A. In other words, we assume that A has n eigenvalues $\lambda_1, \ldots, \lambda_n$ and n linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ with $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for $1 \le i \le n$. This implies that $P^{-1}AP = D$ when we put

$$P = \left(\mathbf{v}_1 \middle| \mathbf{v}_2 \middle| \ldots \middle| \mathbf{v}_n\right), \qquad D = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}$$

Let us introduce new variables $(w_1, w_2, \ldots, w_n)$, which we think of as an n-vector $\mathbf{w}$, with $\mathbf{w} = P^{-1}\mathbf{y}$, or $\mathbf{y} = P\mathbf{w}$. We compute the left- and right-hand side of the equation $\mathbf{y}' = A\mathbf{y}$ in terms of $\mathbf{w}$, and get

$$\mathbf{y}' = (P\mathbf{w})' = P\mathbf{w}', \quad A\mathbf{y} = (PDP^{-1})\mathbf{y} = PD\mathbf{w}$$

This implies that $\mathbf{y}' = A\mathbf{y}$ can be written in the form

$$P\mathbf{w}' = P(D\mathbf{w}) \quad \Rightarrow \quad \mathbf{w}' = D\mathbf{w} \quad \Rightarrow \quad \begin{cases} w_1' = \lambda_1 w_1 \\ w_2' = \lambda_2 w_2 \\ \ldots \\ w_n' = \lambda_n w_n \end{cases}$$

This system is *decoupled* since each first order differential equation just involves one variable. The decoupled system has solution

$$w_1 = C_1 e^{\lambda_1 t}, \quad w_2 = C_2 e^{\lambda_2 t}, \quad \ldots \quad , w_n = C_n e^{\lambda_n t}$$

since it is a linear first order (decoupled) differential equation, which we solve using the methods of Section 1.5. In matrix form, we can write the general solution as

$$\mathbf{w} = \begin{pmatrix} C_1 e^{\lambda_1 t} \\ C_2 e^{\lambda_2 t} \\ \vdots \\ C_n e^{\lambda_n t} \end{pmatrix} \quad \Rightarrow \quad \mathbf{y} = P\mathbf{w} = \big(\mathbf{v}_1 \big| \mathbf{v}_2 \big| \dots \big| \mathbf{v}_n\big) \cdot \begin{pmatrix} C_1 e^{\lambda_1 t} \\ C_2 e^{\lambda_2 t} \\ \vdots \\ C_n e^{\lambda_n t} \end{pmatrix}$$

Multiplication of the matrices in the last equation gives

$$\mathbf{y}(t) = C_1 \mathbf{v}_1 e^{\lambda_1 t} + C_2 \mathbf{v}_2 e^{\lambda_2 t} + \cdots + C_n \mathbf{v}_n e^{\lambda_n t}$$

This is the general solution of the linear system when A is diagonalizable.

General solution of linear systems of differential equations
The linear system $\mathbf{y}' = A \cdot \mathbf{y}$ of first order differential equations has general solution

$$\mathbf{y}(t) = C_1 \mathbf{v}_1 e^{\lambda_1 t} + C_2 \mathbf{v}_2 e^{\lambda_2 t} + \cdots + C_n \mathbf{v}_n e^{\lambda_n t}$$

when A is diagonalizable with n eigenvalues $\lambda_1, \dots, \lambda_n$ and n linearly independent eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ with $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for $1 \le i \le n$.

For an example, let us consider the linear system of differential equations in the variables (y,z) given by

$$\begin{aligned} y' &= 3y + 4z \\ z' &= 4y - 3z \end{aligned}$$

In this case, A is symmetric and therefore diagonalizable, and we can use the method described above. We first compute the eigenvalues of A. The characteristic equation is

$$\det(A - \lambda I) = \lambda^2 - 25 = 0$$

Therefore, the eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = -5$. Next, we find the corresponding eigenvectors. For $\lambda = 5$, the linear system becomes $-2y + 4z = 0$ and $4y - 8z = 0$. Both equations give $y = 2z$ with z free. For $\lambda = -5$, the linear system becomes $8y + 4z = 0$ and $4y + 2z = 0$. Both equations give $z = -2y$ with y free. We may therefore choose eigenvectors in E_5 and E_{-5} given by

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

This gives

$$P = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix}$$

The change of variables are given by $\mathbf{y} = P\mathbf{w}$. This gives

$$\mathbf{w} = \begin{pmatrix} u \\ v \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

In other words, $y = 2u + v$ and $z = u - 2v$. The solutions for u and v are given by

$$\mathbf{w} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} C_1 e^{5t} \\ C_2 e^{-5t} \end{pmatrix}$$

since the decoupled system in u, v is given by $u' = 5u$ and $v' = -5v$. This means that

$$\mathbf{y} = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} C_1 e^{5t} \\ C_2 e^{-5t} \end{pmatrix} = \begin{pmatrix} 2C_1 e^{5t} + C_2 e^{-5t} \\ C_1 e^{5t} - 2C_2 e^{-5t} \end{pmatrix}$$

In other words, the general solution of the system is given by $y(t) = 2C_1 e^{5t} + C_2 e^{-5t}$ and $z(t) = C_1 e^{5t} - 2C_2 e^{-5t}$.

We remark that there are some cases when A is not diagonalizable, for example when there are not enough eigenvalues (that is, some of the eigenvalues of A are not real numbers and involve square roots of negative numbers), or when an eigenvalue λ has multiplicity $m > 1$ and there are not enough eigenvectors in E_λ. Also in these cases, the system of differential equations have solutions that can be found using eigenvalues and eigenvectors. These cases are more complicated, and outside the scope of these notes.

Second order differential equations and characteristic equations. Let us consider a homogeneous linear second order differential equation $y'' + ay' + by = 0$, where a, b are constants. This differential equation can be written as $y'' = -by - ay'$, and when we set $u = y$ and $v = y'$, we obtain the planar system $u' = y' = v$ and $v' = y'' = -by - ay' = -bu - av$, which can be written in the form

$$\begin{aligned} u' &= v \\ v' &= -bu - av \end{aligned}$$

This system of differential equations can be written $\mathbf{w}' = A\mathbf{w}$ with

$$\mathbf{w}' = \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} = A \cdot \mathbf{w}$$

The characteristic equation of the matrix A is $\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0$, which gives $\lambda^2 + a\lambda + b = 0$. We notice that this coincides with the characteristic equation of $y'' + ay' + by = 0$ from Section 1.10, given by

$$r^2 + ar + b = 0$$

when $r = \lambda$. This is the reason why $r^2 + ar + b = 0$ is called the characteristic equation of the homogeneous second order linear differential equation.

Problems

2.3. Consider the planar system of differential equations given by $\mathbf{y}' = A\mathbf{y}$, where

$$A = \begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}$$

Is A diagonalizable? If it is, solve the system using the eigenvalues and eigenvectors of A.

2.4. Consider the planar system of differential equations given by $\mathbf{y}' = A\mathbf{y}$, where

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

Find the general solution of the system.

2.3 Equilibrium states and stability

Let us consider an autonomous coupled system of first order differential equations in $(y_1, y_2, \ldots, y_n)$, written as

$$\begin{aligned} y_1' &= F_1(y_1, \ldots, y_n) \\ y_2' &= F_2(y_1, \ldots, y_n) \\ &\vdots \\ y_n' &= F_n(y_1, \ldots, y_n) \end{aligned}$$

An *equilibrium state* of the system of differential equations, or a *steady state*, is a state vector $(y_1^*, y_2^*, \ldots, y_n^*)$ such that

$$F_1(y_1^*, y_2^*, \ldots, y_n^*) = F_2(y_1^*, y_2^*, \ldots, y_n^*) = \cdots = F_n(y_1^*, y_2^*, \ldots, y_n^*) = 0$$

If the system starts in an equilibrium state, it will remain there as time passes since $y_1' = y_2' = \cdots = y_n' = 0$. However, if the initial condition is $\mathbf{y}(0) = \mathbf{b}$, and $\mathbf{b}$ is close to the steady state $(y_1^*, y_2^*, \ldots, y_n^*)$, the state of the system can either approach the steady state or move away from it as time passes. We say that the steady state is *stable* in the first case, and *unstable* in the second case. It is called *globally asymptotically stable* if the state of the system approaches the steady state no matter what the initial condition is.

For example, we have solved the following linear system of differential equations in the previous section:

$$\begin{aligned} y' &= 3y + 4z \\ z' &= 4y - 3z \end{aligned}$$

We found the general solution $y(t) = 2C_1e^{5t} + C_2e^{-5t}$ and $z(t) = C_1e^{5t} - 2C_2e^{-5t}$. This system has only one steady state $(y, z) = (0, 0)$ since $\det(A) = -25 \neq 0$, which means that $A\mathbf{y} = \mathbf{0}$ has a unique solution $\mathbf{y} = \mathbf{0}$. Given an initial condition $y(0) = b_1$ and $z(0) = b_2$, we can determine values for C_1 and C_2. We notice that if $C_1 \neq 0$, then

$$y(t) = 2C_1e^{5t} + C_2e^{-5t} \to \pm\infty, \qquad z(t) = C_1e^{5t} - 2C_2e^{-5t} \to \pm\infty$$

as $t \to \infty$. There are points close to the steady state $(0,0)$ with $C_1 \neq 0$. For example, if $h > 0$ is small and the initial condition is $y(0) = 2h$ and $z(0) = h$, then $C_1 = h > 0$ and $C_2 = 0$. This means that the steady state $(0,0)$ is unstable.

Let $y'' = F(y, y')$ be a second order autonomous differential equation, and consider the corresponding system of differential equations

$$\begin{aligned} u' &= v \\ v' &= F(u, v) \end{aligned}$$

given by $u = y$ and $v = y'$. We say that $y = y_e$ is an equilibrium state of $y'' = F(y, y')$ if $(u, v) = (y_e, 0)$ is an equilibrium state of the system of differential equations. This means that y_e is an equilibrium state if and only if $F(y_e, 0) = 0$. Moreover, we define the stability of $y = y_e$ to be the stability of $(u, v) = (y_e, 0)$ as a steady state of this system.

Problems

2.5. Consider the linear system of differential equations given by $\mathbf{y}' = A\mathbf{y}$. Show that if A is a diagonalizable matrix where all the eigenvalues are negative, then $\mathbf{0}$ is a globally asymptotically stable steady state of the system.

2.6. Consider the second order differential equation $y'' + 7y' + 12y = 3$. Find the equilibrium states of this system, and determine their stability. Are the steady states globally asymptotically stable?

Appendix A
Indefinite integrals

A.1 The indefinite integral

An *antiderivative* of a function $f(t)$ is a function $F(t)$ such that $F'(t) = f(t)$. For example, the function $f(t) = 2t$ has antiderivative $F(t) = t^2$ since $(t^2)' = 2t$. Since the derivative of any constant is zero, it follows that $t^2 + C$ is also an antiderivative of $f(t) = 2t$ for any constant C.

Antiderivatives
If f is a continuous function defined on an interval I, then there exists an antiderivative $F(t)$ of $f(t)$. Moreover, all antiderivatives of $f(t)$ have the form $F(t) + C$ for a constant C.

We call $F(t) + C$ the general antiderivative of $f(t)$ in this situation. The general antiderivative is in fact an infinite family of functions, one for each value of C. We use the notation

$$\int f(t)\,\mathrm{d}t = F(t) + C$$

for the general antiderivative, and call this an *indefinite integral*. The symbol $\int$ is called the integration symbol, and the symbol dt is a formalism that means that t is the integration variable. For example, we have that

$$\int 2t\,\mathrm{d}t = t^2 + C$$

Notice that the *integration constant* C appears in all indefinite integrals. It is an undetermined coefficient, and this is why the integral is called indefinite.

A.2 Computing indefinite integrals

We can compute many indefinite integrals using *integration rules*. We start with the simplest rules, given below. Since $\int f(t)\,\mathrm{d}t = F(t) + C$ if and only if $F'(t) = f(t)$, it is easy to check that these integration rules hold by computing $F'(t)$ in each case.

Power rule
We have that

$$\int t^n \,\mathrm{d}t = \frac{1}{n+1} t^{n+1} + C \qquad \text{for } n \neq -1$$

The power rule can be used to integrate t^n when $n = 1, 2, \ldots$ is a positive integer. For example, in the case $n = 2$, we have that

$$\int t^2 \,\mathrm{d}t = \frac{1}{3} t^3 + C$$

But it can also be used to integrate t^n when n is zero, a negative integer with $n \neq -1$, or a rational number (a fraction of integers). For example, we have that

$$\int \frac{1}{t^2} \,\mathrm{d}t = \int t^{-2} \,\mathrm{d}t = \frac{1}{-1} t^{-1} + C = -\frac{1}{t} + C$$
$$\int \sqrt{t} \,\mathrm{d}t = \int t^{1/2} \,\mathrm{d}t = \frac{1}{3/2} t^{3/2} + C = \frac{2}{3} t\sqrt{t} + C$$

for $n = -2$ and $n = 1/2$. In general, it is often useful to rewrite a function as a power to integrate it. For $n = -1$, we have the following integration rule for $t^{-1} = 1/t$:

We have that

$$\int \frac{1}{t} \,\mathrm{d}t = \ln|t| + C$$

Notice that the function $1/t$ is defined for $t \neq 0$. An antiderivative should therefore also defined for $t \neq 0$. For $t > 0$, we have that $\ln t$ is an antiderivative of $1/t$, since $(\ln t)' = 1/t$. For $t < 0$, the function $\ln(-t)$ is defined since $-t > 0$, and since

$$(\ln(-t))' = \frac{1}{(-t)} \cdot (-1) = \frac{1}{t}$$

the function $\ln(-t)$ is an antiderivative of $1/t$ for $t < 0$. For $t \neq 0$, it follows that $\ln|t|$ is an antiderivative of $1/t$, since

$$\ln|t| = \begin{cases} \ln(t), & t > 0 \\ \ln(-t), & t < 0 \end{cases}$$

Integrals of linear combinations
For all expressions $u(t)$, $v(t)$ and all constants c, we have that

$$\int [u(t)+v(t)]\,\mathrm{d}t = \int u(t)\,\mathrm{d}t + \int v(t)\,\mathrm{d}t$$
$$\int [u(t)-v(t)]\,\mathrm{d}t = \int u(t)\,\mathrm{d}t - \int v(t)\,\mathrm{d}t$$
$$\int c\cdot u(t)\,\mathrm{d}t = c\cdot \int u(t)\,\mathrm{d}t$$

This means that we may integrate term by term, just as we differentiate term by term. For example, we have that

$$\int t^2-3t+2\,\mathrm{d}t = \int t^2\,\mathrm{d}t - 3\int t\,\mathrm{d}t + 2\int 1\,\mathrm{d}t = \frac{1}{3}t^3 - 3\cdot\frac{1}{2}t^2 + 2\cdot t + C$$
$$= \frac{1}{3}t^3 - \frac{3}{2}t^2 + 2t + C$$

We have computed the integrals of t^2, t and $1 = t^0$ using the power rule.

Integrals of exponential functions
We have that

$$\int \mathrm{e}^t\,\mathrm{d}t = \mathrm{e}^t + C$$
$$\int a^t\,\mathrm{d}t = a^t\cdot\frac{1}{\ln(a)} + C \qquad \text{for } a>0$$

Even though many indefinite integrals can be computed using the integration rules above, we must sometimes use more advanced integration techniques. In the next sections, we go through some of the most useful techniques.

Problems

A.1. Compute the indefinite integrals:
a) $\int(3t^2-12t)\,\mathrm{d}t$ *b)* $\int(2\mathrm{e}^t-t)\,\mathrm{d}t$ *c)* $\int t\sqrt{t}\,\mathrm{d}t$ *d)* $\int 1/t^3\,\mathrm{d}t$ *e)* $\int(t-1)^2\,\mathrm{d}t$

A.2. Compute the indefinite integral

$$\int \frac{t^3-t^2+1}{t}\,\mathrm{d}t$$

A.3 Integration by parts

To differentiate a product, we use the product rule $(uv)' = u'v + uv'$. The product rule means that uv, which is an antiderivative of $(uv)'$ by definition, is an antiderivative of the right-hand side $u'v + uv'$, or that

$$\int u'v\,\mathrm{d}t + \int uv'\,\mathrm{d}t = uv + C$$

When we solve this equation for the first integral on the left-hand side, we obtain the following formula:

Integration by parts
For any expressions $u = u(t),\ v = v(t)$, we have that

$$\int u'v\,\mathrm{d}t = uv - \int uv'\,\mathrm{d}t$$

We shall show how to use this formula to solve an indefinite integral of a product, such as the integral

$$\int t \cdot \mathrm{e}^t\,\mathrm{d}t$$

We let u' and v be the factors of the product in the integral. In the example above, we can for instance let $u' = t$ and $v = \mathrm{e}^t$. This would give $u = t^2/2$ and $v' = \mathrm{e}^t$, as we need to integrate the first factor and differentiate the other factor to find u and v'. Therefore, we get

$$\int t \cdot \mathrm{e}^t\,\mathrm{d}t = uv - \int uv'\,\mathrm{d}t = \frac{1}{2}t^2\mathrm{e}^t - \int \frac{1}{2}t^2\mathrm{e}^t\,\mathrm{d}t$$

Notice that we may choose u to be any antiderivative of t.

The method is called *integration by parts* since we replace one integral with another. The idea is of course that the new integral, on the right-hand side, should be simpler to compute than the original integral. This is not the case in the example above. However, it is possible to switch the order of the factors in the integral and choose $u' = \mathrm{e}^t$ and $v = t$, since

$$\int t \cdot \mathrm{e}^t\,\mathrm{d}t = \int \mathrm{e}^t \cdot t\,\mathrm{d}t$$

This would give $u = \mathrm{e}^t$ and $v' = 1$, and therefore

$$\int t\mathrm{e}^t\,\mathrm{d}t = t\mathrm{e}^t - \int 1 \cdot \mathrm{e}^t\,\mathrm{d}t = t\mathrm{e}^t - \int \mathrm{e}^t\,\mathrm{d}t = t\mathrm{e}^t - \mathrm{e}^t + C$$

The method works well in this case, since the new integral is easier to solve.

Another example is the integral of $\ln(t)$, which is important in itself since we often need to integrate logarithms. We may use integration by parts by rewriting the integral as

$$\int \ln(t)\,\mathrm{d}t = \int 1\cdot\ln(t)\,\mathrm{d}t$$

since this is the integral of a product. We let $u' = 1$ and $v = \ln(t)$, which gives $u = t$ and $v' = 1/t$, and therefore

$$\int \ln(t)\,\mathrm{d}t = \int 1\cdot\ln(t)\,\mathrm{d}t = t\ln(t) - \int t\cdot\frac{1}{t}\,\mathrm{d}t = t\ln(t) - \int 1\,\mathrm{d}t = t\ln(t) - t + C$$

The method works well since integrating 1 and differentiating $\ln(t)$ gives a simple integral that we can compute.

Problems

A.3. Compute the indefinite integrals:

$$a)\ \int t\ln(t)\,\mathrm{d}t \quad b)\ \int te^{t}\,\mathrm{d}t \quad c)\ \int t^2e^{t}\,\mathrm{d}t \quad d)\ \int \frac{\ln(t)}{t^2}\,\mathrm{d}t \quad e)\ \int \sqrt{t}\ln(t)\,\mathrm{d}t$$

A.4. Use integration by parts and recursion to compute the indefinite integral

$$\int \frac{\ln(t)}{t}\,\mathrm{d}t$$

A.4 Integration by substitution

To integrate a function such as $f(t) = e^{2t-3}$, it is tempting to make a change of variables $u = 2t-3$ to simplify the function to e^u. When we make this change of variables, we consider the function f as the composite function

$$f(t) = e^{2t-3} = e^u \quad \text{with } u = 2t-3$$

with *kernel* or inner function $u = u(t) = 2t-3$, and outer function e^u. Recall that to differentiate a composite function, we use the chain rule

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t} \quad \text{or} \quad f'(t) = f'(u)\cdot u'(t)$$

In the example, this would give us $(e^{2t-3})' = e^u\cdot u' = e^{2t-3}\cdot 2 = 2e^{2t-3}$ since the derivative of e^u with respect to u is e^u. Notice that we must multiply with the derivative $u' = u'(t)$ of the kernel.

If we make a substitution $u = u(t)$ in an integral, we should also take the derivative $u' = u'(t)$ of the kernel into account, and we use the formalism

$$\boxed{\mathrm{d}u = u' \cdot \mathrm{d}t} \quad \text{with } u' = u'(t)$$

to do this. For example, to compute $\int \mathrm{e}^{2t-3}\,\mathrm{d}t$, we let $u = 2t-3$. Since $u' = 2$, the formalism above gives $\mathrm{d}u = 2\cdot\mathrm{d}t$, or $\mathrm{d}t = \mathrm{d}u/2$. The integral can then be computed by substitution:

$$\int \mathrm{e}^{2t-3}\,\mathrm{d}t = \int \mathrm{e}^{u}\cdot\frac{\mathrm{d}u}{2} = \frac{1}{2}\int \mathrm{e}^{u}\mathrm{d}u = \frac{1}{2}\mathrm{e}^{u} + C = \frac{1}{2}\mathrm{e}^{2t-3} + C$$

Notice that we use the equations $u = 2t-3$ and $\mathrm{d}u = 2\cdot\mathrm{d}t$ to write the integral as an integral in the new variable u, and this means that we *divide* by $u' = u'(t)$ in the integral.

In general, we use a substitution $u = u(t)$ in an integral to transform the integral in t into one in the new variable u. This means to transform it from the form $\int f(t)\,\mathrm{d}t$ into the form $\int g(u)\,\mathrm{d}u$. The idea is of course that the integral in u should be simpler to compute.

Integration by substitution
When we use the substitution $u = u(t)$ to solve the integral $\int f(t)\,\mathrm{d}t$, we use the equations $u = u(t)$ and $\mathrm{d}u = u'\cdot\mathrm{d}t$ with $u' = u'(t)$ to rewrite the integral

$$\int f(t)\,\mathrm{d}t = \int g(u)\mathrm{d}u$$

When using this method, we try to find a substitution $u = u(t)$ such that the integral $\int g(u)\,\mathrm{d}u$ in the new variable u is simpler to solve.

Let us try to use substitution to solve the integral $\int t\ln(t^2+1)\,\mathrm{d}t$. In this case, we choose $u = t^2+1$, since this is the inner function in the last factor. The formalism $\mathrm{d}u = u'\,\mathrm{d}t$ then gives $\mathrm{d}u = 2t\,\mathrm{d}t$. We therefore obtain

$$\int t\ln(t^2+1)\,\mathrm{d}t = \int t\ln(u)\,\frac{\mathrm{d}u}{2t} = \frac{1}{2}\int \ln(u)\,\mathrm{d}u = \frac{1}{2}\,(u\ln(u) - u) + C$$

We first replace t^2+1 with u in the logarithm, and then replace $\mathrm{d}t$ with $\mathrm{d}u/u'$ using $\mathrm{d}u = u'\,\mathrm{d}t$ and $u' = 2t$. It can be difficult to know from the start if the substitution will give an integral that is easier to solve, but it is the case here it since all factors with t cancel. We use that $\int \ln(t)\,\mathrm{d}t = t\ln(t) - t + C$ from the previous section to solve the integral $\int \ln(u)\,\mathrm{d}u$. It is usual to write the answer in terms of t, the original variable, and using $u = t^2+1$, we get

$$\int t\ln(t^2+1)\,\mathrm{d}t = \frac{1}{2}\,(u\ln(u) - u) + C = \frac{1}{2}(t^2+1)\ln(t^2+1) - \frac{1}{2}(t^2+1) + C$$

Problems

A.5. Compute the integral $\int e^{1-t}\,\mathrm{d}t$, and check your answer by differentiating it.

A.6. Compute the integral when a, b are constants with $a \neq 0$:

$$\int \frac{1}{at+b}\,\mathrm{d}t$$

A.7. Compute the indefinite integrals:

$$a)\ \int 3t\sqrt{t^2+1}\,\mathrm{d}t \quad b)\ \int \frac{t}{t^2-1}\,\mathrm{d}t \quad c)\ \int 5t(t^2-1)^3\,\mathrm{d}t \quad d)\ \int \frac{2t+3}{t^2+3t+2}\,\mathrm{d}t$$

A.8. Compute the indefinite integrals

$$a)\ \int t\,\mathrm{e}^{t^2}\,\mathrm{d}t \quad b)\ \int t^3 e^{t^2}\,\mathrm{d}t \quad c)\ \int e^{\sqrt{t}}\,\mathrm{d}t \quad d)\ \int \sqrt{t}\,e^{\sqrt{t}}\,\mathrm{d}t \quad e)\ \int \frac{2\mathrm{e}^t}{\mathrm{e}^t+\mathrm{e}^{-t}}\,\mathrm{d}t$$

A.5 Integration of rational functions

A rational function is a function of the form $f(t) = p(t)/q(t)$, where $p(t), q(t)$ are polynomials in t. Examples of rational functions are

$$f(t) = \frac{t+1}{t^2-3t+2}, \qquad g(t) = \frac{t+1}{t-7}$$

There are several techniques for integrating rational functions. We shall explain how to use polynomial division and partial fraction decompositions to simplify rational expressions and make them easier to integrate.

Polynomial division. A rational function can often be simplified by polynomial division (if $n \geq m$, where n is the degree of the numerator $p(t)$ and m is the degree of the denominator $q(t)$ in the rational expression). In the example $g(t)$ above, we can write

$$g(t) = \frac{t+1}{t-7} = \frac{t-7+8}{t-7} = \frac{t-7}{t-7} + \frac{8}{t-7} = 1 + \frac{8}{t-7}$$

We say that this polynomial division has quotient 1 and remainder 8. We can use this to integrate $g(t)$, since we have

$$\int \frac{t+1}{t-7}\,\mathrm{d}t = \int \left(1 + \frac{8}{t-7}\right)\mathrm{d}t = t + 8\ln|t-7| + C$$

To integrate $8/(t-7)$, we have used the substitution $u = t-7$ with $\mathrm{d}u = \mathrm{d}t$.

More complicated polynomial divisions are often written in another form, similar to the way we write long division of integers. As an example, let us consider the rational expression

$$\frac{t^2-5}{t+3}$$

We can simplify this expression using polynomial division, which we write in the following way:

$$\begin{array}{l} \big(\ \ t^2 \qquad\ \ -5\big) : \big(t+3\big) = t-3+\dfrac{4}{t+3} \\ \underline{-t^2-3t} \\ \qquad -3t-5 \\ \qquad\ \ \underline{\ 3t+9} \\ \qquad\qquad\ 4 \end{array}$$

In general, to perform a polynomial division $p(t) : q(t)$, we divide the monomial in $p(t)$ of highest degree by the monomial in $q(t)$ of highest degree, which gives $t^2 : t = t$ in this case. This is the first term of the quotient. Next, we multiply this term with $q(t)$, and subtract the product from $p(t)$. In this case, this gives the result $(t^2-5)-t(t+3) = -3t-5$. This is the remainder so far in the process. Finally, we repeat the process, with $p(t)$ replaced by the remainder so far, until we have obtained a remainder of smaller degree than $q(t)$. In this case, the next step is to divide $-3t$ by t, which gives -3, and then multiply this term back and subtract it from $-3t-5$. This gives $-3t-5-(-3)(t+3) = 4$. Since this remainder has smaller degree than $q(t) = t+3$, the remainder is 4, and the quotient is $t-3$. Finally, we can compute the integral using the result of the polynomial division:

$$\int \frac{t^2-5}{t+3}\,\mathrm{d}t = \int \left(t-3+\frac{4}{t+3}\right)\mathrm{d}t = \frac{1}{2}t^2-3t+4\ln|t+3|+C$$

Again, we use a substitution to solve the last integral $\int 4/(t+3)\,\mathrm{d}t$. Integrals like this often occurs, and we have the following formula:

The case of linear denominator
For constants A, a, b with $a \neq 0$, we have that

$$\int \frac{A}{at+b}\,\mathrm{d}t = \frac{A}{a}\cdot\ln|at+b|+C$$

Integration by partial fraction decomposition. We must use additional methods to compute the integral of a rational expression when its denominator has higher degree than one. Sometimes, it is possible to use substitution. For example, we get

$$\int \frac{2t+3}{t^2+3t+2}\,\mathrm{d}t = \int \frac{2t+3}{u}\,\frac{\mathrm{d}u}{2t+3} = \int \frac{1}{u}\,\mathrm{d}u = \ln|u|+C = \ln|t^2+3t+2|+C$$

using the substitution $u = t^2+3t+2$ and $\mathrm{d}u = (2t+3)\mathrm{d}t$. This substitution works well because $2t+3$ in the numerator cancels against $u' = 2t+3$ in the denominator.

It would not work equally well in other cases, such as the example

$$\int \frac{3}{t^2+3t+2}\,\mathrm{d}t$$

In such cases, we simplify the integral using a *partial fraction decomposition* of the rational function: We factorize the denominator as $t^2+3t+2=(t+1)(t+2)$, and use this to find a decomposition

$$\frac{3}{t^2+3t+2}=\frac{A}{t+1}+\frac{B}{t+2}$$

for constants A,B. To find A,B such that this decomposition is valid, we multiply by the common denominator $(t+1)(t+2)$, and get

$$3=A(t+2)+B(t+1)$$

The right-hand side equals $(A+B)t+(2A+B)$. Since the equation should hold for all values of t, we need the linear expression on the right-hand side to have the same coefficients as the one on the right. Hence $A+B=0$ and $2A+B=3$. This gives $B=-A$ and $A=3$, and therefore $B=-3$. It follows that the partial fractions decomposition is

$$\frac{3}{t^2+3t+2}=\frac{3}{t+1}-\frac{3}{t+2}$$

We know how to integrate each partial fraction, and therefore we can compute the integral as

$$\int \frac{3}{t^2+3t+2}\,\mathrm{d}t=\int\left(\frac{3}{t+1}-\frac{3}{t+2}\right)\mathrm{d}t=3\ln|t+1|-3\ln|t+2|+C$$

It is usual to rewrite this answer using the fact that $\ln(a)-\ln(b)=\ln(a/b)$, and this gives

$$\int \frac{3}{t^2+3t+2}\,\mathrm{d}t=3\ln|t+1|-3\ln|t+2|+C=3\ln\left|\frac{t+1}{t+2}\right|+C$$

If the denominator $q(t)$ has higher degree than two, the integral can be solved in a similar way using partial fractions if we are able to factorize $q(t)$. Sometimes this is difficult, such as in the case $q(t)=t^3-2t+1$, and sometimes it is not possible using real numbers, such as in the case $q(t)=t^2+1$.

The case of irreducible quadratic factors in the denominator. A polynomial at^2+bt+c is called *irreducible* if it has no roots among the real numbers, and this happens if and only if $b^2-4ac<0$. The irreducible quadratic polynomials are exactly the quadratic polynomials that cannot be factorized in linear factors with real coefficients.

When the denominator $q(t)$ in a rational expression $p(t)/q(t)$ has factors that are irreducible quadratic polynomials, we must use other methods, in addition to partial fraction decompositions, to solve the integral $\int p(t)/q(t)\,\mathrm{d}t$. A typical irreducible quadratic polynomial is t^2+1, and we mention that the integral

$$\int \frac{1}{t^2+1}\,\mathrm{d}t = \arctan(t) + C$$

is given in terms of the inverse trigonometric function $\arctan(t)$. In general, when the denominator is an irreducible quadratic polynomial, we have the following formula:

The case of irreducible quadratic denominator
For constants A, a, b, c with $a \neq 0$ and $b^2 - 4ac < 0$, we have that

$$\int \frac{A}{at^2+bt+c}\,\mathrm{d}t = \frac{2A}{\sqrt{4ac-b^2}} \cdot \arctan\left(\frac{2a}{\sqrt{4ac-b^2}}t + \frac{b}{\sqrt{4ac-b^2}}\right) + C$$

For example, we have that $t^2+4t+7 = (t+2)^2+3$ is an irreducible quadratic polynomial, since $t^2+4t+7=0$ gives $(t+2)^2=-3$, and this equation has no (real) solutions. Using the formula above, we find that

$$\int \frac{1}{t^2+4t+7}\,\mathrm{d}t = \frac{2}{\sqrt{12}} \arctan\left(\frac{2t+4}{\sqrt{12}}\right) + C$$

since $4ac-b^2 = 12$ in this case.

Problems

A.9. Use the polynomial division $(t^2-3t+7) : (t-4)$ to compute the integral

$$\int \frac{t^2-3t+7}{t-4}\,\mathrm{d}t$$

A.10. Compute the indefinite integrals:

$$a)\ \int \frac{t^2-3}{t+4}\,\mathrm{d}t \quad b)\ \int \frac{t+1}{t^2+2t+4}\,\mathrm{d}t \quad c)\ \int \frac{t}{t^2-4}\,\mathrm{d}t \quad d)\ \int \frac{3}{t(3-t)}\,\mathrm{d}t$$