FORELESNING 17

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JAN 27, 2016

MET 1180 — MATEMATIKK

BI

Plan:

1. Gjennomgang: Eks MET11803 12/2015 Opps I.
2. Repetisjon: Substitusjon
3. Delvis integrasjon
4. Delbrøksoppspalting

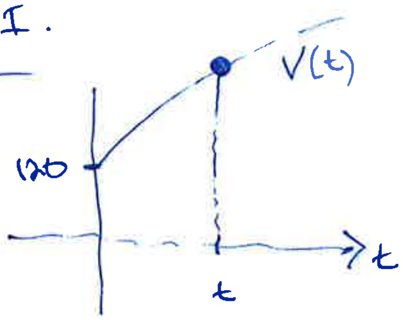
Pensum:

[SJ] 9.6 - 9.7, 9.9

[EJ] 5.3 - 5.5

① Eksamen MET11803, 12/15, Opps. I.

$$V(t) = 120 \cdot e^{\sqrt{t}/5} \qquad r = 0{,}04$$



a) $$N(t) = \frac{120 \cdot e^{\sqrt{t}/5}}{e^{rt}}$$

$$= 120 \cdot e^{\sqrt{t}/5} \cdot e^{-rt} = 120 \cdot e^{\sqrt{t}/5 - rt} = 120 e^{u},$$

$$u = \sqrt{t}/5 - rt$$

$$N'(t) = 120 \cdot e^{u} \cdot u'$$

$$= \underbrace{120 \cdot e^{\sqrt{t}/5 - rt}}_{\text{positive}} \cdot \left(\frac{1}{5} \cdot \frac{1}{2\sqrt{t}} - r \right) = 0$$

$$\frac{1}{10\sqrt{t}} - r = 0$$

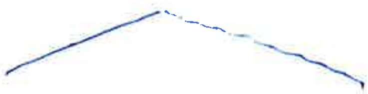
$$\frac{1}{10\sqrt{t}} = r = 0{,}04$$

$$2{,}5 = \frac{1}{0{,}4} = \frac{1}{10 \cdot 0{,}04} = \sqrt{t}$$

$$t = 2{,}5^2 = \underline{6{,}25}$$

t=6,25

N' ++++++ 0 —— − —— t



$t = 6{,}25$ gir maksimal nåverdi

b) $V(T) = 240$ $\qquad 2 \times 120 = 2V(0)$

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$$\frac{\cancel{120} e^{\sqrt{T}/5}}{\cancel{120}} = \frac{240}{120} = 2$$

$$\sqrt{T}/5 = \ln(2)$$

$$\sqrt{T} = 5 \cdot \ln(2)$$

$$T = 25 \cdot (\ln 2)^2 \simeq \underline{12{,}0 \text{ år}}$$

$$V(t) = 480$$

$$\frac{\cancel{120} \cdot e^{\sqrt{t}/5}}{\cancel{120}} = \frac{480}{120} = 4$$

$$\ln\left(e^{\sqrt{t}/5}\right) = \ln(4)$$

$$\sqrt{t}/5 = \ln(4)$$

$$\sqrt{t} = 5 \cdot \ln(4)$$

$$t = 25 \cdot (\ln 4)^2 \simeq \underline{48 \text{ år}}$$

$$= 25 \cdot (\ln 2^2)^2 = 25 \cdot (2 \ln 2)^2$$

$$= 25 \cdot 4 \cdot (\ln 2)^2 = \underline{4T}$$

Skal komme fram til

$$t = T + 3T = 4T$$

$$= 4 \cdot 25 (\ln 2)^2$$

$$\simeq 48 \text{ år}$$

Stemmer!

② Repetisjon: Substitusjon

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Eks:

$$\int \frac{2x}{x^2-1}\,dx =$$

$$\boxed{\begin{aligned} u &= x^2-1 \\ du &= 2x\,dx \end{aligned}}$$

Formel:
$du = u' \cdot dx$

$$= \int \frac{\cancel{2x}}{u} \cdot \frac{du}{\cancel{2x}} = \int \frac{1}{u}\,du$$

Huskeregel:
$\frac{du}{dx} = u'$

$$= \ln|u| + C = \underline{\ln|x^2-1| + C}$$

Substitusjon:

① Velg $u = u(x)$

② Likning: $du = u'\,dx$

③ Skriv om integralet til formen $\int g(u)\,du$

Eks:

$$\int x \cdot \ln(x^2+1)\,dx = \int \cancel{x} \cdot \ln(u) \cdot \frac{du}{2\cancel{x}}$$

$$\boxed{\begin{aligned} u &= x^2+1 \\ du &= 2x \cdot dx \end{aligned}}$$

$$= \int \frac{1}{2}\ln(u)\,du = \frac{1}{2}\int \ln(u)\,du = \ldots$$

Ex:

$$\int \frac{x}{1+\sqrt{x}}\,dx = \int \frac{x}{1+u}\cdot 2\sqrt{x}\,du$$

$x = u^2$ $\longrightarrow$ $\boxed{\begin{aligned} u &= \sqrt{x} \\ du &= \frac{1}{2\sqrt{x}}\,dx \end{aligned}}$ $\qquad dx = \frac{du}{u'}$

$$= \int \frac{2\sqrt{x}\cdot x}{1+u}\,du = \int \frac{2u\cdot u^2}{1+u}\,du$$

$$= \int \frac{2u^3}{1+u}\,du = \ldots$$

Alt: $\boxed{\begin{aligned} u &= \sqrt{x}+1 = 1+\sqrt{x} \\ du &= \frac{1}{2\sqrt{x}}\,dx \end{aligned}}$ $\longrightarrow$ $\begin{aligned} u-1 &= \sqrt{x} \\ x &= (u-1)^2 \end{aligned}$

$$\int \frac{x}{1+\sqrt{x}}\,dx = \int \frac{x}{u}\cdot 2\sqrt{x}\,du = \int \frac{2x\sqrt{x}}{u}\,du$$

$$= \int \frac{2\cdot(u-1)^2(u-1)}{u}\,du = 2\int \frac{(u-1)^3}{u}\,du$$

$$= 2\int \frac{u^3-3u^2+3u-1}{u}\,du = 2\int \left(u^2-3u+3-\frac{1}{u}\right)du$$

$$= 2\left(\frac{1}{3}u^3 - \frac{3}{2}u^2 + 3u - \ln|u|\right) + C$$

$$= \frac{2}{3}u^3 - 3u^2 + 6u - 2\ln|u| + C$$

$$= \frac{2}{3}(\sqrt{x}+1)^3 - 3(\sqrt{x}+1)^2 + 6(\sqrt{x}+1) - 2\ln|\sqrt{x}+1| + C$$

Newton's binomial formel:

$$(u-1)^3 = u^3 + 3u^2\cdot(-1) + 3u(-1)^2 + (-1)^3$$
$$= u^3 - 3u^2 + 3u - 1$$

③ Delvis integrasjon

$$\boxed{\int u' \cdot v \, dx = u \cdot v - \int u \cdot v' \, dx}$$

Produktregelen for derivasjon "baklengs":

$$(u \cdot v)' = u'v + uv'$$

$$uv = \int u'v \, dx + \int uv' dx$$

$$\Downarrow$$

$$\int u'v \, dx = uv - \int uv' dx$$

Eks:

$$\int xe^x dx = uv - \int uv' dx$$

$$\cancel{\boxed{\begin{matrix} u = \frac{1}{2}x^2 & v = e^x \\ u' = x & v' = e^x \end{matrix}}}$$

$$\cancel{= \frac{1}{2}x^2 e^x - \int \frac{1}{2}x^2 e^x dx}$$

$$= \int e^x \cdot x \, dx = uv - \int uv' dx$$

$$\boxed{\begin{matrix} u = e^x & v = x \\ u' = e^x & v' = 1 \end{matrix}}$$

$$= e^x \cdot x - \int e^x \cdot 1 \, dx$$

$$= xe^x - \int e^x dx$$

$$\underline{\underline{= xe^x - e^x + C}}$$

Eks:

$$\int x \cdot \ln x \, dx = uv - \int u \cdot v' \, dx$$

$$\boxed{\begin{array}{ll} u = \frac{1}{2}x^2 & v = \ln x \\ u' = x & v' = 1/x \end{array}}$$

$$= \frac{1}{2}x^2 \cdot \ln x - \int \frac{1}{2}x^2 \cdot \frac{1}{x} \, dx$$

$$= \frac{1}{2}x^2 \cdot \ln x - \frac{1}{2} \int x \, dx$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2} \cdot \frac{1}{2}x^2 + C$$

$$= \underline{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C}$$

Eks:

$$\int x \cdot \ln x \, dx = \int x \cdot u \cdot x \cdot du$$

$$\boxed{\begin{array}{l} u = \ln x \\ du = \frac{1}{x} dx \end{array}} \quad x = e^u$$

$$= \int x^2 u \, du = \int (e^u)^2 \cdot u \, du$$

$$= \int u \cdot e^{2u} \, du = \frac{1}{2}ue^{2u} - \frac{1}{4}e^{2u} + C$$

$$= \frac{1}{2}\ln x \, e^{2\ln x} - \frac{1}{4}e^{2\ln x} + C = \underline{\frac{1}{2}x^2 \cdot \ln x - \frac{1}{4}x^2 + C}$$

Eks:

$$\int x \cdot e^{2x} \, dx = \frac{1}{2}e^{2x} \cdot x - \int \frac{1}{2}e^{2x} \cdot 1 \, dx$$

$$\boxed{\begin{array}{ll} u = \frac{1}{2}e^{2x} & v = x \\ u' = e^{2x} & v' = 1 \end{array}}$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x} dx = \frac{1}{2}xe^{2x} - \frac{1}{2} \cdot \left(\frac{1}{2}e^{2x}\right) + C$$

$$= \underline{\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C}$$

Eks:

$$\int \ln x \, dx = \int 1 \cdot \ln x \, dx$$

$$= x \ln x - \int x \cdot \frac{1}{x} \, dx = x \cdot \ln x - \int 1 \, dx$$

$$\boxed{\begin{array}{ll} u = x & v = \ln x \\ u' = 1 & v' = 1/x \end{array}}$$

$$= \underline{x \ln x - x + C}$$

$$\boxed{\int \ln x \, dx = x \ln x - x + C}$$

Eks:

$$\int x^2 e^x \, dx = x^2 e^x - \int 2x \cdot e^x \, dx$$

$$\boxed{\begin{array}{ll} u = e^x & v = x^2 \\ u' = e^x & v' = 2x \end{array}}$$

$$= x^2 e^x - \left(2x e^x - \int 2 e^x \, dx \right)$$

$$\boxed{\begin{array}{ll} u = e^x & v = 2x \\ u' = e^x & v' = 2 \end{array}}$$

$$= \underline{x^2 e^x - 2x e^x + 2 e^x + C}$$

$$= \underline{(x^2 - 2x + 2) e^x + C}$$

Eks: $\int \frac{\ln x}{x}\,dx =$

Alt I: $\int \frac{1}{x}\cdot \ln x\,dx = (\ln x)^2 - \int \frac{1}{x}\cdot \ln x\,dx$

$$\boxed{\begin{array}{ll} u = \ln x & v = \ln x \\ u' = 1/x & v' = 1/x \end{array}}$$

$$2\cdot \int \frac{1}{x}\ln x\,dx = (\ln(x))^2 + C$$

$$\int \frac{1}{x}\ln x\,dx = \frac{(\ln x)^2}{2} + C$$

Alt 2: $\int \frac{1}{x}\ln x\,dx = \int \cancel{\frac{1}{x}}\cdot u\cdot \cancel{x}\,du$

$$\boxed{\begin{array}{l} u = \ln x \\ du = \frac{1}{x}\,dx \end{array}}$$

$$= \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C$$

④ Delbrøksoppspaltning

(Integrasjon av rasjonale funksjoner)

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Derivasjon:

$(uv)' = u'v + uv'$

$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

Integrasjon:

$\int u'v\,dx = uv - \int uv'\,dx$

?

Eks: $\int \frac{2u^3}{1+u}\,du = ?$

Konklusjon: Vi må bruke andre metoder for å integrere rasjonale funksjoner..

i) Polynomdivisjon.

Eks: $\int \frac{2x^3}{1+x}\,dx = \int \left(2x^2 - 2x + 2 + \frac{-2}{x+1}\right) dx$

$$
\begin{array}{l}
\quad 2x^3 \quad : \quad (x+1) = \underline{2x^2 - 2x + 2} \\
\underline{-(2x^3 + 2x^2)} \qquad\qquad\quad \text{kvotient} \\
\quad -2x^2 \\
\underline{-(-2x^2 - 2x)} \\
\qquad 2x \\
\quad \underline{-(2x + 2)} \\
\qquad -2 \quad \text{Rest}
\end{array}
$$

$$= 2 \cdot \frac{1}{3}x^3 - 2 \cdot \frac{1}{2}x^2 + 2x + \int \frac{-2}{x+1}\,dx$$

$$\boxed{\begin{array}{l} u = x+1 \\ du = 1 \cdot dx \end{array}}$$

$$= \frac{2}{3}x^3 - x^2 + 2x + \int \frac{-2}{u} \cdot \frac{du}{1}$$

$$= \frac{2}{3}x^3 - x^2 + 2x - 2 \int \frac{1}{u}\,du$$

$$= \frac{2}{3}x^3 - x^2 + 2x - 2\ln|u| + C$$

$$= \underline{\frac{2}{3}x^3 - x^2 + 2x - 2\ln|x+1| + C}$$

Polynomdivisjon:

$\int \frac{p(x)}{q(x)}\,dx$ kan forenkles ved polynomdivisjon dersom grad til $p(x)$ er minst like høy som graden til $q(x)$.

$\Rightarrow$ Kan redusere integrasjon til integral av

$\boxed{\text{Polynom}}$ + $\boxed{\text{rasjonal funksjon med graden til teller mindre enn graden til nevner.}}$

↑ lett å integrere

↑ ?

Nevner er lineær = første grad:

$$\int \frac{A}{ax+b}\,dx = \int \frac{A}{u} \cdot \frac{du}{a} = \frac{A}{a} \cdot \int \frac{1}{u}\,du$$

$$\boxed{\begin{array}{l} u = ax+b \\ du = a \cdot dx \end{array}}$$

$$= \frac{A}{a} \ln|u| + C$$

$$= \underline{\frac{A}{a} \ln|ax+b| + C}$$

$$\boxed{\begin{gathered}\underline{\text{Formel:}} \quad A, a, b \text{ konst}, \quad a \neq 0 \\ \int \frac{A}{ax+b}\,dx = \frac{A}{a}\cdot \ln|ax+b| + C\end{gathered}}$$

Nevner har grad større enn 1:

Eks:

$$\int \frac{2}{1-x^2}\,dx = \cancel{\int \frac{2}{u}\cdot\frac{du}{-2x} = \int \frac{1}{xu}\,du}$$

$$\boxed{\begin{gathered}u = 1-x^2 \\ du = -2x\,dx\end{gathered}}$$

Delbrøksoppspalting:

$$\boxed{\begin{gathered}\underline{\text{Faktoriserer nevner:}} \\ 1-x^2 = (1+x)(1-x)\end{gathered}}$$

$$\frac{2}{1-x^2} = \frac{2}{(1+x)(1-x)} = \frac{A}{1+x} + \frac{B}{1-x}$$

(A, B ukjente konstanter)

Hvordan finne A og B:

- Multiplikasjon med fellesnevner $(1+x)(1-x)$

$$\frac{2}{(1+x)(1-x)} = \frac{A}{1+x} + \frac{B}{1-x} \quad \Big| \cdot (1+x)(1-x)$$

$$2 = A(1-x) + B(1+x)$$

← skal være like uttrykk for alle verdier av x. (identitet)

$$= -Ax + Bx + A + B$$

$$0\cdot x + 2 = \underbrace{(-A+B)}_{=0}x + \underbrace{(A+B)}_{=2}$$

- sammenligner koeffisienter

$$\boxed{\begin{gathered}-A+B = 0 \\ A+B = 2\end{gathered}}$$

Innsettingsmetoden:

$$-A+B=0 \qquad \Rightarrow \quad B=A \qquad \underline{B=1}$$
$$A+B=2 \qquad\qquad \Downarrow$$
$$\overline{}\qquad A+(A)=2$$
$$2B=2 \qquad\qquad 2A=2$$
$$\underline{B=1} \qquad\qquad \underline{A=1}$$
$$A=1$$

(eliminasjonsmetode)

$$\frac{2}{1-x^2} = \frac{2}{(1+x)(1-x)} = \frac{A}{1+x} + \frac{B}{1-x} = \underline{\frac{1}{1+x} + \frac{1}{1-x}}$$

delbrøksoppspaltning

$$\frac{1}{1+x} + \frac{1}{1-x} = \frac{1\cdot(1-x)}{(1+x)(1-x)} + \frac{1\cdot(1+x)}{(1-x)(1+x)} = \frac{1-\cancel{x}+1+\cancel{x}}{(1+x)(1-x)}$$

$$= \underline{\frac{2}{1-x^2}}$$

$$\int \frac{2}{1-x^2}\,dx = \int\left(\frac{1}{1+x} + \frac{1}{1-x}\right)dx = \underline{\underline{\ln|1+x| - \ln|1-x| + C}}$$

Alt. metode:

(delbrøksoppspaltning)

$$\frac{2}{1-x^2} = \frac{2}{(1+x)(1-x)} = \frac{A}{1+x} + \frac{B}{1-x} \quad \Big|\cdot(1+x)(1-x)$$

$$2 = A\cdot(1-x) + B(1+x)$$

Uttrykkene skal være like for alle x, så spesielt for x=1,-1.

Bruker to x-verdier fordi vi trenger to ligninger og x=-1, x=1 gir enkelt regning

Sett inn x=1,-1:

$x=1$: $2 = A\cdot 0 + B\cdot 2 = 2B \Rightarrow \underline{B=1}$

$x=-1$: $2 = A\cdot 2 + B\cdot 0 = 2A \Rightarrow \underline{A=1}$

Sjekk: Nødvendig når vi bruker denne metoden

$$\frac{1}{1+x} + \frac{1}{1-x} = \frac{1\cdot(1-x)+1\cdot(1+x)}{(1+x)(1-x)} = \frac{1-x+1+x}{1-x^2} = \frac{2}{1-x^2} \quad \text{ok.}$$